Recent developments on pluripotential theory

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Outline

Let X be a real manifold of dimension n. Let $0 \leqslant p \leqslant n$ and $k \in \mathbb{R}^+$.

$$D_c := \Big\{ \mathcal{C}^k \text{ (differential) } (n-p)\text{-forms } \phi \text{ with compact support on } X \Big\}.$$

ϕ is called a test form of class $\mathcal{C}^k$ and of degree $n-p$ on X.

$$D' := \text{ the dual space } = \Big\{ T : D_c \to \mathbb{C} \text{ linear and continuous} \Big\}.$$

T is called a current of order k and of degree p and dimension $n-p$ on X.

The value of T at ϕ is denoted by

$$T(\phi) \qquad \text{or} \qquad \langle T, \phi \rangle \qquad \text{or} \qquad \langle \phi, T \rangle.$$

Remark

$$p \longleftarrow \textit{dual} \longrightarrow n-p$$

compactness not assumed $\longleftarrow$ *dual* $\longrightarrow$ *compactness assumed*

$$\mathcal{C}^{-k} \longleftarrow \textit{dual} \longrightarrow \mathcal{C}^k$$

The bigger test space is, the more regular currents are.

Examples

Example

Let ψ be a p-form with locally L^1 coefficients. Then it defines a p-current of order 0

$$\phi \mapsto \int_X \psi \wedge \phi$$

for ϕ a test continuous $(n-p)$-form with compact support.

Example

Let V be a manifold of dimension $n-p$ in X, not necessarily closed. Assume that the p-dimensional volume of V is locally finite. Then V defines a p-current of order 0, denoted by $[V]$,

$$\phi \mapsto \int_V \phi.$$

Remark

If V is just a point a, then we get the Dirac mass δ_a (a probability measure)

$$\langle \delta_a, \phi \rangle = \int_X \phi \, d\delta_a := \phi(a) \qquad \text{for } \phi \text{ a continuous function.}$$

Calculus

Weak topology : $T_k \to T$ iff $\langle T_k, \phi\rangle \to \langle T, \phi\rangle$ for every test form ϕ.

Theorem (convolutions + partition of unity)

For every T, there are T_k smooth, $T_k \to T$.

Operator d : if T is a p-current, then dT is a $(p+1)$-current

$$\langle dT, \phi\rangle := (-1)^{p+1}\langle T, d\phi\rangle.$$

Example (Stokes)

It T is given by a $\mathcal{C}^1$ p-form ψ, then dT is given by the $\mathcal{C}^0$ $(p+1)$-form $d\psi$. If $T = [V]$ with a smooth manifold V having smooth boundary bV then $dT = [bV]$.

Theorem (de Rham, suitable convolutions)

If T is d-exact (i.e. $T = dS$ for some S) or d-closed (i.e. $dT = 0$), then there are T_k smooth with the same property and $T_k \to T$.

Problem

Regularize currents keeping some properties such as closedness, positivity, good singularities, norm controls ....

Calculus

Wedge-product or intersection : if α is a smooth (or regular enough) q-form, then $T \wedge \alpha$ is a $(p+q)$-current given by

$$\langle T \wedge \alpha, \phi \rangle := \langle T, \alpha \wedge \phi \rangle.$$

If V, V' are manifolds which intersect transversally, then

$$[V] \wedge [V'] := [V \cap V'].$$

Tensor or cartesian product : If T, T' are currents on X, X' then $T \otimes T'$ is a current on $X \times X'$

$$\big\langle T \otimes T', \phi(x) \wedge \phi'(x') \big\rangle := \langle T, \phi \rangle \langle T', \phi' \rangle.$$

Weierstrass : the forms $\phi(x) \wedge \phi'(x')$ span a dense space of test forms.

Problem (not always solvable)

Define $T \wedge S$ when T, S are not regular enough. Good continuity required.

Remark

Idea 1 : $T \wedge S := (T \otimes S) \wedge [\Delta]$ where $\Delta \simeq X$ is the diagonal of X^2.
Idea 2 : if $T = dR$ is exact and S is closed, then $T \wedge S := d(R \wedge S)$.
Complex case : bidegree (p, q), $d = \partial + \bar\partial$, $\partial, \bar\partial, \partial\bar\partial$ closed, exact, positivity...

Outline

Subharmonic functions

Consider the function $z \mapsto \log|z-a|$ on $\mathbb{C}$ which is harmonic on $\mathbb{C}\setminus\{a\}$ and

$$dd^c \log|z-a| = \delta_a \quad \text{where} \quad dd^c := \frac{i}{\pi}\partial\overline{\partial} \simeq \text{Laplacian}.$$

If u is subharmonic on an open set U in $\mathbb{C}$ then $\mu := dd^c u$ is a locally finite positive measure and by reducing the domain U, we have

$$u(z) = \boxed{\int \log|z-a| d\mu(a)} + \text{a harmonic function} =: \widetilde{u} + \text{a harmonic function}.$$

Theorem (sh functions are almost L^∞)

For every compact set $K \subset U$, $e^{\alpha|u|}$ is in $L^1(K)$ with respect to the Lebesgue measure for some $\alpha > 0$. In particular, u is locally L^p for every $1 \leqslant p < \infty$.

Proof.

Since u is locally bounded above, we can replace $e^{\alpha|u|}$ with $e^{-\alpha u}$ or $e^{-\alpha\widetilde{u}}$. The theorem is clear for $\log|z-a|$. The case of $\widetilde{u}$ is then a consequence of Jensen's inequality ($1/\alpha$ is the mass of μ):

$$\exp\int \leqslant \int \exp \quad \text{integrals wrt a probability measure.}$$

One can also get an uniform estimate for compact families of functions. □

Plurisubharmonic (psh) functions

Definition (Lelong-Oka)

A function $u : X \to \mathbb{R} \cup \{-\infty\}$ on a complex manifold X, with $u \not\equiv 0$, is psh if it is subharmonic or $\equiv -\infty$ on each Riemann surface in X.

Theorem (Skoda, exponential estimate, Fubini theorem + 1D case)

For every compact set $K \subset X$, there is $\alpha > 0$ such that $e^{\alpha|u|}$ is in $L^1(K)$. In particular, u is locally L^p for every $1 \leqslant p < \infty$.

Theorem (strong compactness)

If a family of psh functions is bounded in L^1_{loc}, then it is relatively compact in L^p_{loc} for $1 \leqslant p < \infty$. There is a uniform exponential estimate for it.

Theorem (D-Nguyen-Sibony, Kaufmann, Hiep, Vu)

Let μ be a positive measure which is "regular" enough (e.g. Hölder super-potentials $\Longleftrightarrow$ Hölder Monge-Ampère potentials, Lebesgue on X or on CR submanifolds). Let $\mathcal{F}$ be a compact family of psh functions. Then

$$\int e^{\alpha|u|} d\mu \leqslant c \quad \text{for some positive constants } c, \alpha \text{ and for } u \in \mathcal{F}.$$

Positive closed currents (Lelong-Oka)

For simplicity, assume X is a compact Kähler manifold, e.g. $\mathbb{P}^n$

Positive closed currents allow us to compactify the space of analytic cycles of codimension p in X.

For the case of maximal bidegree $p = n$, a positive closed (n,n)-current of mass 1 is just a probability measure :

$$
\begin{aligned}
\{\text{effective 0-cycles of degree 1}\} \quad &\hookrightarrow \quad \{\text{probability measures on } X\} \\
&\qquad \text{which is a } \boxed{\text{compact}} \text{ space} \\
&\subset \quad \{\text{continuous functions on } X\}^{\star} \\
&=: \quad \{\text{measures on } X\}
\end{aligned}
$$

$$
\boxed{\sum \lambda_i a_i \quad \longleftrightarrow \quad \sum \lambda_i \delta_{a_i}}
$$

Positive closed currents (Lelong-Oka)

$\{$effective cycles of degree 1 and codimension p of $X\}$

$$\hookrightarrow \quad \{\text{positive closed } (p,p)\text{-currents of mass 1 on } X\}$$

which is a $\boxed{\text{compact}}$ space

$$\subset \quad \{\text{continuous differential } (n-p,n-p)\text{-forms } \phi \text{ on } X\}^\star$$

$$=: \quad \{(p,p)\text{-currents of order 0 on } X\}.$$

$$\boxed{\sum \lambda_i V_i \quad \longleftrightarrow \quad \sum \lambda_i \int_{z\in V_i}} \qquad \varphi \mapsto \sum \lambda_i \int_{z\in V_i} \phi(z)$$

So for currents, we need to test differential forms instead of functions for measures.

Positive closed currents are global objects and can be seen as generalized submanifolds. They can be defined on non-compact manifolds X.

Example of positive closed currents

Example

If V is an analytic subset of co-dimension p of X, then $[V]$ is a positive closed (p,p)-current. If ω is a Kähler form on X, then ω^p is a positive closed (p,p)-form/current.

Example (Lelong-Poincaré equation)

If u is psh, then $dd^c u$ is a positive closed $(1,1)$-current. If $u = \log|f|$ with f holomorphic, then

$$dd^c u = [f = 0].$$

Example (Chern-Levin-Nirenberg, Bedford-Taylor, Demailly, Fornaess-Sibony)

If S is a positive closed (p,p)-current and u is psh which is locally integrable with respect to S (e.g. u locally bounded), then

$$dd^c u \wedge S := dd^c(uS)$$

is a positive closed $(p+1,p+1)$-current. In particular, if u is bounded, the Monge-Ampère measure $(dd^c u)^n$ is well-defined.

Exponential estimate (revisited)

Theorem (Vu)

Let M be a CR submanifold of X whose tangent space at each point is not contained in a complex hyperplane. Let μ be a positive measure on M bounded by a constant times the Lebesgue measure. Then μ is locally a Monge-Ampère measure $(dd^c u)^n$ with a Hölder continuous potential u (globally if X is compact).

The proof is difficult and uses CR geometry and theory of super-potentials for currents.

Theorem (D-Nguyen-Sibony, Kaufmann)

Let μ be a positive measure on X which is locally a Monge-Ampère measure with Hölder continuous potential ($\Longleftrightarrow$ Hölder super-potential). If u is psh on X and $K \subset X$ is compact, then

$$\int_K e^{\alpha|u|} d\mu < \infty \text{ for some constant } \alpha > 0.$$

The proof is inspired by the theory of interpolation between Banach spaces. It also uses Skoda's estimate when μ is the Lebesgue measure.

Exponential estimate : interpolation and sketch of proof

Theorem (interpolation, classical, very particular case)

Let $L : \mathcal{C}^0 \to \mathbb{C}$ *be a continuous linear form on the space continuous functions on a compact manifold for simplicity. So* $L : \mathcal{C}^k \to \mathbb{C}$ *is also linear and continuous. Then for every* $0 \leqslant l \leqslant k$ *there is a constant* c *independent of* L *such that*

$$\|L\|_{\mathcal{C}^l} \leqslant c\|L\|_{\mathcal{C}^0}^{1-l/k}\|L\|_{\mathcal{C}^k}^{l/k}.$$

Proof (sketch).

Let u be a $\mathcal{C}^l$ function. We need to bound $|L(u)|$. Write

$$u = u_\epsilon + (u - u_\epsilon) \quad \text{with} \quad \|u_\epsilon\|_{\mathcal{C}^k} \lesssim \epsilon^{-(k-l)} \text{ and } \|u - u_\epsilon\|_{\mathcal{C}^0} \lesssim \epsilon$$

$$|L(u)| \leqslant |L(u_\epsilon)| + |L(u - u_\epsilon)| \lesssim \|L\|_{\mathcal{C}^k}\epsilon^{-(k-l)} + \|L\|_{\mathcal{C}^0}\epsilon.$$

A good ϵ gives some estimate. The best one requires more arguments. □

Proof of exponential estimate (sketch).

Consider $\mu = (dd^c u)^n$. If u is $\mathcal{C}^2$, then μ is bounded by a constant times the Lebesgue measure and Skoda's estimate gives the result. For u Hölder, we expand $\mu = [dd^c u_\epsilon + dd^c(u - u_\epsilon)]^n = \cdots$ and apply the $\mathcal{C}^2$ case + integration by parts (Chern-Levin-Nirenberg) in order to use $\|u - u_\epsilon\|_{\mathcal{C}^0}$. □

Quasi-plurisubharmonic functions

Let X be compact Kähler, e.g. $\mathbb{P}^n$. There is no non-constant psh functions.

Definition (Yau)

A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is quasi-psh if it is locally the sum of a psh function and a smooth one.

Let ω be a Kähler form on X. Then $dd^c u \geqslant -c\omega$ for some constant $c > 0$ and hence $dd^c u + c\omega$ is a positive closed $(1,1)$-current.

Theorem (Yau, many applications)

Let μ be a smooth volume form of the same mass as ω^n. Then

$$(dd^c u + \omega)^n = \mu$$

has a smooth ω-psh solution ($c = 1$), unique up to an additive constant.

Theorem (D-Nguyen, the proof is technical)

Let μ be a positive measure with the same mass as ω^n. Then the above Monge-Ampère equation has a Hölder ω-psh solution if and only if μ has a Hölder super-potential (linear property, always unique up to a constant).

Related works : Bedford-Taylor, Berman-$\cdots$, Demailly, Kolodziedj, Vu...

Quasi-psh envelops

Definition (on compact Kähler manifolds (X, ω))

Let $K \subset X$ be a subset and h a function on K. Define the envelop

$$u_{K,h} := \sup\Big\{u \ \omega\text{-psh such that } u \leqslant h \text{ on } K\Big\}^*,$$

where the star is the upper semi-continuous regularization.

This function and the associated Monge-Ampère measure play important roles in complex (differential) geometry and problems from mathematical physics such as the study of Fekete points, random polynomials.

Theorem (Berman, Demailly-Berman, Tosatti...)

If $K = X$ and h is smooth, then $u_{K,h}$ is $\mathcal{C}^{1,1}$ (best regularity).

Theorem (D-Nguyen-Ma, Vu)

Assume that K is a smooth CR manifold with smooth boundary whose tangent space at each point is not contained in a complex hyperplane. Assume also that h is Hölder continuous. Then $u_{K,h}$ is Hölder continuous.

There are more classical versions for domains of $\mathbb{C}^n$.

Outline

Super-potentials : starting point

Let X be a compact Kähler manifold. For simplicity, assume $X = \mathbb{P}^n$.

Let ω be the Fubini-Study form of mass 1 on $\mathbb{P}^n$.

Let T be a positive closed (p,p)-current of mass 1.

If $p = 1$ (hypersurface case) there is a unique quasi-psh function u_T such that

$$dd^c u_T = T - \omega \qquad \text{and} \qquad \langle \omega^n, u_T \rangle = 0.$$

u_T is defined everywhere (u_T is called normalized quasi-potential).

If $p > 1$ there are solutions to

$$dd^c u_T = T - \omega^p \qquad \text{and} \qquad \langle u_T, \omega^{n-p+1} \rangle = 0$$

but u_T is not unique and not defined everywhere.

Super-potentials of T are canonical functions $\mathcal{U}_T : \mathcal{C}_{n-p+1} \to \mathbb{R} \cup \{-\infty\}$.

They are defined in some compact spaces of infinite dimension $\mathcal{C}_{n-p+1}$.

They play the role of quasi-potentials in bi-degree $(1,1)$ case.

Super-potentials : definition (D-Sibony)

Let $\mathcal{C}_{n-p+1}$ be the space of all positive closed currents R of bi-degree $(n-p+1, n-p+1)$ and mass 1 on $\mathbb{P}^n$.

For $R \in \mathcal{C}_{n-p+1}$ there is an $(n-p, n-p)$-current u_R such that

$$dd^c u_R = R - \omega^{n-p+1} \qquad \text{and} \qquad \langle u_R, \omega^p \rangle = 0.$$

Define

$$\mathcal{U}_T(R) := \langle T, u_R \rangle.$$

Theorem (D-Sibony)

$\mathcal{U}_T$ is well-defined, upper semi-continuous, independent of the choice of u_R and quasi-psh in some sense when R varies "holomorphically".

Some arguments of proof.

When R is not smooth, we need to regularize R in a suitable way.

Assume R smooth and choose u_R smooth. If u_T is a solution of

$$dd^c u_T = T - \omega^p \qquad \text{and} \qquad \langle u_T, \omega^{n-p+1} \rangle = 0$$

then

$$\mathcal{U}_T(R) = \langle T, u_R \rangle = \langle dd^c u_T + \omega^p, u_R \rangle = \langle dd^c u_T, R \rangle$$
$$= \langle u_T, dd^c u_R \rangle = \langle u_T, R - \omega^{n-p+1} \rangle = \langle u_T, R \rangle.$$

So we see that $\mathcal{U}_T(R)$ is independent of the choice of u_T, u_R. □

Let T be a positive closed current of bi-degree $(1, 1)$ and mass 1.

Let u_T be the normalized quasi-potential of T :

$$dd^c u_T = T - \omega \qquad \text{and} \qquad \langle \omega^n, u_T \rangle = 0.$$

$\mathcal{U}_T : \mathcal{C}_n \to \mathbb{R} \cup \{-\infty\}$ with $\mathcal{C}_n$ is the space of all probability measures on $\mathbb{P}^n$

This is a compact simplex of infinite dimension and

$$\big\{\text{extremal points}\big\} = \big\{\text{Dirac masses}\big\} \simeq \mathbb{P}^n.$$

We have

$$\mathcal{U}_T(\delta_a) = \langle u_T, \delta_a \rangle = u_T(a).$$

So $\mathcal{U}_T$ is equal to u_T on the extremal points of $\mathcal{C}_n$.

We also have for any probability measure $\mu \in \mathcal{C}_n$

$$\mathcal{U}_T(\mu) = \langle u_T, \mu \rangle = \int u_T d\mu.$$

So $\mathcal{U}_T$ is the extension by linearity of u_T to $\mathcal{C}_n$.

Super-potentials : exponential estimate

Theorem (Skoda)

When T is of bi-degree $(1,1)$ and u_T is its normalized quasi-potential, we have

$$\int e^{|u_T|} d\mathrm{Leb} \leqslant c$$

for some constant $c > 0$ independent of T.

The estimate is "exponentially" stronger than L^1 estimate.

Theorem (D-Sibony)

When T is of bidegree (p,p) and $\mathcal{U}_T$ is its super-potential, we have

$$|\mathcal{U}_T(R)| \lesssim 1 + \log^+ \|R\|_{\mathcal{C}^1}.$$

This is "exponentially" stronger than $|\mathcal{U}_T(R)| \lesssim \|R\|_{\mathcal{C}^1}$ which is easy to obtain.

For the proof, we consider the restriction of $\mathcal{U}_T$ to a suitable "holomorphic" family of R and use the exponential estimate in finite dimension.

Super-potentials : an application

Theorem (D-Sibony, Ahn, unique ergodicity)

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ *be a generic holomorphic map of algebraic degree* $d > 1$*. Then for every positive closed* (p,p)*-current* S *of mass* 1 *we have*

$$d^{-pk}(f^k)^*(S) \to T_p$$

exponentially fast, where T_p *is the dynamical Green* (p,p)*-current of* f*.*

Sketch of proof.

The invariant positive closed (p,p)-current T_p was constructed by Fornæss-Sibony. Its super-potential $\mathcal{U}$ is bounded.
Define $S_k := d^{-kp}(f^k)^*(S)$, $\mathcal{U}_k$ its super-potential and $\mathcal{U}'_k := \mathcal{U}_k - \mathcal{U}$.
We want to show $\mathcal{U}'_k(R) \to 0$ for R smooth which implies $S_k \to T_p$.
Define $\Lambda := d^{-k(p-1)}(f^k)_* : \mathcal{C}_{n-p+1} \to \mathcal{C}_{n-p+1}$. We have

$$\boxed{\mathcal{U}'_k(R) = \frac{1}{d^k}\mathcal{U}'_0(\Lambda^k(R))}.$$

$\Lambda^k(R)$ is smooth outside the critical values of f^k. We need to assume that f is generic so that the critical values have no big multiplicity. Then we can still apply the "exponential estimate" and get the result. □

Intersection for higher bi-degrees : Demailly's problem

Bi-degree $(1, 1)$: recall that if S is positive closed and locally $T = dd^c u$ with u psh S-integrable then we can define

$$T \wedge S := dd^c(uT).$$

Assume now that T is of bi-degree (p, p) and S of bi-degree (q, q) with $p + q \leqslant n$.

Assume that $\boxed{\mathcal{U}_T(S \wedge \omega^{n-p-q+1}) \neq -\infty}$. Then we can define for ϕ smooth

$$\langle T \wedge S, \phi\rangle := \mathcal{U}_T(S \wedge dd^c\phi) + \langle \omega^p \wedge S, \phi\rangle.$$

Note that $dd^c\phi$ is closed, not positive but it is the difference of two closed positive forms.

The definition is good : we only have finite numbers in the last identity and when T or S smooth

$$\begin{aligned} \text{RHS} &= \langle u_T, S \wedge dd^c\phi\rangle + \langle \omega^p \wedge S, \phi\rangle \\ &= \langle dd^c u_T, S \wedge \phi\rangle + \langle \omega^p \wedge S, \phi\rangle \\ &= \langle T, S \wedge \phi\rangle = \langle T \wedge S, \phi\rangle \text{ with the standard wedge-product.} \end{aligned}$$

Other good properties of wedge-product are also proved.

Hölder continuous super-potentials

Recall that $\mathcal{C}_{n-p+1}$ is the space of all positive closed currents of bi-degree $(n-p+1, n-p+1)$ and mass 1. It is compact for the weak topology.

For $\gamma > 0$, define the distance dist_γ on $\mathcal{C}_{n-p+1}$, related to Kantorovich-Wasserstein distance,

$$\text{dist}_\gamma(R, R') := \sup_{\|\phi\|_{\mathcal{C}^\gamma}=1} |\langle R - R', \phi\rangle|.$$

By interpolation between Banach spaces, if $0 < \gamma < \gamma'$ then

$$\text{dist}_{\gamma'} \leqslant \text{dist}_\gamma \lesssim [\text{dist}_{\gamma'}]^{\gamma/\gamma'}.$$

So $\mathcal{U}_T$ is Hölder for one dist_γ, it is Hölder for all dist_γ.

Theorem (D-Sibony, D-Nguyen)

- *The wedge-products of currents with Hölder super-potentials have Hölder super-potentials.*
- *A positive measure has a Hölder Monge-Ampère quasi-potential if and only if it has a Hölder super-potential (linear property).*

Outline

Starting point : intersection with dimension excess

Let T, S be positive closed currents of bi-degree (p,p) and (q,q).

If $T \wedge S$ is well-defined, it is of bi-degree $(p+q, p+q)$, zero if $p+q > n$.

So if V, W are submanifolds of codimension p and q with $p+q > n$, then $V \cap W$ cannot be seen using classical intersection of currents.

Generically, this intersection is empty. But it may be sometimes non-empty.

More generally, if $V \cap W \neq \varnothing$, we have

$$\text{codim} V \cap W \leqslant \text{codim} V + \text{codim} W.$$

If the inequality is strict, then we have the phenomenon of dimension excess.

Aim of the theory of densities : to get a more general theory of intersection of currents and to measure the dimension excess.

Idea :

- Consider the intersection between $T \otimes S$ and the diagonal Δ of X^2.
- The intersection doesn't change if we dilate the coordinates in the normal directions to Δ.
- Consider the limits when the factor of dilation tends to infinity.

Relative density between two currents

Let $\pi : E \to \Delta$ be the normal bundle to Δ in X^2. We identify Δ with the section zero of E. Let $A_\lambda : E \to E$ be the fiberwise multiplication by λ.

$T \otimes S$ is defined on X^2 and there is no holomorphic identification between neighbourhoods of Δ in X^2 and neighbourhoods of Δ in E.

Consider a diffeomorphism τ from a neighbourhood of Δ in X^2 to a neighbourhood of Δ in E which is admissible :

$$\tau : \Delta \to \Delta \ \text{ is identity} \qquad \text{and} \qquad d\tau : E \to E \ \text{ is identity.}$$

Theorem (D-Sibony)

- *The family of currents $(A_\lambda)_* \tau_*(T \otimes S)$ is relatively compact.*
- *When $\lambda \to \infty$, any limit value of $(A_\lambda)_* \tau_*(T \otimes S)$ is positive closed in $\overline{E}$.*
- *All these tangent currents are independent of τ and are in the same cohomology class, denoted by $\Theta(T, S)$.*
- *If S is the Dirac mass at a, then the degree of this cohomology class is the Lelong number of T at a.*
- *(upper semi-continuity) If $T_k \to T$ and $S_k \to S$ then any limit value of $\Theta(T_k, S_k)$ is smaller or equal to $\Theta(T, S)$.*

Some applications

Theorem (D-Sibony, D-Nguyen-Truong, D-Nguyen-Vu)

Let $f : X \to X$ *be a dominant meromorphic map or correspondence. Let* P_k *be the set of isolated periodic points of period* k*. Then*

- f *is Artin-Mazur :* $\#P_k$ *grows at most exponentially fast. Moreover,*

$$\limsup_{k\to\infty} \frac{1}{k} \log \#P_k \leqslant h_a(f) \text{ the algebraic entropy.}$$

- *In several cases, e.g. for Hénon maps in any dimension,* P_k *is equidistributed with respect to a dynamical probability measure.*

Remark

Kaloshin and al. : wrong for real smooth maps.

A key point.

P_k can be identified to a part the intersection $\Gamma_k \cap \Delta$ between the graph of f^k and the diagonal.
We use the theory of densities to control the positive dimension part and the transversality of the intersection when $k \to \infty$ □

Some applications

Theorem (Fornæss-Sibony, D-Sibony, unique ergodicity for foliation)

Let $\mathcal{F}$ be a foliation by Riemann surfaces in $\mathbb{P}^2$ such that all singularities are hyperbolic (generic situation). Then

- either $\mathcal{F}$ has a unique directed harmonic current, non-closed, of mass 1, and no algebraic leaf;

- or all directed harmonic currents are closed, supported by algebraic leaves.

A key point.

We use the theory of densities for harmonic currents. □

Theorem (D-Sibony, wedge-product with likely weakest condition)

Assume that $T \otimes S$ has only one tangent current $\mathcal{R}$ along Δ and there is no dimension excess. If $\pi : E \to \Delta$ is the canonical projection, then there is a unique positive closed $(p+q, p+q)$-current R on $\Delta \simeq X$ such that

$$\mathcal{R} = \pi^*(R).$$

We then define

$$T \curlywedge S := R.$$

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– Thank You –