K-Stability (Busan)
Winter School

1. History:

existence of Kähler-Einstein metric.

$$\mathrm{Ric}\,\omega = \lambda\omega \quad \text{since } [\mathrm{Ric}\,\omega] = c_1(T_X)$$

we know K_X has to be >0, 0, <0.

>0 (Aubin, Yau)
$=0$ (Yau) } always exists.

<0 not always exists. let X be a Fano

(Futaki) $\mathrm{Aut}(X)$ is reductive.

Siu: if $|\mathrm{Aut}(X)| < \infty$. X has KE?

(Wrong)

Yau: if T_X is polystable in GIT sense

+ reductive auto $\Longrightarrow$ KE ?

(Not known)

(Tian 97) formulated an (analytic) condition called K-stable, and conjecture

K-polystable $\Longleftrightarrow$ KE.

(Donaldson 01) gives an algebraic formulat call it K-stable

Yau-Tian-Donaldson Conj:

X Fano X has KE iff X is K-polystable.

Proved by Chen-D-Sun, and Tian

Algebraic study: (New World!)

- Ross-Thomas: completely alg treatment deformation to normal cone

- birational geometry (MMP)

  Odaka, Li-Xu, Fujita and others....

2 Definitions: (X, L) L very ample, X proj

$$X \hookrightarrow \mathbb{P}^N \quad f^*\mathcal{O}(1) = L$$

Check whether (X, L) is GIT, i.e. $[X]$ is GIT stable under $PGL(N+1)$

Hilbert-Mumford Criterion: $\mathbb{C}^*$ acts on $[X]$

$\lim_{t\to 0} [X_t] = [X_0]$. the Chow weight or Hilbert weight ≥ 0

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbb{C} \end{array} \qquad H^0(\mathcal{X}, \mathcal{L}^k) / tH^0(\mathcal{X}, \mathcal{L}^k)$$

$$\cong H^0(X_0, L^k)$$

total weight is $w(k)$

$$w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$$

rank $P(k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$

$$r w(k) P(1) - k P(k) w(1)$$

replace L by L^r $\quad w(k) r P(r) - k P(k) w(r)$

$$= \sum_{i=0}^{n+1} e_i(r) k^r$$

e_i of degree $n+1$, e_{n+1} of degree n.

e_{n+1} leading term $=$ coeff of $r^n k^{n+1}$

$= a_0 b_1 - a_1 b_0$

$:= \mathrm{Fut}(\mathcal{X}, \mathcal{L})$ (or $\mathrm{DF}(\mathcal{X}, \mathcal{L})$)

Fix r s.t $\forall$ $W_{r,k}$ for $k \gg 0$. Hilbert stab

$r \gg 0$ $\forall W_{r,k} \geq 0$ asymto Hilbert stab

Fix r $\forall$ $e_{n+1}(r) \geq 0$ Chow stability

$r \gg 0$ $\forall$ $e_{r+1}(r) \geq 0$ asym Chow

$\mathrm{Fut} \geq 0$ K-semistable

$\mathrm{Fut} \geq 0$ and $\mathrm{Fut} = 0 \iff \mathcal{X} \simeq X \times \mathbb{A}^1$ outside codim 2

K-polystable

K-stable $=$ K-poly $+$ $|\mathrm{Aut}(X, L)| < \infty$

3 Intersection formula (Wang, Odaka)

$$P(m) = H^0(X, L^m)$$

$$= \frac{L^n}{n!} m^n - \frac{L^{n-1} \cdot K_X}{2(n-1)!} m^{n-1} + O(m^{n-2})$$

$$b_0 = \frac{L^n}{n!} \qquad b_1 = - \frac{L^{n-1} \cdot K_X}{2(n-1)!}$$

$$X \times \mathbb{P}^1 \setminus \{0\} \cup \mathcal{X} \quad \text{glue along } X \times \mathbb{C}^* \text{ to get } \overline{\mathcal{X}}$$

$$\downarrow$$

$$\mathbb{P}^1 \setminus \{0\} \cup \mathbb{C}$$

Ex: $t \cdot ([z_0, z_1], \lambda(z_0, z_1)) \quad (\in (\mathbb{P}^1, \mathcal{O}(-1)))$

$$= ([z_0, tz_1], \lambda(z_0, tz_1))$$

weights $\mathcal{O}_{\mathbb{P}^1}(-1)|_{0} = 0$

$\mathcal{O}_{\mathbb{P}^1}(-1)|_{\infty} = 1 \qquad \mathcal{M} = L + N f^{-1}(\infty)$

$$0 \to H^0(\overline{\mathcal{X}}, \overline{\mathcal{M}}^{\otimes k}(-\overline{X}_0)) \to H^0(\overline{\mathcal{X}}, \overline{\mathcal{M}}^{\otimes k}) \to H^0(\overline{\mathcal{X}}_0, \overline{\mathcal{M}}^{\otimes k}|_{X_0}) \to 0$$

$$\underset{A}{\parallel} \qquad \underset{B}{\parallel} \qquad \underset{C}{\parallel}$$

$$0 \to H^0(\overline{\mathcal{X}}, \overline{\mathcal{M}}^{\otimes k}(-\overline{X}_\infty)) \to H^0(\overline{\mathcal{X}}, \overline{\mathcal{M}}^{\otimes k}) \to H^0(\overline{\mathcal{X}}_\infty, \overline{\mathcal{M}}^{\otimes k}|_{\overline{X}_\infty}) \to 0$$

$$\underset{A}{\parallel} \qquad \underset{B}{\parallel} \qquad \underset{D}{\parallel}$$

$$w_C = d_A + w_D \qquad d_A = d_B - d_C$$

$$w_D = -kN \dim H^0(X, L^{\otimes k})$$

$$a_0 = \frac{\overline{\mathcal{M}}^{n+1}}{(n+1)!} - N b_0 = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!}$$

$$a_1 = \frac{1}{2}\left(-K_{\overline{\mathcal{X}}}\right) \cdot \frac{\overline{\mathcal{M}}^{n}}{n!} - N b_1 - b_0 = \frac{1}{2} \frac{(-K_{\overline{\mathcal{X}}}) \cdot \overline{\mathcal{L}}^{n}}{n!} - b_0$$

$$a_0b_1 - a_1b_0 = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!}\frac{1}{2}\frac{(-K_X)\cdot L^{n-1}}{(n-1)!} - \frac{L^n}{n!}\left(\frac{1}{2}\frac{(-K_{\mathcal{X}})\mathcal{L}^n}{n!} - \frac{L^n}{n!}\right)$$

$$= \frac{1}{2\,n!\,(n+1)!}\left(n\,\overline{\mathcal{L}}^{n+1}(-K_X)\cdot L^{n-1} + (n+1)\left(K_{\overline{\mathcal{X}}/\mathbb{P}^1}\cdot\overline{\mathcal{L}}^n\right)\cdot L^n\right)$$

if $L = -rK_X$

$$= \frac{L^n}{2\,n!\,(n+1)!}\left(\frac{n}{r}\,\overline{\mathcal{L}}^{n+1} + (n+1)K_{\overline{\mathcal{X}}/\mathbb{P}^1}\overline{\mathcal{L}}^n\right)$$

/C projective smooth curve Paul-Tian

$$CM(\mathcal{X}/C, \mathcal{L}) = \frac{L^n}{2\,n!\,(n+1)!}\left(\frac{n}{r}\,\overline{\mathcal{L}}^{n+1} + (n+1)K_{\overline{\mathcal{X}}/C}\overline{\mathcal{L}}^n\right)$$

Philosophy: fix $\begin{array}{c}\mathcal{X}^0\\ \downarrow \\ C^0 = C\setminus\{0\}\end{array}$ consider $\begin{array}{ccc}\mathcal{X} & \supset & \mathcal{X}^0\\ \downarrow & & \downarrow\\ C & \supset & C^0\end{array}$

$CM(\mathcal{X}, C)$ smaller, X_0 more stable

Conjecture: if X_η is K-semistable. $\exists$ a unique S-equ class X_0 s.t $CM(\mathcal{X}/C, -K_{\mathcal{X}}) \leq CM(\mathcal{X}'/C, \mathcal{L}')$ (after a suitable base change of C)

"=" iff X_0 and X_0' are K-semistable and they must be S-equivalent.

Ex: consider the conj $\mathbb{C}^*$-equivariant family.
for

Question: using K-stability to construct a moduli space (algebraically !!).

5. Birational geometry

Goal: change the models to lower Fut/CM.

minimal model Program / birational geometry.

Def of sing. X normal. $\mathbb{Q}$-gorenstein. i.e

$\exists\ m>0$ s.t mK_X is Cartier

$f: Y \to X$ s.t $(Y, Ex(f))$ is snc.

$$f^*K_X + \Sigma a_i E_i = K_Y$$

if $\begin{cases} a_i > -1 & \text{klt} \\ a_i \geq -1 & \text{log canonical} \end{cases}$

One MMP construction: lc modification

(X, Δ) X normal coef of $\Delta \in [0,1]$

Find $Y \to X$ s.t $\mathrm{Ex}(f) = E$

$$\begin{cases} K_Y + E + f_*^{-1}\Delta \text{ ample}/X \\ (Y, E + f_*^{-1}\Delta) \text{ log can} \end{cases}$$

(Odaka-X.) if $K_X + \Delta$ $\mathbb{Q}$-Cartier. s.t Y exists.

called lc modification. it's unique

Thm (Odaka)

- If (X, L) is K-semistable. X is normal $\mathbb{Q}$-gorenstein, then X is lc
- If X is K-semistable Fano $\Rightarrow$ X is klt

pf: $Y \to X$ lc modifi

$$\underbrace{K_Y + E}_{\text{ample}/X} \sim \sum \underbrace{(1 + a_i(X; E_i))E_i}_{F}$$

$f_*(mF) = I$ m sufficiently divisible

$Bl_I X = Y$

consider $\overline{I + t^k}$ blow-up

$$\begin{array}{ccc} \mathcal{X} \to X \times A^1 & \to & X \times A^1 \\ \downarrow & & \downarrow \\ A^1 & \xrightarrow[t^k]{} & A^1 \end{array}$$

we choose k s.t the fiber is reduced.

to compute $(\mathcal{X}, \underbrace{(f^* p_1^* L - tS)}_{L_t})$

$$\mathrm{Fut} = n\overline{L}_t^{n+1}\left(-K_X\cdot L^{n-1}\right) + (n+1)\left(K_{\mathcal{X}/\mathbb{P}^1}\cdot L_t^n\right)\cdot L^n$$

$\operatorname{codim}_X f(S) = a$

$$= O(t^{a+1}) + t^a (n+1) L^n \, G\cdot(-S)^a (f^* L)^{n-a}$$

$$< 0$$

$$K_{\mathcal{X}/X\times\mathbb{A}^1} = \sum (a_i+1) G_i$$

$$= G < 0$$

$$\operatorname{supp}(G) = \operatorname{supp}(S)$$

Ex: Show K-semistable Fanos are klt.

6. special degeneration

Tian and Donaldson's definitions.

Def: if $\mathcal{L} = -rK_{\mathcal{X}}$ and X_0 is klt.

we call $\mathcal{X}$ a special TC.

Thm: (Li-X, Tian's Conj): to test K-(semi,poly)

we only need to test on special tc.

More precisely: $\forall$ TC $(\mathcal{X}, \mathcal{L})$ $\exists$ a special TC $(\mathcal{X}^s, -K_{\mathcal{X}^s})$

s.t. $\mathrm{Fut}(\mathcal{X}^s) \le m\,\mathrm{Fut}(\mathcal{X}, \mathcal{L})$

and "=" holds iff $(\mathcal{X}, \mathcal{L})^{\mathrm{norm}}$ is a special TC.

idea: take a sequence of birational modifications

$\mathcal{X} \dashrightarrow \mathcal{X}^s$ s.t that we can track the change of Fut

step 0.

Pf: $\mathcal{X}' \longrightarrow \mathcal{X} \times_{A'} A'$ s.t $\mathcal{X}'_0$ is red

$$\mathrm{Fut}(\mathcal{X}', \mathcal{L}') \leq m \cdot \mathrm{Fut}(\mathcal{X}, \mathcal{L})$$

("=" holds if $\mathcal{X}_0$ is reduced)

step 1.

$\mathcal{X}$ take the lc modification of $(\mathcal{X}, \mathcal{X}_0)$

$\mathcal{X}^{lc}$ replace by $f^*\mathcal{L} + tE$ $\quad E = K_{\mathcal{X}^{lc}} + f^*\mathcal{L}$

$$\mathrm{Fut} \doteq n\mathcal{L}_t^{n+1} + K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \mathcal{L}_t^n (n+1)$$

$$\frac{d\,\mathrm{Fut}}{dt} = n(n+1)\mathcal{L}_t^{n-1} E (\mathcal{L}_t + K_{\bar{\mathcal{X}}/\mathbb{P}^1})$$

$$= (1+t)n(n+1)\mathcal{L}_t^{n-1} E^2 < 0 \quad (\text{if } E \neq 0)$$

$$\Rightarrow \mathrm{Fut}(\mathcal{X}^{lc}, \mathcal{L}_t) \leq \mathrm{Fut}(\mathcal{X}, \mathcal{L})$$

$(\mathcal{X}^{lc}, \mathcal{X}^{lc}_0)$ is log canonical

Step 2. We want to run MMP s.t X_t doesn't change.

run MMP with scaling

$H = aL - K_X \quad a \gg 0$

run MMP with scaling by H / A'

$t_0 = 1 > t_1 \geq \cdots \geq t_m$ with

$X_0 = X^{lc} \dashrightarrow X_1 \cdots \dashrightarrow X_m$

s.t $K_{X_i} + t_i H_i$ is nef on X_i

So $t_m = \tau(K_{X_m}, H_m)$

$\Rightarrow K_{X_m} + t_m H_m \sim 0$ / A'

we assume $t_{m-1} > t_m$

$\Rightarrow$ on X_m and H_m is big and nef

take $\mathrm{proj}(X/A', H_m)$

$$\begin{cases} (X^{an}, X_0^{an}) \text{ is } lc \\ -K_{X^{an}} = L \end{cases}$$

The same calculation shows Fut $\geq$.

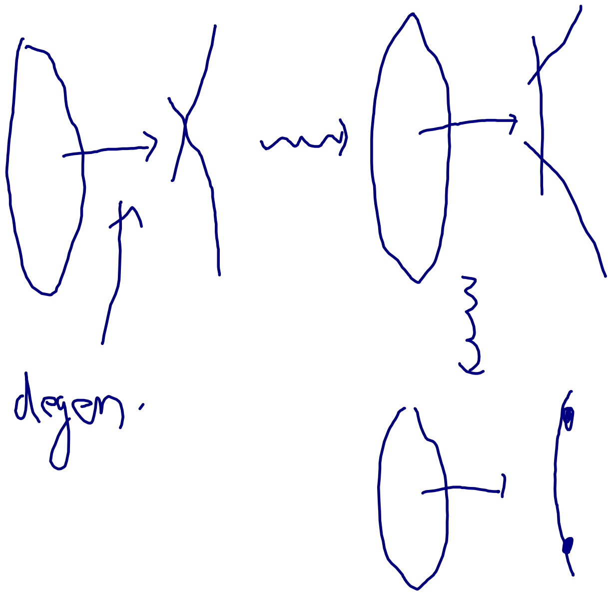
Step 3. a finer analysis after base change

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, dashed] & \\ X^{an} & & X^{s} \end{tikzcd}$$

s.t. (X^s, X_0^s) is plt i.e.

$$\begin{cases} X_0^s \text{ is klt} \\ -K_{X^s} \text{ is ample} \\ a(X_0^s; \mathcal{X}^{an}, X_0^{an}) = -1 \end{cases}$$

$\Leftrightarrow a(X_0^s; \mathcal{X}^{an}) = 0$



Question: Check $\mathbb{P}^n$ (or $n=2$) is K-semistable (poly).

7. Fujita's β-invariant

for any D divisorial valuation over X

$$\beta(D) = A_X(D) L^n - \int_0^\infty \mathrm{Vol}(L - xD)\, dx \qquad Y \xrightarrow{f} X$$

$$\| $$

$$\int_0^\infty \mathrm{Vol}_Y(f^*L - xD)\, dx$$

Thm (Fujita, Li) X is K-semistable iff $\beta(D) \geq 0$. $\forall D$.

RK: $\exists$ a uniform version

Corollary (Fujita): if X is K-semistable then $\mathrm{Vol}(-K_X) \leq (n+1)^n$

pf: blow up a smooth x

$$n(-K_X)^n \geq \int \mathrm{vol}(-K_X - xE)\,dx$$

$$\geq \int_0^{(-K_X)^{\frac{1}{n}}} \left((-K_X)^n - x^n \right) dx$$

$$= (-K_X)^{\frac{n+1}{n}} - \frac{(-K_X)^{\frac{n+1}{n}}}{n+1}$$

$$\Longrightarrow (-K_X)^n \leq (n+1)^n \quad \square$$

Def of Ding-stability: $D \sim_{\mathbb{Q}} -K_{\mathcal{X}} - \mathcal{L}$ supports on X_0

$$\mathrm{Ding}(\mathcal{X}) = \frac{-\mathcal{L}^{n+1}}{(n+1)L^n} - 1 + \mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0)$$

Ding-semistable if $\mathrm{Ding}(\mathcal{X}) \geq 0$

Ding-polystable if "semi" + " = $\Rightarrow$ trivial outside codim 2"!

$$
\begin{tikzcd}
\text{K-semistability} \arrow[rr, Leftrightarrow] & & \text{K-semist for special TC} \arrow[d, equal] \\
\text{Ding-semistability} \arrow[u, Rightarrow, "1"'] \arrow[rr, Leftrightarrow] \arrow[d, Rightarrow, "3"'] & & \text{D-semi for special TC} \arrow[d, equal, "2"'] \\
\beta \geq 0 \text{ for any divisorial valuation} \arrow[rr, Rightarrow] & & \beta \geq 0 \text{ for div induced by special TC}
\end{tikzcd}
$$

Lemma : $\mathrm{Fut} \geq \mathrm{Ding}$
(Berman)

Pf: $$\mathrm{LHS} - \mathrm{RHS} = \frac{\bar{\mathcal{L}}^{n+1}}{L^n} + \frac{\bar{\mathcal{L}}^n \cdot K_{\mathcal{X}/\mathbb{P}^1}}{L^n} + 1 - \mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0)$$

$$= -\frac{\bar{\mathcal{L}}^n \cdot D}{L^n} + 1 - \mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0) \qquad \left(L^n = \bar{\mathcal{L}}^n \cdot \mathcal{X}_0\right)$$

$D = \sum d_i E_i \quad \mathcal{X}_0 = \sum b_i E_i \quad b_i \in \mathbb{N}$

$$\mathrm{lct} \leq \min\left\{1 - \frac{d_i}{b_i}\right\}$$

$$1 - \mathrm{lct} \geq \max\left\{\frac{b_i - 1 + d_i}{b_i}\right\}$$

$$\mathrm{DF} - \mathrm{Ding} \geq \frac{1}{L^n}\left(\bar{\mathcal{L}}^n \sum_i \left(\max\left\{\frac{b_i - 1 + d_i}{b_i}\right\} b_i - d_i\right) E_i\right)$$

$$\geq \frac{1}{L^n}\left(\bar{\mathcal{L}}^n \sum (b_i - 1) E_i\right) \geq 0$$

"=" $b_i = 1$, $(\mathcal{X}, \mathcal{X}_0)$ is lc. $D = d\,\mathcal{X}_0$. i.e. $\mathcal{L} \sim -K_{\mathcal{X}}$ $\square$

② Filtered linear series (Witt-Nyström, Székelyhidi, Boucksom-Hisamoto-Jonsson)

Def: $V_\cdot = \bigoplus_{k=0}^{\infty} H^0(X, kL) = \bigoplus V_k$

- Filtration of $V_\cdot$. $x \in \mathbb{R}$. $F^x V_k$ decreasing
- $F^x V_k \otimes F^{x'} V_{k'} \longmapsto F^{x+x'} V_{k+k'}$

given $\mathcal{X}$ test conf.

$$F^{x} V_k = \{ t \in V_k \mid t^{\lfloor -x \rfloor} f \in H^0(\mathcal{X}, k\mathcal{L}) \}$$

$$= \{ D \in |kL| \mid \pi^* p_1^* D + kB + \lfloor -x \rfloor \mathcal{Y}_0 \geq 0 \}$$

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "\pi"'] \arrow[dr, "\theta"] & \\ X \times \mathbb{A}^1 & & \mathcal{X} \end{tikzcd}$$

$$B = \theta^* \mathcal{L} - \pi^* p_1 L$$

apply to $\mathcal{X}$ special test configu

the valuation X_0 determines a valuation V_{X_0} on $K(X) \subset K(X \times \mathbb{A}^1)$

Lemma: (BHJ) $A(V_{X_0}) = A(X_0; X \times \mathbb{A}_1)$

$$= \{ f \in V_k \mid V_{X_0}(f) \geq k \cdot A(V_{X_0}) + x \}$$

look at the filtration

$$\lambda^{(k)}_{min} \simeq -kA_x(V)_{x_0} \qquad \lambda^{(k)}_{max} \cong -kA_x(V)_{x_0} + \tau$$

$$W(k) = \int_{\lambda^{(k)}_{min}}^{\lambda^{(k)}_{max}} \dim F^{\lambda} V_k \left(= h^0\left(X, kL - (kA_x(V)+\lambda) V_{x_0}\right)\right) + \lambda^{k}_{min} \cdot h^0(X, mL) = \sum j \, F^j / F^{j+1}$$

total weight

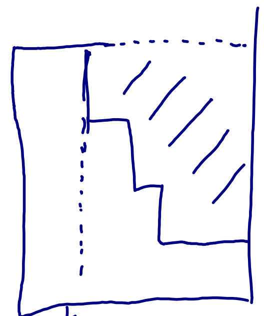
$$\lim_{k\to\infty} W(k) = \left(\int \mathrm{Vol}(kL - xV)\, dx - L^n A_x(V_{x_0}) \right) \frac{k^{n+1}}{n!} + O(k^n)$$

Prop (BHJ): $\boxed{W(k) = \bar{\mathcal{L}}^{n+1} \cdot \frac{k^{n+1}}{(n+1)!} + O(k^n)}$ for any test conf.

(the second term is $-\frac{1}{2} \frac{K_{\mathcal{X}/\mathbb{P}^1} \mathcal{L}^n}{n!} k^n$)

Ex. For blow-up of J.

$$W(k) = -\dim\left(H^0(X \times \mathbb{P}^1, p^* \mathcal{L}^k) \big/ H^0(X \times \mathbb{P}^1, p^* \mathcal{L}^k \otimes J^k) \right)$$



λ^k_{min} λ^k_{max}

③ Prop: Ding $\geq 0 \Longrightarrow \beta \geq 0$

Step 1. Constr of TC from F

$$I_{(k,j)} = \mathrm{Im}\left(H^0(X, kL - jF) \otimes \mathcal{O}_X(-kL) \xrightarrow{ev} \mathcal{O}_X\right)$$

Choose $e \geq \tau$

$$J_k = I_{(k,ke)} + \cdots + I_{(k,1)} t^{ke-1} + t^{ke}$$

$\Pi_k : \mathcal{X}_k \longrightarrow X \times \mathbb{A}^1$ normalized blow-up of J_k

$$\mathcal{O}_{\mathcal{X}_k}(-E_k) = J_k \mathcal{O}_{\mathcal{X}_k} \qquad \mathcal{L}_k = \Pi_k^* p^* L - \frac{1}{k} E_k$$

Step 2: calculation of $\lim_k \lambda_{max}$, $\lim_k \overline{\mathcal{L}_k}^{n+1}$

$\cong k \cdot \tau(F)$

$$\lambda_{max}(\mathcal{X}_k, \mathcal{L}_k) = \frac{1}{k}\left(-ke + \max\{ j \mid I_{(k,j)} \neq 0 \}\right)$$

$$\lim_k \left(\overline{\mathcal{L}_k}^{n+1}\right) = (n+1)\left(L^n(-e) + \int_0^\infty \mathrm{vol}(L - xF)\,dx\right)$$

Step 3 take the limit

Ding implies $\left(X\times\mathbb{A}^1, \mathcal{J}_k^{\frac{1}{k}}(t)^{d_k}\right)$ is sub. lc. where

$$d_k = 1 + \frac{L_k^{n+1}}{(n+1)L^n}$$

For $\sigma: Y \to X$, then $\left(Y\times\mathbb{A}^1, \Pi(-F_1\times\mathbb{A}^1)^{(-A_1(F)+1)}(\mathcal{J}_k\mathcal{O}_{Y\times\mathbb{A}_1})^{\frac{1}{k}}, t^{d_k}\right)$ is sub. lc.

$$\cap$$

$$(\mathcal{O}(-F)+t)^e$$

Blow up $F\times\{0\}$ to get E_F

$$0 \le 1 + \mathrm{ord}_{E_F}(K_{Y}/_{Y\times\mathbb{A}^1}) - (-A(F)+1) - e - d_\infty$$

$$= A_F - \frac{1}{L^n}\int_0^\infty \mathrm{vol}(L-xF)\,dx.$$

8. local version

$x \in X$ klt sing. consider a valuation $v \in Val_{x,X} = \{$ valuations with center $x \}$

($X = \mathrm{Spec}\, R$)

$\mathfrak{a}_k = \{ f \mid v(f) \geq k \}$ $\quad vol(v) = \lim_{k \to \infty} \dfrac{leng(R/\mathfrak{a}_k)}{k^n/n!}$ $\quad \left(vol(\lambda v) = \dfrac{1}{\lambda^n} vol(v) \right)$

$A(v)$

if $v = \lambda \cdot ord_D$ $\quad A(v) = \lambda A(D)$

v quasi-monomial, i.e. toroidal over $(Y, E) \to X$ with coordinates $(\beta_1, \cdots, \beta_i)$

then $A(v) = \sum \beta_i A(E_i)$

Define $\widehat{vol}(v) = A^n(v)\, vol(v)$, Goal to find the minimizer of $\widehat{vol}(v)$

(Chi Li) :

called normalized volume

(Blum) a minimizer always exists. $\quad$ Def: $\widehat{vol}(v) = vol(x \in X)$

Stable degeneration Conj: for any klt $x \in X$

Li
1. the minimizer v of $\widehat{\mathrm{vol}}$ is unique up to scaling
2. it is quasi-monomial
3. $R_0 = \mathrm{gr}_v(R)$ is f.g.

Li-X.
4. $\operatorname{Spec} R_0$ is klt
   $\quad \| \quad$
   X_0
5. (X_0, v) is K-semistable

K-semistable valuation

Consider a $T = (\mathbb{C}^*)^r$ action on $\operatorname{Spec} R$

$R = \bigoplus R_\alpha$ $\quad \alpha \in \operatorname{Hom}(T, \mathbb{C}^*) = M \quad R_0 = \mathbb{C}$

$\Gamma = \{\alpha \mid R_\alpha \neq 0\}$. $\quad t^+ = \{ v \in N_{\mathbb{R}},\ \langle \alpha, v \rangle > 0 \}$

v gives a valuation $f = \bigoplus f_\alpha$ $\quad v(f) = \min\limits_{f_\alpha \neq 0} \langle v, \alpha \rangle$

Spec R is KLT, $\quad \forall\ v \in t^+_{\mathbb{Q}}$ $\quad R/\mathbb{C}\cdot v = (S, \Delta)$ is log Fano

so (X, v) is K-semistable $\iff$ (S, Δ) is K-semistable

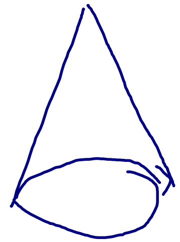
we can define (X, v) K-semi even $v \in t^+_{\mathbb{R}}$ (Colins-Szekelyhidi)

- degen has a $T = (\mathbb{C}^*)^r$ $\quad r = \mathbb{Q}$-rank of v

Γ = valuation semi group of v

$v: \alpha \longmapsto v(\alpha)$ is a canonical vector.

History: Sasaki, search for Sasaki-Einstein metric at a link of a sing. $T \hookrightarrow X$



- (Martelli-Sparks-Yau): $\widehat{\mathrm{vol}}(v)$ for $\{v \in t^+\}$ is strictly convex.
  if X has SE metric along v^* then v^* is a minimum.

- Li generalizes the definition of $\widehat{\mathrm{vol}}$ from t^+ to $\mathrm{Val}_x(X)$
  so it can be defined for any klt.

- SDC suggests any $x \in X$ has a unique degeneration to K-semistable sasakian case.

- The relation with K-stability: $\dfrac{d\,\widehat{\mathrm{vol}}(v+t\xi)}{dt} = \mathrm{Fut}(X, v; \xi)$ for $v, \xi \in t^+$

$$\left(= \mathrm{Fut}(X/C.(v); \xi)\right)$$

Some known results: (Yuchen Liu) $\mathrm{vol}(x \in X) = \inf_{a} \mathrm{lct}(a)^n \mathrm{mult}(a)$

Ex: (Li, Li-Liu, Li-X.). $X = C(V, -rK_V)$ the blow-up divisor is a minimizer iff V is K-semistable ($\Longrightarrow$ T-equi K-semi = K-semi)

Thm. (Liu-X.) $\mathrm{vol}(x, X) \le n^n$ "=" iff $x \in X$ is smooth

Thm. (Li-X.) (rational rank 1) assume v is divisorial valuation then v is a Kollár component i.e.

$$\begin{array}{ccc} Y & \supset & S \\ \downarrow & & \downarrow \\ X & \ni & pt \end{array}$$

$K_Y + S|_S = K_S + \Delta_S$ is anti ample

and (S, Δ_S) is K-semistable.

Furthermore: $\widehat{\mathrm{vol}}(S) < \widehat{\mathrm{vol}}(E)$ for any $E \ne S$.

Thm (Li-X.) (rational rank arbitrary) assume v is quasi-monomial, $\overset{R_0}{\overset{\shortparallel}{\mathrm{gr}}}(v)$ is f.g

Then $\mathrm{spec}(R_0)$ is klt $(\mathrm{Spec}(R_0), v)$ is K-semistable.

$\widehat{\mathrm{vol}}(v) < \widehat{\mathrm{vol}}(v')$ any v' quasi-monomial $\ne v$.

Corollary: K-semistable valuation is unique. (Conj by Donaldson-Sun)

Application: Cubic 3-folds (Liu-X.)

Liu's generalization of Fujita's result: X is K-semistable

$$\mathrm{vol}(-K_X) \leq \left(\frac{n+1}{n}\right)^n \cdot \widehat{\mathrm{vol}}_x(v) \quad \forall x, \ \forall v \in \mathrm{Val}_x(X)$$

apply to K-semistable limits X of cubic 3-folds

using 3-fold sing classi.

$\widehat{\mathrm{vol}}(x) \geq \frac{81}{8} \quad \forall x \in X \Longrightarrow$ sing $x \in X$ $\quad \mathrm{Cl}^{tor}(x \in X) = \{0\}$

$\Longrightarrow -K_X = 2H$. H cartier $\overset{\text{T. Fujita}}{\Longrightarrow}$ X is a cubic

$\Longrightarrow$ the moduli space of K-stable Fano X deform to a cubic

$=$ GIT moduli of cubic 3-fold (explicit known by Allcock)

- Question: how about cubic fourfolds?

9. moduli spaces of K-stable

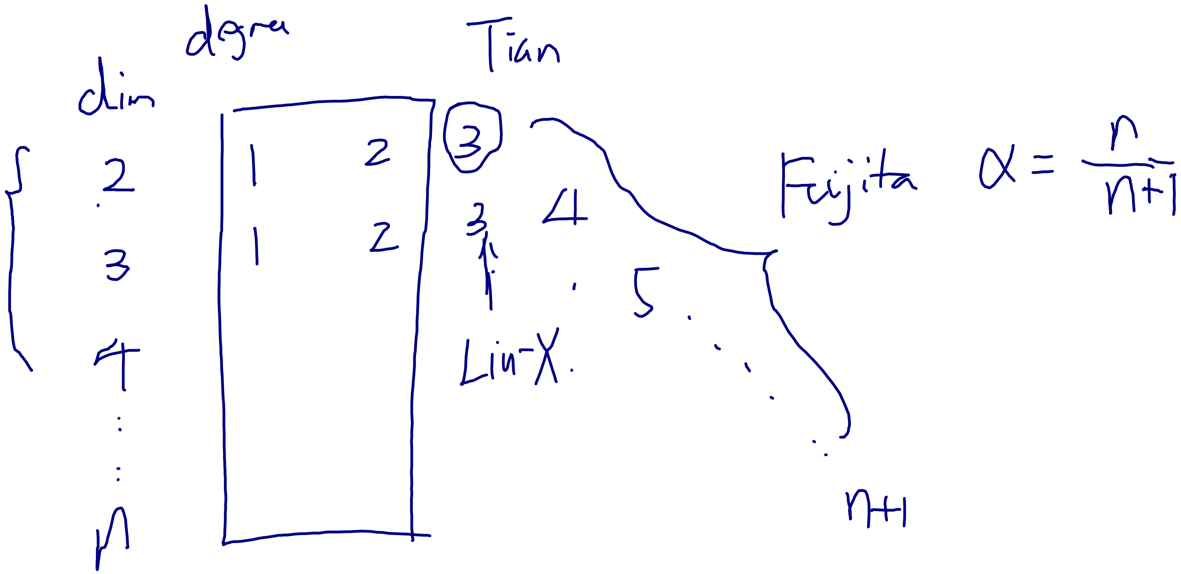
	smoothable (analytic)	$\mathbb{Q}$-Fano (algebraic)
boundedness	CDS, Tian	Jiang (using Birkar's BAB)
Openess	CDS, Tian	??
compactness	CDS, Tian	???
separatedness	Li-Wang-X., (Spotti-Sun-Yao)	??
other conditions $[Z/PGL]$ has a good moduli	L-W-X (Odaka some part)	?
projectivity	partially L-W-X for CM line bundle	??

check: K-stable moduli coincides with KSBA moduli if K_X is ample. CM degree minimizes.

10. Questions:

to verify K-stability.

- 3-folds. $g=12$ all smooth objects K-semistable ?
- cubic 4 (n)-folds. ?
- all smooth hypersurfaces.



Ques. On normalized volume

- quasi-monomial? (implied by Jonsson-Mustaţă conj)
- finite generation for all quasi-monomial.
- finite degree: $y \in Y \longrightarrow x \in X$ $\pi^* K_X = K_Y$

$$\deg \cdot \mathrm{Vol}(x) = \mathrm{Vol}(Y)$$

- second largest volume $2(n-1)^n$ (implies Li-X. for higher dim)
- only accumulation pts are 0?
- affine cone over hypersurfaces $f_d(x_1, \dots, x_n) \subset \mathbb{C}^n$.

Thank You!!