Discrete groups of automorphisms of the ball

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References:

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1) Complex hyperbolic geometry
2) Discrete groups
3) Effective aspects.

1) Complex hyperbolic geometry.

$\mathbb{B}^n \subset \mathbb{C}^n$ unit ball $\{(z_1,\dots,z_n)\in\mathbb{C}^n \mid |z_1|^2+\dots+|z_n|^2<1\}$

Bihol$(\mathbb{B}^n)$ acts transitively on $\mathbb{B}^n$

$\exists!$ Kähler metric on $\mathbb{B}^n$ that is invariant under Bihol$(\mathbb{B}^n)$ (Bergman metric)

Kähler form $\omega = i\,\partial\bar\partial \log(1-\|z\|^2)$

Note $0\in\mathbb{B}^n$ is an isolated fixed point of the biholomorphism $z\mapsto -z$

$\rightsquigarrow$ get a homogeneous Riemannian manifold s.t. every point is an isolated fixed point of an isometry, i.e. a symmetric space $H^n_{\mathbb{C}}$

Concrete description? view $\mathbb{C}^n \supset \mathbb{B}^n$ as the affine chart $z_{n+1} \neq 0$ in $V = \mathbb{C}^{n+1}$, with coordinates $\left(\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_{n+1}}\right)$.

$$\left|\frac{z_1}{z_{n+1}}\right|^2 + \dots + \left|\frac{z_n}{z_{n+1}}\right|^2 < 1 \iff |z_1|^2 + \dots + |z_n|^2 - |z_{n+1}|^2 < 0$$

$\rightsquigarrow$ consider the Hermitian form on V given by

$$\langle v, w \rangle = v_1 \bar{w}_1 + \dots + v_n \bar{w}_n - v_{n+1} \bar{w}_{n+1}$$
$$= w^* H v$$

$\left(H = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} \right.$ [std Lorentz Hermitian form], or more generally any Hermitian form with signature $(n,1)$ $\Big)$

$V_- = \{ v \in V \mid \langle v, v \rangle < 0 \}$
$V_0 = \{ v \in V \mid \langle v, v \rangle = 0 \}$ are scaling invariant

$$\mathbb{P}: V \setminus \{0\} \longrightarrow \mathbb{P}(V) = \mathbb{P}^n_{\mathbb{C}}$$
(set of complex 1-dim. subspaces of V)

the unit ball $\mathbb{B}^n$ corresponds to $\mathbb{P}(V_-)$
$\partial \mathbb{B}^n$ ——— $\mathbb{P}(V_0)$.

$U(H) = \{ A \in GL(V) \mid \underbrace{A^*}_{\text{conjugate-transpose}} HA = H \}$ preserves V_- and V_0.

(if H is the std Lorentz Hermitian form above, write $U(n,1)$).

To get an effective action on $\mathbb{P}^n_{\mathbb{C}}$, consider

$$PU(H) = U(H) / \text{scalar matrices}$$

$\rightsquigarrow$ $PU(n,1)$ acts as a group of biholomorphisms of $\mathbb{B}^n$.

<u>Facts</u>: 1) $PU(n,1)$ acts on $\mathbb{B}^n$ by biholomorphisms, and every biholo comes from some $\boxed{A} \in PU(n,1)$

2) The action of $PU(n,1)$ is transitive on $\mathbb{B}^n$, and also on $\partial\mathbb{B}^n$.

(Gramm-Schmidt to complete a vector to a suitable basis).

in particular, $\mathbb{B}^n \simeq PU(n,1) / \mathrm{Stab}_{PU(n,1)}\{pt\}$

$$\cong PU(n,1) / \underbrace{P(U(n) \times U(1))}_{\text{stab of } (0,\dots,0,1) \in \mathbb{C}^n}$$

$$= G/K$$

which corresponds to the origin in $\mathbb{B}^n$;

and K is <u>compact</u>.

From this it follows that there exists a G-invariant metric on G/K, and that metric is in fact unique (up to scale) because K acts irreducibly on $T_0\mathbb{B}^n \simeq \mathbb{C}$ (cf. $U(n)$ acts irreducibly on $\mathbb{C}^n$).

$\rightsquigarrow$ get a symmetric metric on G/K

$\mathbb{B}^n$ with that metric is called <u>complex hyperbolic space</u> denoted by $H^n_{\mathbb{C}}$.

$\partial\mathbb{B}^n$ corresponds to the <u>visual</u> boundary (also called <u>ideal</u> boundary) of $H^n_{\mathbb{C}}$, sometimes denoted by

$$\partial_\infty H^n_{\mathbb{C}}.$$

Note the above description makes sense to define hyperbolic spaces $H^n_{\mathbb{K}}$ over other fields

$\mathbb{K} = \mathbb{R}$ $\rightsquigarrow$ real hyperbolic space
$H^n_{\mathbb{R}} = PO(n,1) / P(O(n) \times O(1))$

$\mathbb{K} = \mathbb{H}$ (quaternions) $\rightsquigarrow$ quaternionic hyperb. space
$H^n_{\mathbb{H}} = PSp(n,1) / Sp(n)$

$\mathbb{K} = \mathbb{O}$ (octonions) when $n=2$ $H^2_{\mathbb{O}} = F_4^{-20} / K$.

In ball coordinates,

$$g = \frac{4}{(1-\|z\|^2)^2}\left\{ \sum_j z^j d\bar{z}^j \sum_k \bar{z}^k dz^k + (1-\|z\|^2)\sum_j dz^j d\bar{z}^j \right\}$$

→ gives nice normalization of the curvature, the holomorphic sectional curvature is $\equiv -1$.

distance function:

$$\cosh\left(\tfrac{1}{2} d(z,w)\right) = \frac{|\langle \tilde{z}, \tilde{w}\rangle|}{\sqrt{\langle \tilde{z},\tilde{z}\rangle \langle \tilde{w},\tilde{w}\rangle}}$$

where $\tilde{z}, \tilde{w}$ are (any) lifts in $\mathbb{C}^{n+1}$ of $z, w \in \mathbb{B}^n \subset \mathbb{P}^n_{\mathbb{C}}$.

- The metric is complete
- $\partial_\infty H^n_{\mathbb{C}}$ is at "infinite distance" from any point in $H_{\mathbb{C}}$

The real sectional curvature of a real 2-plane Π in $T_z\mathbb{B}^n$ is given by the formula

$$K(\Pi) = -\frac{1+3\cos^2\alpha}{4}$$

where $\alpha = \measuredangle(\Pi, J\Pi)$ (angle of holomorphy)
↳ complex structure on $\mathbb{B}^n \subset \mathbb{C}^n$!

So the real sectional curvatures satisfy ($\frac{1}{4}$-pinching)

$$-1 \leq K_{sect} \leq -\frac{1}{4}$$

correspond to real 2-planes tangent to complex lines

↳ correspond to "totally real 2-planes, $\Pi \perp J\Pi$
example: $\mathbb{B}^2 \cap \mathbb{R}^2 \subset \mathbb{B}^2$.

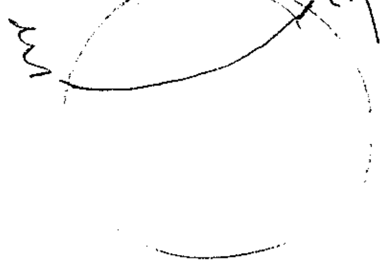
In particular, there is no zero curvature, i.e. this symmetric space has **rank 1** (rank = largest dimension of a euclidean subspace).

Negative curvature implies that the distance function is strictly convex; given two geodesics $\gamma_1, \gamma_2 : I \to H^n_{\mathbb{C}}$, the function $d(t) = d_{H^n_{\mathbb{C}}}(\gamma_1(t), \gamma_2(t))$ is convex

(this follows from elementary comparison theorems in Riemannian geometry, see $K_{sect} \leq -\frac{1}{4}$)

In particular:

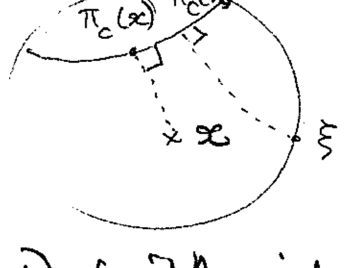
- any $x \neq y \in H^n_{\mathbb{C}}$ are joined by a unique geodesic, and this is also true for points $\xi, \eta \in \partial_\infty H^n_{\mathbb{C}}$.



("visibility property", not true in higher rank)

- Given any (non-empty) convex subset $C \subset H^n_{\mathbb{C}}$, there is a well-defined nearest point projection
$$\pi_C : H^n_{\mathbb{C}} \longrightarrow C$$
(which extends continuously to $H^n_{\mathbb{C}} \cup \partial_\infty H^n_{\mathbb{C}}$).

(proper map)



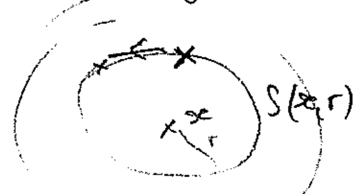
Remark: $\mathrm{Isom}\, H^n_{\mathbb{C}}$ contains $PU(n,1)$ with index 2, the full isometry group is generated by $PU(n,1)$ and complex conjugation.

Isometries extend to homeomorphisms of the closed ball $\overline{H^n_{\mathbb{C}}} = H^n_{\mathbb{C}} \cup \partial_\infty H^n_{\mathbb{C}}$.

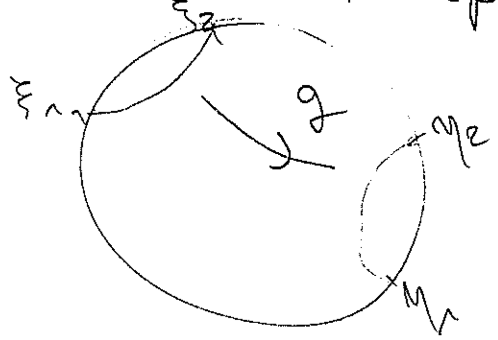
Homogeneity properties of isometry group?

1) Given $x_1, y_1, x_2, y_2 \in H^n_{\mathbb{C}}$, there exists an isometry $g \in PU(n,1) = G$ s.t. $g(x_j) = y_j$
$\iff d(x_1, x_2) = d(y_1, y_2)$

In particular, the action of $Stab_G(x)$ is transitive on any sphere $S(x, r)$, $r > 0$



2) Given $\xi_1, \eta_1, \xi_2, \eta_2 \in \partial_\infty H^n_{\mathbb{C}}$, there exists
$(\xi_1 \neq \xi_2,\ \eta_1 \neq \eta_2)$
a $g \in G$ s.t. $g(\xi_j) = \eta_j \ \forall j = 1, 2$
(action is transitive on the space of geodesics)



3) The action is not transitive on the set of triples of points in $\partial_\infty H^n_{\mathbb{C}}$, there is an invariant, called the Cartan angular invariant,

$$\arg\left(\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle \langle \tilde{\xi}_2, \tilde{\xi}_3 \rangle \langle \tilde{\xi}_3, \tilde{\xi}_1 \rangle\right)$$

($\tilde{\xi}_j \in \mathbb{C}^{n+1}$ lifts of pts in $\partial_\infty H^n_{\mathbb{C}}$, in particular $\langle \tilde{\xi}_i, \tilde{\xi}_j \rangle = 0$).

(and this is actually a complete invariant).

Classification of isometries

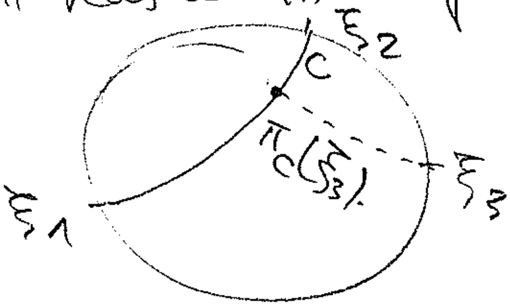
Any isometry induces a continuous self-map of the closed ball $\overline{H}^n_{\mathbb{C}}$, by Brouwer it must have at least one fixed point.

$[A] \in PU(n,1)$ is called

- elliptic if it has at least one fixed pt in $H^n_{\mathbb{C}}$
- parabolic if it has exactly one fixed point, which is in $\partial_\infty H^n_{\mathbb{C}}$
- loxodromic if it has exactly two fixed points, which are in $\partial_\infty H^n_{\mathbb{C}}$.

fact: these three classes cover all isometries!

If A has three different fixed points in $\partial_\infty H^n_{\mathbb{C}}$, then it has a fixed point in $H^n_{\mathbb{C}}$:

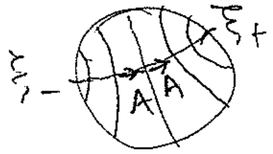


the geodesic C from ξ_1 to ξ_2 is A-invariant, so $\pi_C(\xi_3)$ is fixed.

These classes can be characterized by linear algebra (fixed pts correspond to eigenvectors).

One easily sees that loxodromic isometries A have nice dynamical properties: if $\xi_-, \xi_+ \in \partial_\infty H^n_{\mathbb{C}}$ are its fixed points, the geodesic from ξ_- to ξ_+ is invariant, and A acts as a translation on that geodesic in particular for every $x \in]\xi_-, \xi_+[$, $A^k x \xrightarrow[k \to \pm\infty]{} \xi_\pm$

In fact, for every $x \in \overline{H}^n_{\mathbb{C}}$ with $x \neq \xi_-$, $A^k x \xrightarrow{k \to \infty} \xi_+$



$x \neq \xi_+$, $A^k x \xrightarrow{k \to -\infty} \xi_-$

We will say that the map $\overline{H}^n_{\mathbb{C}} \setminus \{\xi_-\} \longrightarrow \overline{H}^n_{\mathbb{C}}$, $x \longmapsto \xi_+$

is a "quasi-constant".

The previous discussion shows that for a loxodromic A, A^k converges uniformly on compact sets of $\overline{H}^n_{\mathbb{C}} \setminus \{\xi_-\}$ to a quasi-constant, as $k \to \infty$.
(and similarly for $k \to -\infty$).

If A is parabolic, something similar occurs (but it converges to constant, not just a quasi-constant)

<u>Remark</u>: special case of loxodromic element: product of two "half-turns" (isometries conjugate to $z \mapsto -z$)
these are called <u>hyperbolic</u> isometries.
($\Leftrightarrow$ a suitable matrix representative has real eigenvalues).

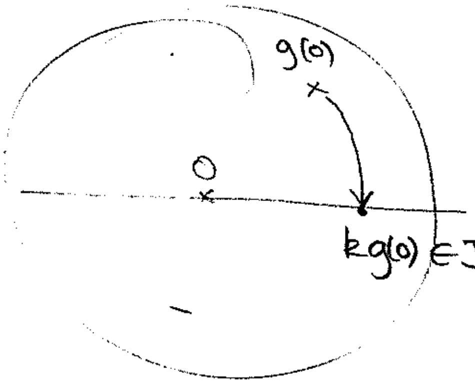
The behavior of loxodromic isometries can be strengthened to get some kind of compactness property for $\mathrm{Isom}\, H^n_{\mathbb{C}} = G$

> For every sequence $(\gamma_i)_{i \in \mathbb{N}}$ of elements in G, there is a subsequence that converges, either to some $\gamma \in G$ or to a quasiconstant (and the convergence is uniform on compact sets of $\overline{H}^n_{\mathbb{C}} \setminus \{pt\}$).

(we call this the "<u>convergence property</u>")

pf of the convergence property relies on the Cartan decomposition $G = KAK$, where $K = Stab_G(0)$

$A =$ group of hyperbolic transformations along a fixed geodesic through 0. (γ)



$\rightsquigarrow akg(0) = 0$ ($\cong \mathbb{R}, +$ as a group)

ie. $k'akg = id$.

Given a sequence $\gamma_j = k_j a_j k_j'$, we may assume (up to extraction) that $k_j \xrightarrow{j\to\infty} k$

$k_j' \xrightarrow{j\to\infty} k'$

if no subsequence of a_j converges, we may assume it converges to a quasi-constant

$x_y : \overline{H}^n_{\mathbb{C}} \setminus \{y\} \to \overline{H}^n_{\mathbb{C}}$

$u \longmapsto x$

(with $x \neq y$)

then $k_j a_j k_j' \xrightarrow{j\to\infty} k x$ (uniformly on compact sets away from $k'^{-1}(y)$).

Classification of totally geodesic submanifolds.

Come from the obvious Lie group inclusions

yield holomorphic submanifolds $\rightarrow$ $SU(k,1) \subset SU(n,1)$ $(k \leq n)$

$SO(k,1) \subset SO(n,1) \subset SU(n,1)$

yield "totally real" submanifolds.

Note: Holomorphic totally geodesic submanifolds are obtained by taking (the intersection with $\mathbb{B}^n$) of complex affine subspaces in $\mathbb{C}^n$; this is obvious in homogeneous coordinates, they correspond to complex linear subspaces $W \subset V$ such that $H|_W$ has signature $(k,1)$.
(restriction of the Hermitian form

An important special case of complex totally geodesic subspace is given by those of dimension 1 ($\dim_{\mathbb{C}} W = 2$ in the above description, $H|_W$ with signature $(1,1)$)
These are called complex geodesics in $\mathbb{B}^n$.
Note that given $x \neq y \in \mathbb{B}^n$, there is a unique complex geodesic containing x and y.
Complex geodesics are copies of the unit disk, the induced metric is the usual Poincaré metric on that disk.

Totally real submanifolds (of maximal dimension) can be obtained by taking $\mathbb{B}^n_{\mathbb{R}} = \mathbb{B}^n \cap \mathbb{R}^n$, which is the fixed point set of the isometry $z \mapsto \bar{z}$.
(note that the fixed point set of an isometry is always totally geodesic).
Another description: take a totally real subspace $W_{\mathbb{R}} \subset V$, ie. a real linear subspace such that $H|_{W_{\mathbb{R}}}$ is real and has signature $(k,1)$.
equivalently, there exists an $\mathbb{R}$-basis $e_1, \dots, e_{k+1}$ such that $\langle e_j, e_k \rangle \in \mathbb{R}\ \forall j,k$.

Then $\mathbb{P}(W_{\mathbb{R}} \cap V_-)$ is a totally geodesic copy of $H^k_{\mathbb{R}}$.

The special case $k=1$ corresponds to real geodesics. In particular, to describe the real geodesic between z and $w \in \mathbb{B}^n$, consider the real span of $\tilde{z}$ and $\alpha\tilde{w}$ where $\tilde{z} = (z,1)$, $\tilde{w} = (w,1)$ and $\alpha \in \mathbb{C}$ is chosen so that $\langle \tilde{z}, \tilde{w} \rangle \in \mathbb{R}$ eg $\alpha = \langle \tilde{z}, \tilde{w} \rangle$.

Thm: every (complete) totally geodesic submanifold of $\mathbb{B}^n$ is among the two families just described.

In particular, there are no totally geodesic submanifolds of real codimension 1, which makes the construction of fundamental domains very difficult!

A "reasonable" substitute is given by so called bisectors.

$(z \neq w)$ $B(z,w) = \{x \in \mathbb{B}^n \mid d(z,x) = d(x,w)\}$.

↳ this is globally preserved (but NOT fixed) by the isometries that exchange z and w.

If Σ = complex geodesic through z and w (called the complex spine of the bisector)

σ = real geodesic in Σ equidistant of z and w

Then $B(z,w) = \pi_\Sigma^{-1}(\sigma)$, where $\pi_\Sigma : H^n_{\mathbb{C}} \to \Sigma$ denotes orthogonal projection.

In particular, if $B(z_1,w_1) = B(z_2,w_2)$, then the complex lines Σ_1 through z_1, w_1, Σ_2 through z_2, w_2 must coincide, unlike what happens in constant curvature!

$B(z,w)$ is a topological ball, and its trace at infinity $\partial_\infty B(z,w)$ is a topological sphere, but intersection of pairs of bisectors can be pretty awful (in general not smooth, not connected).

Bisectors are not convex, but they are <u>quasi-convex</u>, i.e. there is a $R>0$ such that for every bisector B and $x,y \in B$, the geodesic segment $[x,y]$ is at distance $< R$ from B.

(this follows from comparison theorems in Riemannian geometry and δ-hyperbolicity of $H^2_{\mathbb{R}}$).

<u>The Heisenberg group</u>.

Appears naturally when studying the stabilizer in $PU(n,1)$ of a point at infinity $\xi \in \partial_\infty H^n_{\mathbb{C}}$.

To see this, use a different Hermitian form

$$\langle v,w\rangle = w^* J v$$

where
$$J \doteq \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & I_{n-1} & 0 \\ \hline 1 & 0 & 0 \end{array}\right)$$

$$\langle v,w\rangle = v_1 \bar{w}_{n+1} + v_{n+1}\bar{w}_1 + \sum_{j=2}^{n} v_j \bar{w}_j$$

$$\langle v,v\rangle = 2\,\mathrm{Re}(v_1 \bar{v}_{n+1}) + \sum_{i=2}^{n} |v_i|^2$$

J has signature $(n,1)$, so we have another model of $H^n_{\mathbb{C}}$, called the Siegel half space, obtained once again in the chart $z_{n+1} \neq 0$

$$U = \left\{ v \in \mathbb{C}^n \;\middle|\; 2\,\mathrm{Re}(v_1) + \sum_{j=2}^{n} |v_j|^2 < 0 \right\}$$

gives an unbounded domain biholomorphic to the ball

the boundary ∂U under this biholomorphism corresponds to $\partial B \setminus \{pt\}$

ie we sent one point of ∂B off to infinity.

(In homogeneous coordinates, that point is $p_\infty = (1,0,0,\dots,0)$).

A large part of the stabilizer of p_∞ in $U(J)$ consists of unipotent upper triangular matrices.

One checks that every unipotent element that stabilizes p_∞ can be written (uniquely) as

$$M(\tau,t) = \left(\begin{array}{c|c|c} 1 & -\tau^* & \frac{1}{2}(it - |\tau|^2) \\ \hline 0 & I_{n-1} & \tau \\ \hline 0 & 0 & 1 \end{array} \right)$$

for some $\tau \in \mathbb{C}_{n-1}$, $t \in \mathbb{R}$

column vector

These act transitively on ∂U ($\simeq \partial B \setminus \{pt\}$).

One checks that

$$M(\tau,t)\,M(\sigma,s) = M(\tau+\sigma,\; t+s+2\,\mathrm{Im}(\tau\cdot\bar{\sigma}))$$

usual dot product

which defines a group law on $N = \mathbb{C}_{n-1} \times \mathbb{R}$

$$(\tau,t) * (\sigma,s) = (\tau+\sigma,\; t+s+2\,\mathrm{Im}(\tau\cdot\bar{\sigma}))$$

called the Heisenberg group law.

The full stabilizer of p_∞ in $PU(J)$ is larger, it also contains "rotations" and "dilations",

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad A \in U(n-1)$$

(elliptic elements)

$$\left(\begin{array}{c|c|c} \lambda & 0 & 0 \\ \hline 0 & I_{n-1} & 0 \\ \hline 0 & 0 & \frac{1}{\lambda} \end{array}\right), \quad \lambda \in \mathbb{R}_+^*$$

(loxodromic elements)
hyperbolic

fact: these generate the stabilizer of p_∞, which is a semi-direct product

$$\left(\mathbb{R}_+^* \times U(n-1)\right) \ltimes N,$$

Heisenberg.

with the obvious action of rotation on $N = \mathbb{C}^{n-1} \times \mathbb{R}$

$$A \in U_{n-1} \qquad (\tau, t) \mapsto (A\tau, t)$$

and action of hyperbolic element

$$\lambda \in \mathbb{R}_+^* \qquad (\tau, t) \mapsto (\lambda\tau, \lambda^2 t).$$

There is a natural contact distribution on $N = \mathbb{C}^{n-1} \times \mathbb{R}$ that comes from the fact that it is a real hypersurface S in $\mathbb{C}^n$, take $TS \cap J(TS)$ for J the std complex structure on $\mathbb{C}^n$

For $n=2$, this gives the distribution in $\mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$
$(z, t) \simeq (x, y, t)$

spanned by $\partial_x + 2y\,\partial_t$ and $\partial_y - 2x\,\partial_t$

at $(0,0)$ it is just the horizontal plane, but otherwise it is "tilted".

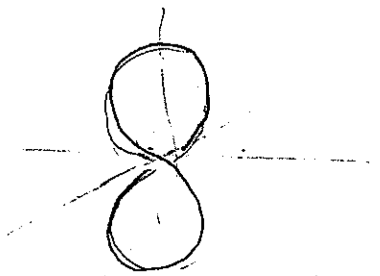
Note $\partial_\infty H^1_{\mathbb{C}}$, $\partial_\infty H^2_{\mathbb{R}}$ are topological circles in $\partial_\infty H^2_{\mathbb{C}}$, so they give curves in $N = \mathbb{C} \times \mathbb{R}$ (Heisenberg).

the corresponding curves are called

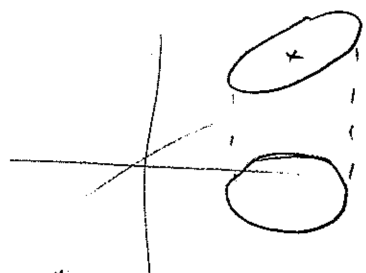
$\mathbb{R}$- circles (for $\partial_\infty H^2_{\mathbb{R}}$)

and $\mathbb{C}$- circles (for $\partial_\infty H^1_{\mathbb{C}}$).

$\mathbb{R}$-circles are tangent to the contact distribution, whereas $\mathbb{C}$-circles are transverse to that distrib.



typical $\mathbb{R}$-circle (projects to a lemniscate in $\mathbb{C}$)



typical $\mathbb{C}$-circle (ellipse that projects to a round circle in $\mathbb{C}$)

these circles are linked $\Leftrightarrow$ the corresponding totally geodesic submanifolds intersect in $H^2_{\mathbb{C}}$.

(for more details, see Goldman's book)

Horospherical coordinates

$$\underset{\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}_+}{\underset{\rotatebox{90}{$\in$}}{(\zeta, v, u)}} \longrightarrow \begin{pmatrix} \frac{-|\zeta|^2 - u + iv}{2} \\ \zeta \\ 1 \end{pmatrix}.$$

$\underbrace{\mathbb{C}^{n-1}\times\mathbb{R}}_{\text{Heisenberg}}$

↳ negative vector w.r.t $\begin{pmatrix} & & 1 \\ & I_m & \\ 1 & & \end{pmatrix}$ for $u > 0$, null vector for $u = 0$.

metric in these coordinates:

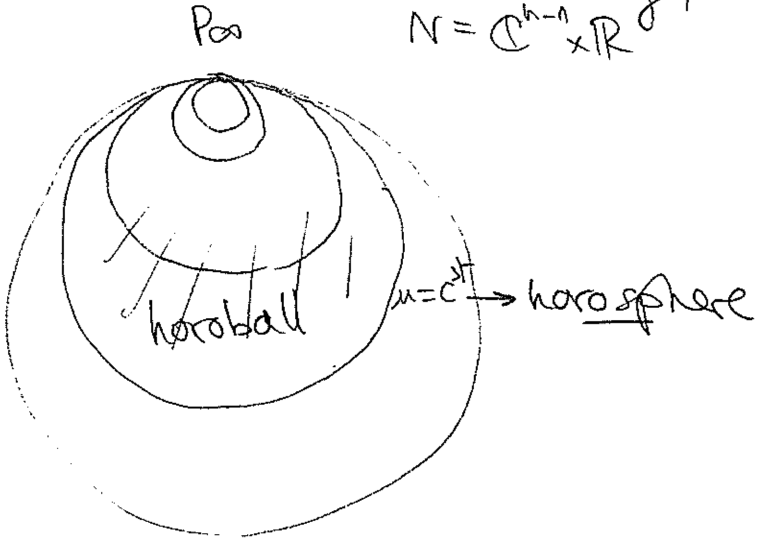
$$g = \frac{du^2 + (dv - 2\,\mathrm{Im}\langle\langle \zeta, d\zeta\rangle\rangle)^2 + 4u\langle\langle d\zeta, d\zeta\rangle\rangle}{u^2}.$$

volume form

$$d\mathrm{vol}_g = \frac{4^{n-1}}{u^{n+1}}\, du\, dv\, \underbrace{d\mathrm{vol}_{\mathbb{C}^{n-1}}}_{\text{Euclidean volume}}$$

In particular, for any compact set M in Heisenberg, the set $M \times [a, \infty[$ has finite volume for every $a > 0$.

$N = \mathbb{C}^{n-1}\times\mathbb{R}$



$u \geq c^{st}$ horoball

$u = c^{st}$ horosphere

2) Discrete groups

$PU(n,1)$ is a linear group (e.g. use the adjoint representation)
$\to$ natural topology ($\Leftarrow$ convergence of some lifts in $U(n,1)$)

A subgroup $\Gamma \subset PU(n,1)$ is discrete if every element in Γ is isolated $\iff$ $e \in \Gamma$ (identity element) is isolated in Γ.

We write $G = PU(n,1)$.

Since $H^n_{\mathbb{C}} = G/K$ and K is compact, the projection $G \to G/K$ is proper, so if Γ is discrete then every orbit Γx, $x \in H^n_{\mathbb{C}}$ is discrete in $G/K = H^n_{\mathbb{C}}$.
(and the converse is clear).

More generally, Γ discrete is equivalent to saying that for every compact set $M \subset H^n_{\mathbb{C}}$,
$$\{\gamma \in \Gamma \mid M \cap \gamma M \neq \emptyset\}$$ is finite,

or equivalently that the action $\Gamma \times H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$, $(\gamma, x) \mapsto \gamma x$ is a proper map, when Γ is equipped with the discrete topology.

This implies that the quotient $\Gamma \backslash H^n_{\mathbb{C}}$ has the structure of an orbifold (locally $H^n_{\mathbb{C}}$ modulo a finite group)

If Γ acts without fixed points on $H^n_{\mathbb{C}}$, ie. there are no elliptic elements in Γ, the quotient is actually a manifold (called a locally symmetric space).

If Γ is discrete, elliptic elements are precisely the elements of Γ of finite order, so for Γ discrete,

no elliptics $\Longleftrightarrow$ torsion-free

(only element of finite order is identity).

Selberg's lemma says that if Γ is finitely generated, then it has a finite index subgroup which is torsion-free, ie there exists $\Gamma_0 \underset{\text{finite index}}{\subset} \Gamma$ such that

$$H^n_{\mathbb{C}}/\Gamma_0 \text{ is a manifold}$$

If $\Gamma \subset PU(n,1)$ is discrete and $\underset{\uparrow}{Vol}(H^n_{\mathbb{C}}/\Gamma) < \infty$,

Riemannian volume coming from the symmetric metric

Γ is called a lattice.

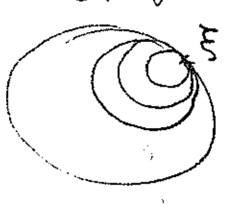
(called "uniform" or "co-compact" lattice if furthermore $H^n_{\mathbb{C}}/\Gamma$ is compact)

Lattices exist for every n (both uniform and non-uniform) due to a result of Borel (in fact they exist in the isometry groups of all symmetric spaces)

* All lattices are finitely generated
* Non-uniform lattices have finitely many ends, and each end has the structure of a cusp, looks topologically like $\mathbb{R}_+ \times M$

where M is a compact quotient of the Heisenberg group.

the $\mathbb{R}_+$ factor corresponds to a parametrization of horospheres based at a pt $\xi \in \partial_\infty H^n_{\mathbb{C}}$



If Γ has no torsion and no torsion at ∞ (the parabolic elements in the group have trivial rotation/unitary component), can use this to construct the so-called
toroidal compactification

(Mumford, Ash-Rappaport-Tsai, Siu, Mok)

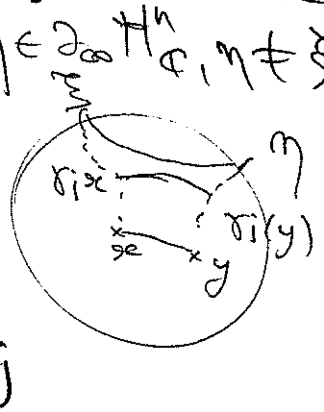
We want to consider non-lattice discrete groups, so we won't assume $\mathrm{Vol}(\mathbb{H}^n_{\mathbb{C}}/\Gamma)<\infty$ in the sequel.

A basic tool to study discrete groups: limit set.

Prop 1: Let (γ_j) be a sequence of elements in $G=PU(n,1)$ such that $\gamma_j x \xrightarrow[j\to\infty]{} \xi$ for some $x\in \mathbb{H}^n_{\mathbb{C}}$, $\xi\in\partial_\infty \mathbb{H}^n_{\mathbb{C}}$.

Then for every $y\in\mathbb{H}^n_{\mathbb{C}}$, $\gamma_j(y)\xrightarrow{j\to\infty}\xi$.

pf. Suppose $\gamma_j(y)$ does not converge to ξ. Then we may assume it converges to some $\eta\in\partial_\infty \mathbb{H}^n_{\mathbb{C}}$, $\eta\neq\xi$ (up to extraction).



Then $d(\gamma_j x,\gamma_j y)\xrightarrow{j\to\infty}\infty$ but this is impossible since
$$d(\gamma_j x,\gamma_j y)=d(x,y)\quad\forall j$$
(γ_j's are isometries).

Consequence: the set of accumulation pts in $\partial_\infty\mathbb{H}^n_{\mathbb{C}}$ of Γx (seen as a sequence in $\overline{\mathbb{H}^n_{\mathbb{C}}}=\mathbb{H}^n_{\mathbb{C}}\cup\partial_\infty\mathbb{H}^n_{\mathbb{C}}$) is independent of $x\in\mathbb{H}^n_{\mathbb{C}}$.

This is called the limit set of Γ, denoted by Λ_Γ.

Prop 2: Let $\Gamma \subset PU(n,1)$ be a subgroup without any global fixedpoint in $\partial_\infty H^n_{\mathbb{C}}$.
Then Λ_Γ is the smallest closed non-empty Γ-invariant subset in $\partial_\infty H^n_{\mathbb{C}}$.

In the sequel, we write $\underline{S = \partial_\infty H^n_{\mathbb{C}}}$.

pf: we reformulate the result as follows:

$$\boxed{\begin{array}{c}\text{Suppose } X \subset S \text{ is a closed } \Gamma\text{-invariant} \\ \text{set with at least two points.} \\ \text{Then } \Lambda_\Gamma \subset X.\end{array}}$$

Now take $\eta_1 \neq \eta_2 \in X$, and $\xi \in \Lambda_\Gamma$, say $\xi = \lim_{j\to\infty} \gamma_j x$

If $\gamma_j \eta_1 \xrightarrow{j\to\infty} \xi$ or $\gamma_j \eta_2 \xrightarrow{j\to\infty} \xi$ then we are done
(because X is closed and Γ-invariant)

If not, we may assume (by extracting) $\gamma_j \eta_1 \xrightarrow{j\to\infty} \xi_1$, $\gamma_j \eta_2 \xrightarrow{j\to\infty} \xi_2$
with $\xi_1 \neq \xi$ and $\xi_2 \neq \xi$.

Pick a point $p \in]\eta_1, \eta_2[$ (geodesic between η_1 and η_2).

We may assume $\gamma_j(p)$ converges to some point in $[\xi_1, \xi_2]$ (again replacing γ_i by a subsequence if necessary)
which contradicts $\gamma_j(p) \xrightarrow{j\to\infty} \xi$ (prop 1).

Since fixed points of loxodromic/parabolic elements in Γ are clearly in the limit set of Γ, we have that

$$\boxed{\Lambda_\Gamma = \text{closure of the orbit of any loxo/parab. fixed point for an elt in } \Gamma.}$$

Prop 3: (i) If $\Gamma_1 \subset \Gamma_2$, then $\Lambda_{\Gamma_1} \subset \Lambda_{\Gamma_2}$

(ii) If $\Gamma_1 \subset \Gamma_2$ is a subgroup of finite index, then $\Lambda_{\Gamma_1} = \Lambda_{\Gamma_2}$.

(iii) If $N \triangleleft \Gamma$ is a normal subgroup such that $\Lambda_N \neq \phi$ and Γ has no global fixed point, then $\Lambda_N = \Lambda_\Gamma$.

pf of (iii): Λ_N is closed and Γ-invariant by normality:

If $n_j \in N$, $n_j \cdot x \xrightarrow{j\to\infty} \xi$ and $\gamma \in \Gamma$,

$$\underbrace{\gamma n_j \gamma^{-1}}_{\in N} \cdot x \xrightarrow{j\to\infty} \gamma \xi$$

so $\gamma \xi \in \Lambda_N$

If Λ_N doesn't have 2 pts, then Γ has a global fixed point!

So $\Lambda_\Gamma \subset \Lambda_N \subset \Lambda_\Gamma$ $\square$

From now on, we consider discrete groups.
Would like criteria to verify that property, but there is no completely satisfactory general method.

Sufficient conditions:

* arithmetic ($\mathbb{Z} \subset \mathbb{R}$ is discrete $\to$ extend this to matrices)
* ping-pong $\to$ free groups (Schottky)
* fundamental domains (painful!)

necessary conditions:

* no loxo/parab with a common fixed pt
* no elliptic element of infinite order
* Jorgensen / Shimizu inequality

necessary and sufficient?

* (Γ non elementary) and every pair of elements satisfies Jorgensen
* every elliptic element has finite order

nice theoretically but useless in practice!

* fundamental domains.

Important notion: domain of discontinuity Ω_Γ of a discrete group, which is by definition $\partial_\infty H^n_{\mathbb{C}} \setminus \Lambda_\Gamma$ (complement of the limit set).

Key property: Ω_Γ is Γ-invariant, and the action of Γ on Ω_Γ is proper.

(this follows from the convergence property).

Easy to see that Ω_Γ is the largest invariant open set of $\partial_\infty H^n_{\mathbb{C}} = S$ where the action is proper.

The quotient Ω_Γ/Γ is an orbifold, called the orbifold at infinity of $H^n_{\mathbb{C}}/\Gamma = X$, denoted $\partial_\infty X$.

(sometimes we call it the orbifold at ∞ of Γ)

When $\Omega_\Gamma \neq \phi$, we get an orbifold

$$\overline{X} = (H^n_{\mathbb{C}} \cup \Omega_\Gamma)/_\Gamma = X \cup \partial_\infty X.$$

with strictly pseudoconvex boundary

Under some extra assumptions on Γ (the elliptic elements in Γ should have isolated fixed points), Ω_Γ/Γ is actually a manifold, called the manifold at infinity of Γ, denoted M_Γ

CR-structure on $\partial_\infty H^n_{\mathbb{C}}$

$S = S^{2n-1} \subset \mathbb{C}^{2n}$ unit sphere

$TS \cap J(TS)$ defines a distribution,

J ↳ std complex structure on $\mathbb{C}^n$

largest distribution w/ complex structure.

Since $G = PU(n,1)$ acts by biholomorphisms, the distribution is G-invariant.

<u>Def</u>: a spherical CR structure on a $(2n-1)$-dimensional manifold (G,X)-structure (in the sense of Ehresmann) with $G = PU(n,1)$, $X = S^{2n-1}$

$\iff$ atlas of charts with values in $S = S^{2n}$ and transition functions given locally by restriction of elts of $PU(n,1)$

Note the stabilizer of a point $x \in S$ in G is <u>not</u> compact, so this is <u>not</u> a metric structure.

(G,X)-structure on M is characterized by its <u>developing map</u> and <u>holonomy representation</u>

$\rho : \pi_1(M) \longrightarrow G$ homomorphism

$\mathrm{Im}(\rho) =$ holonomy group of the structure.

$\mathrm{dev} : \widetilde{M} \longrightarrow X$ ρ-equivariant

ie $\mathrm{dev}(\gamma x) = \rho(\gamma)\,\mathrm{dev}(x)$ $\forall \gamma \in \pi_1(M),\ x \in \widetilde{M}$.

(obtained by "analytic continuation" of charts).

A structure is called <u>complete</u> if dev is a covering (in particular $\mathrm{dev}(\widetilde{M}) = X$, the holonomy group is discrete, and $M = X/\Gamma$).

[See Thurston, Three-Dimensional Geometry and Topology]

Given a discrete subgroup $\Gamma \subset G = PU(n,1)$ of isolated type (i.e. elliptic elements in Γ have isolated fixed pts), the manifold at ∞ Ω_Γ/Γ carries a spherical CR structure (induced by the obvious spherical CR structure on $S = \partial_\infty H^n_{\mathbb{C}}$).
We call such structures uniformizable (or uniformizations)

The uniformization Ω_Γ/Γ has developing map given by the composition

$$\widetilde{\Omega}_\Gamma \xrightarrow[\text{universal cover}]{} \Omega_\Gamma \longrightarrow \Omega_\Gamma/\Gamma$$

the holonomy group $Im(\rho)$ is given by Γ
the kernel of the holonomy representation ρ is given by $\ker\rho = \pi_1(\Omega_\Gamma)$.

In what follows, we restrict to the case $n=2$
ie we consider $\Gamma \subset PU(2,1)$ discrete (of isolated type), and consider $\Omega_\Gamma \subset S^3$ as well as the 3-manifold at infinity Ω_Γ/Γ.

Motivating (open) question(s):

Q1) Which 3-manifolds admit a spherical CR structure?
Q2) Which 3-manifolds admit a spherical CR uniformization

preliminary remark:
|| every open 3-manifold admits an SCR structure
Indeed, every open 3-manifold admits an immersion in $\mathbb{R}^3 \subset S^3$, so we can simply pull-back the standard CR-structure on S^3.

So Q1 is only interesting for compact 3-manifolds.
Q2 is interesting even for open manifolds

Let us start with "obvious" examples.

① Let $\Gamma \subset U(2)$ be a finite group such that every element $\gamma \in \Gamma$ has distinct eigenvalues.
the $\Gamma \subset U(2) \subset PU(2,1)$ has isolated type,
stabilizer of the origin in the ball

and S^3/Γ carries a uniformizable spherical CR structure.

note that $\Lambda_\Gamma = \phi$ ($\Leftrightarrow \Omega_\Gamma = S^3$) in this case, since there are no loxo/parab. element in Γ.

for example, Lens spaces arise in this way, by taking $\Gamma = \left\langle \begin{pmatrix} \zeta_p & 0 \\ 0 & \zeta_p^q \end{pmatrix} \right\rangle$, $\gcd(p,q)=1$, $\zeta_p = e^{2\pi i/p}$.

there are also more complicated finite subgroups of $U(2)$, eg. non Abelian ones, for example the Poincaré dodecahedron space
(finite subgroups of $U(2)$ are classified, see old work of DuVal / Coxeter)

② Natural variation on the previous example by taking the stabilizer of a pt $\xi \in \partial_\infty H^2_{\mathbb{C}}$ instead!

Compact quotients of the Heisenberg group $\rightsquigarrow$ Nil 3-manifolds admit spherical CR uniformizations
eg. $\Gamma = \mathbb{Z}[i] \times \mathbb{Z} \subset \mathbb{C} \times \mathbb{R} = \mathrm{Heis}^3$, seen as a subgroup of $PU(2,1)$.

here $\Lambda_\Gamma = \{\xi\}$, $\Omega_\Gamma = S^3 \setminus \{\xi\} \simeq \mathrm{Heis}^3_{\mathbb{C}}$

(3) One more variation: Use the stabilizer of a point in $\mathbb{P}^2_{\mathbb{C}} \setminus \overline{H^2_{\mathbb{C}}}$, ie a positive line $\mathbb{C}v$ with $\langle v, v\rangle > 0$.

$\pi(v^\perp \cap V_-)$ is a totally geodesic copy of H (a complex line).

The stabilizer of $\pi(v)$ is (almost) given by the obvious inclusion $SU(1,1) \subset SU(2,1)$.

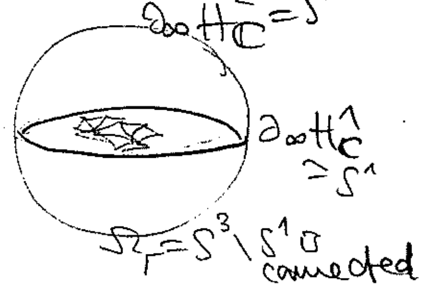
So any discrete subgroup of $SU(1,1) \simeq SL(2,\mathbb{R})$ gives a discrete subgroup of $SU(2,1)$
(we call these $\mathbb{C}$-Fuchsian groups in $PU(2,1)$).

If !! $\Gamma \subset SU(1,1)$ is a lattice, then its limit set in $H^1_{\mathbb{C}}$ is all of $\partial_\infty H^1_{\mathbb{C}}$, so the limit set in $H^2_{\mathbb{C}}$ contains all $\partial_\infty H^1_{\mathbb{C}}$ (= the $\mathbb{C}$-circle at ∞ of the invariant complex line)

By minimality, it is actually equal to $\partial_\infty H^1_{\mathbb{C}}$.

$\Lambda_\Gamma = \partial_\infty H^1_{\mathbb{C}}$

$\Omega_\Gamma = S^3 \setminus (\mathbb{C}\text{-circle})$.



What is the manifold at infinity?

Consider orthogonal projection $H^2_{\mathbb{C}} \xrightarrow{p} H^1_{\mathbb{C}}$ whose fibers are complex lines

This shows that $H^2_{\mathbb{C}}/\Gamma$ is a disk-bundle over $H^1_{\mathbb{C}}/\Gamma$.

$\rightarrow$ $\partial_\infty(H^2_{\mathbb{C}}/\Gamma)$ is a circle-bundle over $H^1_{\mathbb{C}}/\Gamma$.

(It is a manifold provided $\Gamma \subset SU(1,1)$ has no non-trivial elliptic element, which is the case for surface groups for instance)

(4) Can use the stabilizer of $H^2_{\mathbb{R}} \subset H^2_{\mathbb{C}}$ instead!

$$\Gamma \subsetneq SO(2,1) \subset SU(2,1)$$

If we take Γ = surface group in $SO(2,1)$, we get

$$H^2_{\mathbb{C}}/\Gamma = \text{disk bundle over that surface}$$

$$\partial_\infty(H^2_{\mathbb{C}}/\Gamma) = \text{circle-bundle}$$

The limit set is an $\mathbb{R}$-circle

$$\Omega_\Gamma = S^3 \setminus (\mathbb{R}\text{-circle})$$

Examples (3) and (4) look the same, but they are quite different, the bundles are not the same because the normal bundles $N_{H^2_{\mathbb{C}}/H^1_{\mathbb{C}}} \simeq T^{1/2}_{H^1_{\mathbb{C}}}$

$$N_{H^2_{\mathbb{C}}/H^2_{\mathbb{R}}} \simeq T_{H^2_{\mathbb{R}}}$$

are very diffrent.

In ex (4), we get the unit <u>tangent bundle</u> to the surface

In ex (3), get a square-root of the unit tangent bdle

+ Many other disk bundles have been constructed (Ananin-Guserskiy construct a uniformizable structure on a <u>trivial</u> circle bundle)

Negative result — Goldman 1983 (28)

classifies T^2-bundles over S^1 that carry a spherical CR structure: only Nil ones.

In particular, T^3 does not carry any SCR structure (a fortiori no SCR uniformization!)

Goldman asks a question:

‖ can (closed) hyperbolic manifolds admit spherical CR structures/uniformization?

Given a lattice $\Gamma \subset SO(3,1)$, there is no obvious representation of Γ into $SU(2,1)$!

(recall there is no (metric) copy of $H^3_{\mathbb{R}}$ in $H^2_{\mathbb{C}}$)

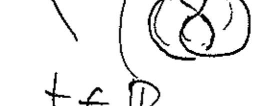
But R. Schwartz constructed explicit discrete groups $\Gamma \subset PU(2,1)$ of isolated type such that $M_\Gamma = \Omega_\Gamma/\Gamma$ is actually a hyperbolic manifold!

His first example gives M_Γ = Whitehead link complement

$= S^3 \setminus ($  $)$

then obtained compact hyperbolic examples.

D-Falbel (2015): $\exists\, \Gamma \subset PU(2,1)$ such that M_Γ = figure eight knot complement

$= S^3 \setminus ($  $)$

D (2016) there is a 1-parameter family Γ_t, $t \in \mathbb{R}$ of pairwise non-conjugate groups of $PU(2,1)$ s.t. M_{Γ_t} = figure eight knot complement

Parker-Will 2017: get another SCR uniformization of the Whitehead link complement

Can go from non-compact hyperbolic (finite volume) manifolds to compact ones by deformation / Dehn filling:

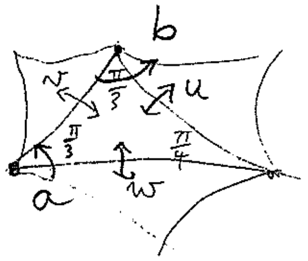
Spherical CR surgery theorem by R. Schwartz (difficult to apply in practice, but some examples where it applies are known)

The spherical CR uniformization of the figure 8 knot compl can be obtained by deforming an $\mathbb{R}$-Fuchsian example (see example ④ above).

From topologists, it is known that there is a Dehn filling of $M = S^3 \setminus (8)$ that is a Seifert fibration over the $(3,3,4)$-triangle group orbifold. By contracting the fibers of this fibration, we get a homomorphism

$$\pi_1(M) \longrightarrow T_{3,3,4} = \langle a, b \mid a^3, b^3, (ab)^4 \rangle$$

$\cap$ index 2



$$\hat{T}_{3,3,4} = \langle u, v, w \mid u^2, v^2, w^2, (uv)^3, (vw)^3, (wu)^4 \rangle.$$

There is a real 1-parameter family of representations of $\hat{T}_{3,3,4}$ such that u, v, w are mapped to complex reflections of order 2.

The mirrors of the reflections are $e_1^\perp, e_2^\perp, e_3^\perp$ for some basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$.

w.r.t. Lorentzian Herm. form.

We may assume $\langle e_j, e_j\rangle = 1 \quad \forall j = 1,2,3$

$$|\langle e_1, e_2\rangle| = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$|\langle e_2, e_3\rangle| = \cos\frac{\pi}{3} = \frac{1}{2}.$$

$$|\langle e_3, e_4\rangle| = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

and also that $\langle e_1, e_2\rangle, \langle e_2, e_3\rangle \in \mathbb{R}$.

(simply replace e_1, e_3 by a multiple such that this is true).

but in general $\langle e_1, e_3\rangle$ will not be real!

so $\langle e_1, e_3\rangle = \frac{\varphi}{\sqrt{2}}$ for some $\varphi \in \mathbb{C}$, $|\varphi| = 1$.

Write the Hermitian form in the basis $\{e_1, e_2, e_3\}$

$$\rightsquigarrow \quad H = \begin{pmatrix} 1 & -1/2 & -\varphi/\sqrt{2} \\ -1/2 & 1 & -1/2 \\ -\bar{\varphi}/\sqrt{2} & -1/2 & 1 \end{pmatrix}$$

has signature $(2,1) \Leftrightarrow \varphi + \bar{\varphi} > 0$

$\Leftrightarrow \varphi = e^{it}, \ |t| < \frac{\pi}{2}$.

This gives groups Γ_t, $t \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, such that Γ_t and Γ_{-t} are complex-conjugate of each other, and Γ_0 is $\mathbb{R}$-fuchsian.

The limit set Λ_{Γ_0} is an $\mathbb{R}$-circle (see example ④), so for small t, we expect Λ_{Γ_t} to be a topological circle.

$$I_1^{(t)} = \begin{pmatrix} -1 & 1 & \sqrt{2}\varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_2^{(t)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_3^{(t)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{2}\bar{\varphi} & 1 & -1 \end{pmatrix}$$

Schwarz $\rightsquigarrow$ there $t_0 \in]0, \frac{\pi}{2}[$ s.t.
that
$$M^t = I_1^{(t)} I_2^{(t)} I_3^{(t)} I_2^{(t)}$$
is loxodromic for $|t| < t_0$
parab. (unipotent) for $|t| = t_0$
elliptic for $t > t_0$.

In fact $\mathrm{tr}(I_1 I_2 I_3 I_2) = \sqrt{2}(\varphi + \bar{\varphi}) + 2$

So $t_0 = \arccos\left(\frac{1}{2\sqrt{2}}\right)$

Theorem (Parker-Xie-Zhang)

Λ_{Γ_t} is a topological circle for all t with $|t| < t_0$.

Theorem (D-Falbel)

Γ_{t_0} is discrete, of isolated type and $M_{\Gamma_{t_0}} = \Omega_{\Gamma_{t_0}} / \Gamma_{t_0}$ is homeomorphic to the figure eight knot complement

Note: for $|t_0| < t < \frac{\pi}{2}$, the group Γ_t is usually not discrete, conjecturally (Schwarz) it is discrete iff
$$\mathrm{tr}(I_1 I_2 I_3 I_2) = 1 + 2\cos\frac{2\pi}{n}$$
for some integer n.

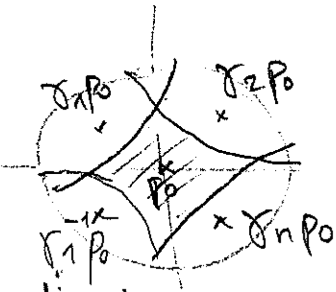
(think of this as saying that an elliptic element has finite order n; in fact it is a bit stronger, angle should be $2\pi/\text{integer}$)

Proof of both theorems above:
Study Dirichlet domains for Γ_{t_1} centered at the fixed point of $I_2 I_3$ (= elliptic element of order 4).

For some (finite) set $S \subset \Gamma$ with $S = S^{-1}$, consider

$$D_S = \{ z \in H^n_{\mathbb{C}} \mid d(z, p_0) \leq d(z, \gamma p_0) \ \forall \gamma \in S \}.$$

NB: by definition Γ is discrete $\Leftrightarrow D_\Gamma$ has nonemty interior
↳ but D_Γ is hard to determine!



this is bounded by bisectors $B(p_0, \gamma p_0), \gamma \in S$

$$\uparrow \gamma$$

$$B(p_0, \gamma^{-1} p_0)$$

We write $\hat{\gamma} = D_S \cap B(p_0, \gamma^{-1} p_0)$ for the corresponding side
note that γ maps $B(p_0, \gamma^{-1} p_0)$ to $B(p_0, \gamma p_0)$,
but it need not map $\hat{\gamma}$ to $\widehat{\gamma^{-1}}$!

Hypothesis 1: there exists a finite set S such that for every $\gamma \in S$, $\gamma^{-1} \in S$ and γ maps $\hat{\gamma}$ to $\widehat{\gamma^{-1}}$.

We then consider the facets of codimension two, ie $\hat{\gamma}_1 \cap \hat{\alpha}$ (when it has dim 2) for some $\alpha, \gamma_1 \in S$

Then $\gamma_1(\hat{\gamma}_1 \cap \hat{\alpha})$ is a codim 2 facet, so it is of the form $\hat{\gamma}_2 \cap \widehat{\gamma_1^{-1}}$ for some $\gamma_2 \in S$

Similarly, $\gamma_2(\hat{\gamma}_2 \cap \widehat{\gamma_1^{-1}}) = \hat{\gamma}_3 \cap \widehat{\gamma_2^{-1}}$ etc...

Since there are finitely many faces, after finitely many steps we come back to the same facet, ie

$$\widehat{\gamma_{n+1}} \cap \widehat{\gamma_n^{-1}} = \widehat{\gamma_1} \cap \widehat{\alpha}$$

for some (minimal) n.

The sequence of facets

$$\widehat{\gamma_1} \cap \widehat{\alpha} \xrightarrow[\gamma_1]{} \widehat{\gamma_2} \cap \widehat{\gamma_1^{-1}} \xrightarrow[\gamma_2]{} \widehat{\gamma_3} \cap \widehat{\gamma_2^{-1}} \longrightarrow \cdots \xrightarrow[\gamma_n]{} \widehat{\gamma_{n+1}} \cap \widehat{\gamma_n^{-1}} = \widehat{\gamma_1} \cap \widehat{\alpha}$$

is called a cycle, and the composition

$$g = \gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$$

is called the corresponding cycle transformation

Hypothesis 2: the cycle transformations have finite order, and the images of the polytope under (partial) cycle transformations tile a neighborhood of each codim 2 facet.

Naively: Think of this condition as saying that g rotates by $\frac{2\pi}{k}$ for some $k \in \mathbb{N}^*$, and the sum of the "angles" of the polytope at facets in the cycle is $\frac{2\pi}{k}$

(but this is more subtle, bisectors don't intersect at constant angle in general!)

Hypothesis 3: Cycle transformations for ideal vertices are all parabolic.

> Thm: (Poincaré Polyhedron Theorem, due to Mostow)
> Under hypotheses H1, H2, H3, the group generated by S is discrete, and $D = D_S$ is a fundamental domain.
> Moreover: • Presentation $\langle S \mid g_e^{r_e} = id,\ e \text{ codim 2 facet} \rangle$
> • We know conjugacy classes of finite subgroups in the group, given by cycles of facets (of every codimension)

This allows to check whether the group is "of isolated type", since we know conjugacy classes of elliptic elements.

To determine the manifold at infinity, we consider the trace at ∞ of the fundamental domain:

$$\partial_\infty D = \underbrace{\overline{D}}_{\text{closure in } \overline{H}^2_{\mathbb{C}} = H^2_{\mathbb{C}} \cup \partial_\infty H^2_{\mathbb{C}}} \cap \partial_\infty H^2_{\mathbb{C}}$$

In general, $\partial_\infty D$ is NOT contained in Ω_Γ, since there may be parabolic elements fixing ideal vertices in $\partial_\infty D$!

$\to$ consider $E = \partial_\infty D \setminus \{$fixed points of (parabolic) cycle transformations$\}$

FACT: $\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma E$, ie the images of E tile the domain of discontinuity

In other words, the manifold at ∞ is homeomorphic to E with identifications on its facets coming from the side-pairing transformations of D.

In particular, we get a description of $M_\Gamma = \Omega_\Gamma / \Gamma$ without knowing Ω_Γ nor Λ_Γ explicitly!!

In the DF spherical CR uniformization of the figure eight knot complement, we know Λ_{Γ_0} is a quotient of the circle (loxodromic element becomes parabolic $\longleftrightarrow$ identify the two fixed pts of (loxo))

Note: the groups we study here are "geometrically finite", thanks to the structure of boundary pts of fund domain.

Bowditch → this implies that the convex core $\mathrm{Hull}(\Lambda_\Gamma)/\Gamma$ has finite volume (in fact its ε-neighborhood have finite volume for all $\varepsilon > 0$).

If we assume the stronger condition that $\mathrm{Hull}(\Lambda_\Gamma)/\Gamma$ is compact (such groups are called convex-cocompact),

$$\overline{X}_\Gamma = (H^2_{\mathbb{C}} \cup \Omega_\Gamma)/\Gamma$$

has strongly pseudo-convex bdy, so (if $\Omega_\Gamma \neq \phi$) it is a proper modification of a Stein space. In particular, Ω_Γ/Γ must be connected.

Question 1: If $\Gamma \subset PU(2,1)$ is convex cocompact, is Ω_Γ connected?
Question 2: If $\Gamma \subset PU(2,1)$, is geometrically finite, is Ω_Γ connected?

Note: Apanosov-Xie construct examples of non-finitely generated groups such that Ω_Γ is not connected.

Question 3: Does every real hyperbolic 3-manifold admit a spherical CR uniformization?

Question 4: Does every compact real hyperbolic 3-manifold admit a spherical CR structure?