

2018-04-26

Masanori Adachi "Weighted Bergman spaces of domains with Levi-flat boundary" at KIAS

Plan

(1) Levi-flat What is? Why? Examples?

(2) Liouville property of the ^{special} example

(3) Holomorphic functions with slow growth on the ^{special} example.

(another example if time permits)

What is?

(1)

Definition

X : a complex manifold

$\Omega \subsetneq X$: a ~~real~~ domain $\partial\Omega \in C^\infty$ $\stackrel{=}{=} M$

$\Omega = \{r < 0\}$, $r: \bar{\Omega} \xrightarrow{C^\infty} \mathbb{R}$, $dr \neq 0$ on $\partial\Omega$

We say M is Levi-flat real hypersurface

if the Levi form on $\partial\Omega$, $i\partial\bar{\partial}r|_{T^{1,0}_M} \equiv 0$ on M .

where $T^{1,0}_M := T^{1,0}_X \cap \mathbb{C}TM \simeq TM \cap J_X TM$

Prop

M is Levi-flat iff M is foliated by complex hypersurfaces of X

proof. (sketch) $i\partial\bar{\partial}r|_{T^{1,0}_M} \equiv 0 \iff [TM \cap J_X TM, TM \cap J_X TM] = 0$

$\iff \exists \mathcal{F} = \{L_\alpha\}_{\alpha \in \Lambda}$, $\iota_\alpha: L_\alpha \hookrightarrow M$
inj. immersion (leaf)
s.t. $M = \coprod_{\alpha \in \Lambda} L_\alpha$

J_X inherits ex. str on $\iota_\alpha(L_\alpha)$. $(\iota_\alpha)_*: TL_\alpha \rightarrow TM \cap J_X TM$
(L_α are tangent to $TM \cap J_X TM$)

Trivial Example $X = \mathbb{C}^n$, $M = \mathbb{C}^{n-1} \times \mathbb{R}$, $\Omega = \mathbb{C}^{n-1} \times \{\operatorname{Im} z_n < 0\}$

$r = \operatorname{Im} z_n$

$\mathcal{F} = \{\mathbb{C}^{n-1} \times \{t\}\}_{t \in \mathbb{R}}$

Fact

M is $C^\omega \Rightarrow M \subset X$ is locally identified with this example.

Why interesting?

Leri problem

X : ex. mfd. $\Omega \in X$: rel cpt, $\partial\Omega \in C^\infty$ for simplicity

Asks how the geometry of $\partial\Omega$ controls holomorphic functions on Ω .

Oka $X = \mathbb{C}^n$. $\partial\Omega$: pseudoconvex (Leri form ≥ 0)

$\Rightarrow \Omega$ is Stein. plenty of holomorphic functions including those diverging at any point on $\partial\Omega$

Grauert X : general, just a ex mfd

$\partial\Omega$: str. pseudoconvex (Leri form > 0)

$\Rightarrow \Omega$ is (a ^{proper} modification of) Stein (space).

may contain compact analytic set

Grauert found examples of $\Omega \in X$ where

$\partial\Omega$: Leri-flat and $\mathcal{O}(\Omega) = \mathbb{C}$.

Examples

Grauert, Diederich-Ohsawa

Σ : a cpt Riemann surface of genus ≥ 2

$\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{PSU}(1, 1) \cong \mathrm{Aut}(\mathbb{D}) \subset \mathrm{Aut}(\mathbb{CP}^1)$

We suspend $\mathbb{CP}^1$ over Σ by ρ :

$$X_\rho := \Sigma \times_\rho \mathbb{CP}^1 := \mathbb{D} \times \mathbb{CP}^1 / \pi_1(\Sigma)$$

$$\begin{array}{ccc} \text{Deck} & & \\ \pi_1(\Sigma) \curvearrowright \mathbb{D} & \xrightarrow{\text{univ. cov.}} & \Sigma \\ & & \uparrow \\ & & (\text{Deck}, \rho) \end{array}$$

$$\Omega_\rho := \Sigma \times_\rho \mathbb{D} = \mathbb{D} \times \mathbb{D} / \pi_1(\Sigma) \quad \text{i.e. } (z, w) \sim (\text{Deck}(\delta)z, \rho(\delta)w) \text{ for } \delta \in \pi_1(\Sigma).$$

(Image $\rho \subset \mathrm{Aut}(\mathbb{D})$)

$$M_\rho := \Sigma \times_\rho \partial\mathbb{D} = \mathbb{D} \times \partial\mathbb{D} / \pi_1(\Sigma) : \text{Leri-flat!}$$

G, D=0

(a ^{proper} modification of)

(D=0) Ω_ρ is Stein (space) if $\pi_1(\Sigma) \cap \rho$ does not have a global fixed point in $\mathbb{D}$. (not a root of unity)

(G) $\mathcal{O}(\Omega_\rho) = \mathbb{C}$

if $\rho : \pi_1(\Sigma) \rightarrow (\text{Rotations of } \mathbb{D})$

4. $\rho(\delta) = \text{id}$

$\Rightarrow \Omega_\rho = \Sigma \times \mathbb{D}$

$\mathcal{O}(\Omega_\rho) = \mathcal{O}(\mathbb{D})$

Why interesting?
(Lohd)

Proof. (sketch)

(G) $p: \pi_1(\Sigma) \rightarrow U(1)$ Image $p \ni$ an inational rot.

$$\Omega_p = \sum_z x_p \mathbb{D}_w \supset \sum_p x_p \partial \mathbb{D}(r) \quad 0 < r < 1.$$

$=: M_r$ (Leri-flat)

$f \in \mathcal{O}(\Omega_p)$ $|f|$ takes maximum on M_r at p .

M_r is Leri-flat $\Rightarrow \exists L \subset M_r$ s.t. $p \in L$
leaf

$\Rightarrow$ f is constant on L
maximum principle

From the assumption, L is dense in $M_r \Rightarrow f$ is constant on M_r .

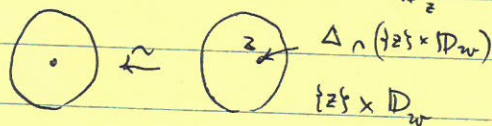
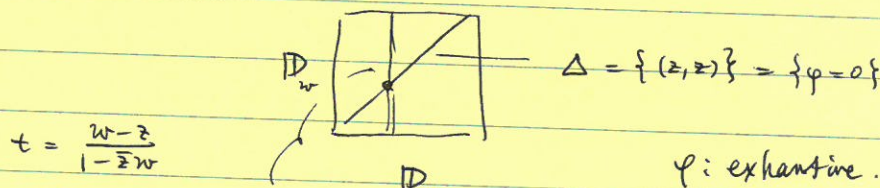
$\Rightarrow f$ is constant on Ω_p

(Di-OR) ~~the~~

When $p = \text{Deck}$ i.e. $\Omega_{\text{Deck}} = \mathbb{D}_z \times \mathbb{D}_w / \pi_1(\Sigma)$ the diagonal act.
(Deck, Deck)

Look at

$$\varphi = -\log \left(1 - \left| \frac{w-z}{1-\bar{z}w} \right|^2 \right)$$



$$\varphi = -\log(1-|t|^2)$$

φ : exhaustive. Ω is a p.m. of Stein sp.

$\varphi \in \text{PSH}$ and

$$\text{SPSH}(\Omega_{\text{Deck}} \setminus D)$$

$$\left(D := \Delta / \pi_1(\Sigma) \right) \simeq \Sigma.$$

If we perturb p a bit, we ~~make~~ use of
harmonic diffeos between Σ and $\mathbb{D} / p(\pi_1(\Sigma))$

//

Why interesting?
(expt.)

In the example, X_p is also foliated.

So M_p is an invariant real hypersurf
of this holomorphic foliation.

$$X_p = \mathbb{D} \times \mathbb{CP}^1 / \pi_1(\Sigma).$$

$$\{ \mathbb{D} \times \gamma \}_{\gamma \in \mathbb{CP}^1}$$

Brazil, Russia
(late 1980s)

Conjecture $X = \mathbb{CP}^2$. $\mathcal{F}$: a holomorphic foliation (singular)

$\Rightarrow \mathcal{F}$ does not have a non-trivial minimal set.
(no non-empty invariant closed set away from singularity)

in particular. X does not have a $\mathbb{C}^\omega$ Leri-flat.

Remark Affirmative in $\mathbb{CP}^{\geq 3}$ (Lins Neto, Sin for $\mathbb{C}^\omega$ case)

Moreover, X : ex. mfd of $\dim \geq 3$ does not admit.
(closed) $\mathbb{C}^\omega$ Leri-flat with ~~no~~ positive holom normal bundle (Brunella, Ohsawa, Bland-Indrajit, Brinkschulte)

The example M
 M_{Deck} has pos. normal bundle
(Brunella) $N_M^{1,0} = (T^1 X|_M)/T_M^{1,0}$
 $\exists$ herm met. curvature on leaves are positive.

Liouville property

(2) Open Problem (Sidney Frankel asked Ohsawa) KIAS 2016.

To classify Leri-flat bounded domains with non-const bounded function. In particular, which Ω_p admits non-const bounded fct?

Very open!!

Goal

Discuss $\varphi = \text{Deck}$ in detail

We showed Ω_{Deck} is a p.m. of Stein sp. using psh exhaustion $-\log(1 - |\frac{z-w}{1-\bar{w}z}|^2)$

Ohsawa-Sibony $0 < \eta \ll 1$ $-\left(1 - \left|\frac{z-w}{1-\bar{w}z}\right|^2\right)^\eta$ is also psh and spsh on $\Omega_{\text{Deck}} \setminus D$
($\leq \frac{1}{2}$) Density Fu-Shaw A-Brinkschulte

$$i\partial\bar{\partial}(-\log \delta) = \frac{i\partial\bar{\partial}(-\delta)}{\delta} + \frac{i\partial\delta \wedge \bar{\partial}\delta}{\delta^2}$$

$$\frac{i\partial\bar{\partial}(-\delta^2)}{\eta\delta^2} = \frac{i\partial\bar{\partial}(-\delta)}{\delta} + (1-\eta) \frac{i\partial\delta \wedge \bar{\partial}\delta}{\delta^2}$$

Ω_{Deck} admits a bounded p.s.h. BUT exhaustion.

Theorem Ω_{Deck} does not admit non-const bdd hol. fct.

1st proof (dynamical system)

We write Ω for Ω_{Deck} in the sequel.

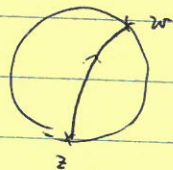
$f \in L^\infty \cap \mathcal{O}(\Omega)$. f can be regarded as $f: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ bdd hol. invariant under the diagonal act of $\pi_1(\Sigma)$.

Fatou type theorem $\overset{\text{non-tangential limit}}{\underset{\text{the boundary value}}{f}} : \underset{\substack{\partial \mathbb{D} \times \partial \mathbb{D} \\ \overline{\mathbb{D}} \times \overline{\mathbb{D}}}}{\overset{\text{Leb-meas.}}{\rightarrow}} \mathbb{C} \quad \pi_1(\Sigma)\text{-invariant}$

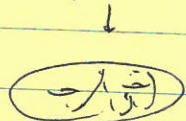
Hopf's ergodicity theorem

The geodesic flow on the unit tangent bundle of Σ is ergodic
In other words,

$E \subset \partial \mathbb{D} \times \partial \mathbb{D}$, $\pi_1(\Sigma)\text{-invariant} \Rightarrow E$: null or full $\overset{\text{Leb-meas.}}{\text{Leb-meas}}$



$(z, w) \in \partial \mathbb{D} \times \partial \mathbb{D} \setminus \text{diag.}$



E is a collection of geodesics on Σ

$\overset{\circ}{f}$ is const almost everywhere in Leb $\Rightarrow f$: const on Ω . //

Remark This proof is not complex analytic! (Although Hopf's theorem has $\overset{\text{complex analytic proof}}{\text{complex analytic proof}}$)
Don't know how to apply this argument to other Ω_p .

2nd proof

2nd proof (potential theoretic) cf. arXiv:1804.09569

Take the "Green function" of $D \subset \Omega$.
" $\Delta / \pi_1(\Sigma)$ "

$$\varphi = \frac{1}{2} \log \frac{1-\sqrt{\delta}}{1+\sqrt{\delta}}$$

$$\delta = 1 - \left| \frac{z-\bar{w}}{1-\bar{z}w} \right|^2$$

$$1) \varphi \in \text{PSH}(\Omega), \varphi: \Omega \rightarrow [-\infty, 0)$$

$$2) (i\partial\bar{\partial}\varphi)^2 = 0 \text{ on } \Omega \setminus D$$

$$3) \varphi \sim \log \text{dist}_D \text{ near } D$$

$$4) \varphi \sim -\sqrt{\delta} \text{ near } \partial\Omega = M$$

$\approx$ essential.

(Diederich-Fornaess index = $\frac{1}{2}$)

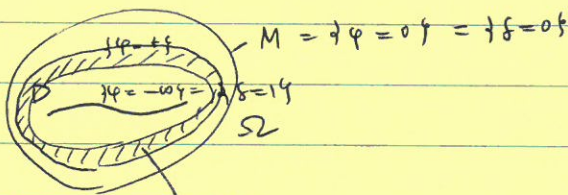
$$f: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C} \quad \bar{f}: \overline{\mathbb{D}} \times \overline{\mathbb{D}} \rightarrow \mathbb{C} \quad \bar{f}|_{\mathbb{D} \times \partial \mathbb{D}_w}$$

Let $f \in L^\infty \cap \mathcal{O}(\Omega)$. Want to show $\bar{f}: M \rightarrow \mathbb{C}$ is leafwise constant.

const in z

Replace z and w

$$\Rightarrow \bar{f}: \partial \mathbb{D} \times \partial \mathbb{D} \rightarrow \mathbb{C} \text{ is constant.}$$



$$\{a \leq \varphi \leq b\} \\ = \{t' \leq \delta \leq a't\}$$

Estimate f on this annular region.

$$\begin{aligned} i\partial\bar{\partial}|f|^2 \wedge d\varphi \wedge d^c\varphi + |f|^2 (i\partial\bar{\partial}\varphi)^2 \\ = d(d^c|f|^2 \wedge i\partial\varphi \wedge \bar{\partial}\varphi + |f|^2 d^c\varphi \wedge i\partial\bar{\partial}\varphi). \end{aligned}$$

(Griffiths.
Demailly's Lelong-Jensen
A. Birkeschulte)

$$\int_{\{a \leq \varphi \leq b\}} i\partial\bar{\partial}|f|^2 \wedge d\varphi \wedge d^c\varphi = \left(\int_{\{\varphi=b\}} - \int_{\{\varphi=a\}} \right) |f|^2 d^c\varphi \wedge i\partial\bar{\partial}\varphi.$$

Plug-in φ .

$$\int_{\{b' \leq \delta \leq a't\}} \frac{i\partial\bar{f} \wedge \bar{\partial}f}{i\partial\bar{\partial}|f|^2} \wedge \frac{1}{4} \frac{d(-\delta) \wedge d^c(-\delta)}{\delta(1-\delta)^2} = \int_{\substack{\varphi=b' \\ \{\delta=b't\}}} - \int_{\substack{\varphi=a't \\ \{\delta=a't\}}} |f|^2 \frac{d^c(-\delta) \wedge \frac{i\partial\bar{\partial}(-\delta)}{\delta}}{4(1-\delta)^2 \delta}.$$

the condition 4) involves here.

We let $b' \nearrow 0$ ($b' \searrow 0$)

$$\int_{b'}^{a'} \frac{dt}{(1-t)^2 t} \left(\int_{\{\delta=t\}} i\partial\bar{f} \wedge \bar{\partial}f \wedge d^c(-\delta) \right) = \frac{1}{4(1-b')^2} \int_{\{\delta=b't\}} |f|^2 d^c(-\delta) \wedge \frac{i\partial\bar{\partial}(-\delta)}{\delta} + (\text{const in } a').$$

RHS remains bounded when $b' \searrow 0$ because M is Levi-flat.

Hence $\limsup_{t \searrow b'} \int_{\{\delta=t\}} i\partial\bar{f} \wedge \bar{\partial}f \wedge d^c(-\delta) = 0$

$$\lim_{t \searrow b'} \int_{\{\delta=t\}} i\partial\bar{f} \wedge \bar{\partial}f \wedge d^c(-\delta)$$

$$\int_M i\partial_b f \wedge \bar{\partial}_b f \wedge d^c(-\delta).$$

(3)

Main Theorem

So far we checked $L^\infty \cap \mathcal{O}(\Omega) = \mathbb{C}$.

Hardy space

If we examine the proof, actually we showed that $A_{-1}^2(\Omega) = \mathbb{C}$ Definition dV on X $\Omega \in X$. $\Omega = \{r < 0\}$. $\delta := -r$

$$A_\alpha^2(\Omega) := \{f \in \mathcal{O}(\Omega) \mid \|f\|_\alpha^2 := \int_\Omega |f|^2 \delta^\alpha dV < \infty\}$$

$\Gamma(\alpha+1)$

weighted Bergman space of order α .

$$\|f\|_{-1}^2 := \lim_{\alpha \rightarrow -1} \|f\|_\alpha^2$$

Question Is $A_\alpha^2(\Omega)$ trivial for $\alpha > -1$?Remark. $\alpha > -\frac{1}{2}$ "Berndtsson-Charpentier trick" applies andwe can solve $\bar{\partial}$ -eq. in $L_{\alpha,*}^2$ and produce sections.Main Theorem Yes.

(in Xir: 1703.08165)

We can construct holom fets with slow growth by

$$I : \bigoplus_{N=0}^{\infty} H^0(\Sigma, K_\Sigma^N) \longrightarrow \mathcal{O}(\Omega) \quad \text{linear}$$

$N=0$ just map the constant.

$$N \geq 1 \quad \gamma_N \in H^0(\Sigma, K_\Sigma^N) \quad \gamma = \gamma_N(t)(dt)^N \quad t \in \mathbb{D}$$

$$I(\gamma_N)(z, w) = \int_z^w \left(\frac{(w-t)(t-z)}{w-z} \right)^{N-1} \gamma_N(t) \frac{dt}{B(N, N)} \in \mathcal{O}(\mathbb{D} \times \mathbb{D})^P$$

Claim I is well-defined, ^{injective,} dense image in compact open topology

$$\|I(\gamma_N)\|_\alpha^2 = \pi \|\gamma_N\|^2 \frac{P(N+1)}{\sum P(N+2+\alpha)} {}_3F_2 \left(\begin{matrix} N+1, N, N \\ 2N, N+2+\alpha \end{matrix}; 1 \right)$$

$< \infty$ if $\alpha > -1$

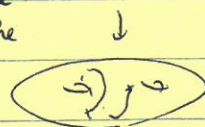
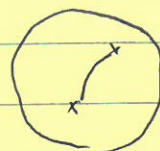
where the norm is computed with $dV = \frac{4}{|1-\bar{z}w|^2} \frac{i}{2} dz_1 d\bar{z}_1 \frac{i}{2} dw_1 d\bar{w}_1$ Cor $A_{-1}^2(\Omega) = \mathbb{C}$ $\approx (\text{Poincaré}) \otimes (\text{Euclid})$

Example 1 $N=1$.

$$\gamma \in H^0(\Sigma, K_\Sigma) \quad \gamma = \gamma(t) dt$$

$$I(\gamma)(z, w) = \int_z^w \gamma(t) dt \in \mathcal{O}(\mathbb{D} \times \mathbb{D})^P$$

we need to multiply the "cross ratios" to show the invariance.



geodesic segment

Open

Problem

Direct proof?

Example 2

$N \geq 2$

$$\gamma = \sum_{\gamma \in \pi_1(\Sigma)} \gamma^* dt^{\otimes N} \in H^0(\Sigma, K_\Sigma^N) \quad (\text{Poincaré})$$

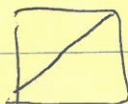
$$I(\gamma) = \sum_{\gamma \in \pi_1(\Sigma)} (\gamma(z) - \gamma(w))^N \quad (\text{Ohsawa KIAS 2016})$$

Proof.

Idea of the proof

$$\bigoplus_{N=0}^{\infty} H^0(\Sigma, K_\Sigma^N) \xrightarrow{\cong} \bigoplus_{N=0}^{\infty} H^0(D, \mathcal{I}_D^N / \mathcal{I}_D^{N+1})$$

$$D = \Delta / \pi_1(\Sigma)$$



$$\mathcal{I}_D(U) = \{ f \in \mathcal{O}(U) \mid f|_D = 0 \}$$

follows from

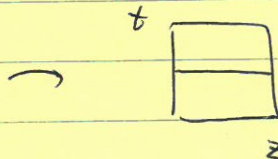
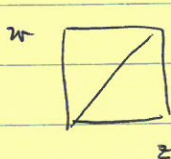
$$K_\Sigma \cong T_\Sigma^* \cong T_D^* \cong N_D^* : \text{conormal bundle.}$$

Remark Similar to the expansion in $\Delta \subset L^*$ near zero but holonomy is non-linear!

$I(\gamma_N)$ gives the holomorphic function that extends the jet $\tilde{j}(\gamma_N)$ and have the minimum L^2 norm among the extension.

We found the optimal extension by hand:

$$I(\gamma_N) = \gamma_N(z) t^N + a_{N+1}(z) t^{N+1} + \dots$$



$$t = \frac{w-z}{1-\bar{z}w}$$

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Put $a_n \left(\frac{\sqrt{z} dz}{1-|z|^2} \right)^{\otimes n} =: \varphi_n \in C^\infty(\Sigma, K_\Sigma^n)$

$$\bar{\partial} \Gamma(\varphi_n) = 0 \iff \bar{\partial} \varphi_n = -\frac{n+1}{\sqrt{z}} \varphi_{n-1} \otimes \frac{2dz \otimes d\bar{z}}{(1-|z|^2)^2}$$

we solved this system of $\bar{\partial}$ -equations on Σ inductively
with by the canonical solution (minimal L^2 norm solution).

Then computed the power series and obtained the expression. //

Open Problem

Direct proof?

Proof by optimal L^2 -jet extension?
of Ohsawa-Takegoshi type

Another example

Remark

We can show $\dim A_\alpha^2(\Omega_\rho) = \infty$ for $\rho = \overline{\text{Deck}}$

i.e. $\rho(\gamma)(w) = \overline{\rho(\gamma)\bar{w}}$.

In fact. $\Omega_{\overline{\text{Deck}}} \simeq X_{\text{Deck}} \setminus \overline{\Omega_{\text{Deck}}}$

By the same argument, we can show $A_{-1}^2(\Omega_{\overline{\text{Deck}}}) = \mathbb{C}$.

To construct holomorphic functions with slow growth,
we use analytic continuation from totally real set
"Lagrangian"

Guillemin-Stenzel
Lempert-Szöke

Gravit tube

complexification.

$$\Omega_{\overline{\text{Deck}}} = \mathbb{D} \times \mathbb{D} / (z, w) \sim (\text{Deck}(\gamma)z, \overline{\text{Deck}(\gamma)w})$$

$$\supset \{(z, \bar{z})\} / \sim =: L \simeq \Sigma$$

anti-holom. diagonal set diffeo

The eigenfunctions of the Laplace-Beltrami op Δ on L extends
on $\Omega_{\overline{\text{Deck}}}$.

$$L^2(\Sigma) \supset \bigoplus_{N=0}^{\infty} \text{Ker}(\Delta - \lambda_N \text{Id}) \hookrightarrow \mathcal{O}(\Omega_{\overline{\text{Deck}}}) //$$