G_2- GEOMETRY IN CONTACT GEOMETRY OF SECOND ORDER

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In his famous "Five variables paper [1]", E. Cartan investigated the (local) contact equivalence problem of two classes of second order partial differential equations for a scalar function in two independent variables. One class consists of overdetermined systems, which are involutive, and the other class consists of single equations of Goursat type, i.e., single equations of parabolic type whose Monge characteristic systems are completely integrable. Especially, in [1], he found out the following facts: the symmetry algebras (i.e., the Lie algebras of infinitesimal contact transformations) of the following overdetermined system (A) and the single Goursat type equation (B) are both isomorphic with the 14-dimensional exceptional simple Lie algebra G_2.

$$(A) \qquad \frac{\partial^2 z}{\partial x^2} = \frac{1}{3}\left(\frac{\partial^2 z}{\partial y^2}\right)^3, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)^2.$$

$$(B) \qquad 9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0,$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

are the classical terminology.

In [7], we observed, for each exceptional simple Lie algebra X_ℓ, we could find the overdetermined system (A_ℓ) and the single equation of Goursat type (B_ℓ), whose symmetry algebras are isomporphic with X_ℓ and formulated this fact as the G_2-geometry. We will first recall this observation in this talk.

The purpose of this talk is to exhibit the (local) models for overdetermined systems (A_ℓ) explicitly for each exceptional simple Lie algebra and also for the classical type analogy. We will also give parametric descriptions of the single equation of Goursat type (B_ℓ).

See [3], for the recent development of this subject. Our basic references are [6], [8], [9].

References

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