ANOMALY CANCELLATION AND GLOBAL CONSISTENCY

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Based on arXiv:1710.07627 with S.J.Rey

2nd IBS-KIAS Joint Workshop at High1
JeongSeon, January 9, 2018
Chiral anomaly

- Chiral anomaly [Adler] [Bell, Jackiw]

\[ D^\mu j^{5,\alpha}_\mu = -g^2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^c \text{tr}(t_\alpha \{ t_b, t_c \}) \]

- Consistency: axial current conservation vs gauge symmetry
- They can be traded, but cannot be simultaneously removed.

- Cancellation: constrains low energy spectrum in terms of charge

\[ \sum = 0 \]
Guiding principle

- Good guiding principle of model building.
- Ex. $SU(3) \times SU(2) \times U(1)$

![Diagram of SU groups]

Constrains the spectrum uniquely
- $(3,2)_{1/6}$, $(3^*,1)_{2/3}$, $(3,1)_{1/3}$, $(1,2)_{1/2}$, $(1,1)_{-1}$, $q$, $u^c$, $d^c$, $l$, $e^c$
- Singlets $(1,1)_0$ not constrained.

- New physics extensions
  - Grand Unification, SUSY breaking, ...
Chiral anomaly

- Chiral anomaly [Adler] [Bell, Jackiw]

\[ D^\mu j_\mu^{5,a} = -g^2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^c \text{tr}(t_a \{t_b, t_c\}) \]

\[ F \wedge F \equiv \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \]

- Consistency, low energy spectrum

- Field theory is the low-energy limit of string theory.
  - If string theory is UV finite (S-matrix, partition function), so is field theory.

- Adler-Bardeen theorem: anomaly is one-loop exact.
Global consistency condition

In string theory, anomaly cancellation is promoted to global consistency condition from one-loop diagram.

- **Closed string:**
  - Vacuum-to-vacuum (torus) diagram - modular invariance
  - Condition between gauge symmetry $F$ and geometry $R$.
  - $F, R$ in the low energy theory.
  - $\text{tr } R \wedge R - \text{tr } F \wedge F = 0$

- **Open string:**
  - Cylinder and its twisted variants – RR tadpole cancellation.
  - Constrains the number of D-branes $n$. 
  - $\begin{align*}
  &+ \quad + \quad + \\
  = & \left( + \right)^2
  \end{align*}$
Modular invariance

- Point particle on a circle of circumference $l$  
  
  \[
  Z_{S_1}(m^2) = V_d \int \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{dl}{2l} \exp[-(k^2 + m^2)l/2] 
  = iV_d \int_0^\infty \frac{dl}{2l} (2\pi l)^{-d/2} \exp(-m^2 l/2) .
  \]

- UV divergent at $l \to 0$.

- String on a circle: no such region $\tau_2 \to 0$ in the moduli space

  \[
  \sum_{i \in \mathcal{H}^\perp} Z_{S_1}(m_i^2) = iV_d \int_\mathcal{R} \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2 \alpha' \tau_2)^{-d/2} \sum_{i \in \mathcal{H}^\perp} q^{h_i-1} \bar{q}^{\bar{h}_i-1}
  \]

- “Modular invariance cuts off the UV divergences without spoiling the space-time gauge invariance.”
Conclusion: partition function and elliptic genus

• Partition function  
  \[ Z(\tau) = \text{Tr} \ q^{\tilde{L}_0} \bar{q}^{L_0} \quad q = e^{2\pi i \tau} \]

• Modular invariance (main topic today) UV finite condition.
• Anomaly free low energy field theory.
• Works well with perturbative heterotic string.

• Strongly coupled heterotic string  
  + non-perturbative objects like N5 (M5 in M-theory)

• Elliptic genus  
  \[ Z(\tau) = \text{Tr} \ q^{\tilde{L}_0} \bar{q}^{L_0} (-1)^F \]
• Twisted version often gives us good enough information. Cf. SUSY \[\text{[Witten]}\]
• Can be calculated using topological string.

• Modular invariance of the elliptic genus gives global consistency condition.  
  \[\text{[KSC, S.J.Rey]}\]
Strongly coupled heterotic string

- M-theory on $S^1 / \mathbb{Z}_2 = \text{heterotic string theory}$ [Horava, Witten]
- At the ends of interval we have separated two $E_8$'s
- Heterotic string = M2-brane stretched between them.
- String tension & gauge coupling is proportional to $R$
Small instantons

- Solution to self-dual equation $F = \ast F : \text{codim}_R = 4, F \wedge F = \text{integer}$
- Global consistency condition $dH = F_1 \wedge F_1 + F_2 \wedge F_2 - R \wedge R = 0$
- $E_8 \rightarrow G + \text{commutant unbroken group}$
  1. Shrinking to zero size = $E_8$ unbroken
  2. Phase transition: NS5 detached into bulk. [Ganor, Hanany]

What will be global consistency condition in this case?
Modular form

- Partition function is quasi-modular form.
- \( SL(2, \mathbb{Z}) \)
  \[
  \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1
  \]
- Generated by
  \[
  T: \tau \rightarrow \tau + 1, \quad S: \tau \rightarrow -\frac{1}{\tau}
  \]
- Modular form
  
  \[
  f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau)
  \]
  of weight \( k \)
- Partition function \( Z(\tau) \) is quasi-modular form. (modular invariance + anomaly)
Bianchi identity

- The partition function/elliptic genus is
- Invariant under $T$

- Non-invariant under $S$
  \[ Z \left( -\frac{1}{\tau} \right) = Z(\tau) \exp \left( \frac{\pi i}{\tau} (\text{tr} R \wedge R - \text{tr} F \wedge F) \right) \].

- The non-invariant piece can be expressed in terms of $F \wedge F$, $R \wedge R$

Anomaly

- Non-invariant piece: Bianchi identity for Kalb-Ramond field
  \[ dH = \text{tr} R \wedge R - \text{tr} F \wedge F \]

- Anomaly cancellation requires it to be zero $dH = 0$. 
Holomorpy vs modular invariance

- The building blocks of partition function/elliptic genus

\[ \vartheta_1(z) = \eta(\tau)^3(2\pi i z) \exp \left( \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(2\pi i z)^{2k} \right), \]

\[ E_{2k} : 2^{k} \text{th Eisenstein series.} \]

- In the holomorphic basis. All is modular form except \( E_2 \).

\[ E_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) - \frac{6ic}{\pi} (c\tau + d) \]

- The two can be traded.

\[ \hat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{6i}{\pi(\tau - \bar{\tau})} \]

\[ \hat{E}_2 \left( \frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = (c\tau + d)^2 \hat{E}_2(\tau, \bar{\tau}). \]

- But homomorphy lost. Cf. anomalous theory.
- Non-invariant phase: consistent relation among \( F, R, n \).
Elliptic genus, with M5-branes

• For strongly coupled heterotic string with M5-branes, we need elliptic genus

\[ Z = \text{Tr} q^{L_0 - \bar{L}_0} (-1)^F \]

• We can calculate it using defect operator via topological vertex [Minahan, Nemeschansky, Vafa, Warner] [Iqbal, Kozcaz, Vafa] [Haghighat et al.]

\[ Z = D_{L,\nu_1}^{M^9} D_{\nu_1 \nu_2}^{M^5} D_{\nu_2 \nu_3}^{M^5} \ldots D_{\nu_n}^{M^9} \]

With probe M2-branes
M9-M2 vertex \( D_{L,\nu}^{M^9}, D_{\nu}^{M^9} \)
M2-M5-M2 vertex \( D_{\nu \mu}^{M^5} \)

To regularize IR divergence we introduce \( \Omega \)-deformation.
Application \cite{KSC, S.J.Rey}

- The number of M5-branes $n$
- The probe M2-branes parameterized by Young tableaux $\nu_a$

\[ Z = D^{M9}_{L,\nu_1} D^{M5}_{\nu_1\nu_2} D^{M5}_{\nu_2\nu_3} \cdots D^{M9}_{\nu_n,R} \]

\[ \prod_{a=1}^{n} D^{M5}_{\nu_a\nu_{a+1}} \left( -\frac{1}{\tau} \right) = \prod_{a=1}^{n} D^{M5}_{\nu_a\nu_{a+1}} (\tau) \exp \left[ \frac{\pi i}{\tau} ((|\nu_a| - |\nu_{a+1}|)^2 \epsilon_1 \epsilon_2 - (|\nu_a| + |\nu_{a+1}|) \epsilon_3 \epsilon_4) \right]. \]

- With

\[ Z \left( -\frac{1}{\tau} \right) = Z(\tau) \exp \left( \frac{\pi i}{\tau} \left( \text{tr} R \wedge R - \text{tr} F \wedge F \right) \right). \]

- this is interpreted as

\[ dH = \text{tr} R \wedge R - \text{tr} F \wedge F = \sum_{i \in \text{NS5}} \delta^4(z - z_i) \]

- cf. NS5 brane from small instanton. No information other than four-form. From $F \wedge F$
Summary

• Anomaly cancellation is lifted to global consistency condition in string theory.

• 1-loop: embedded in partition function / elliptic genus

• We need also the effect of strongly coupled heterotic string and non-perturbative objects like NS5-branes.

• Requiring both holomorpy and modular invariance, we get global consistency condition.

\[ dH = \text{tr} R \wedge R - \text{tr} F \wedge F = \sum_{i \in \text{NS5}} \delta^4(z - z_i) \]