Real submanifolds of maximum complex tangent space at a CR singular point

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May 13, 2018

KIAS, Progress in SCV, 2018

CR singularities of M^n in $\mathbf{C}^n$.

$x_0 \in M$ is a CR singularity, if

$$\dim T_x M \cap iT_x M$$

is not constant near x_0.

Example: Bishop real surface (1965) in $\mathbf{C}^2$:

$$Q_\gamma\colon z_2 = z_1\overline{z}_1 + \gamma(z_1^2 + \overline{z}_1^2) + O(3).$$

The origin is a complex tangent of type

- elliptic, if $0 \leq \gamma < 1/2$;
- parabolic, if $\gamma = 1/2$,
- hyperbolic, if $\gamma > 1/2$.

For $M^n \subset \mathbf{C}^n$ with $n > 2$, it has been studied extensively for

$$\dim_{\mathbf{C}} T_0 M \cap iT_0 M = 1.$$

Moser-Webster theory (1983)

$$M \subset \mathbf{C}^2 \colon z_2 = z_1\overline{z_1} + \gamma(z_1^2 + \overline{z_1}^2) + O(3)$$

<u>is</u>

- holomorphic equivalent to

$$\hat{M} \colon z_2 = z_1\overline{z_1} + (\gamma + \epsilon x_2^{2s})(z_1^2 + \overline{z_1}^2)$$

  if $0 < \gamma < 1/2$.
- formally equivalent to the above $\hat{M}$, if $\gamma > 1/2$ and non-exceptional.

Main problems

We consider

$$\dim_{\mathbf{C}} T_0 M^n \cap i T_0 M^n = \textbf{largest possible dimension} = \frac{n}{2} = p.$$

Therefore, $M^{2p} \subset \mathbf{C}^{2p}$:

$$\begin{aligned} z_{p+j} &= E_j(z', \overline{z}'), \quad 1 \leq j \leq p, \\ E_j &= \underbrace{h_j(z', \overline{z'}) + q_j(\overline{z'})}_{\text{quadratic}} + O(3). \end{aligned}$$

$z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_{2p})$

We want to study real analytic M up to local biholomorphisms that preserve the origin.

Basic invariants

Complexification of $M^{2p} \subset \mathbf{C}^{2p}$:

$$\mathcal{M}\colon \begin{cases} z_{p+j} = E_j(z', w') = h_j(z', w') + q_j(w') + O(3), \\ \\ w_{p+j} = \overline{E_j}(w', z'), \qquad j = 1, \dots, p. \end{cases}$$

$\mathcal{M}$ is a complex submanifold in $\mathbf{C}^{2p} \times \mathbf{C}^{2p}$.

There are two projections from $\mathcal{M}^{2p}$ onto $\mathbf{C}^{2p}$:

$$\pi_1\colon (z, w) \to z, \quad \pi_2(z, w) \to w, \quad \pi_2 = \overline{\pi_1 \circ \rho},$$
$$\rho(z, w) = (\overline{w}, \overline{z}).$$

Condition B: $q^{-1}(0) = \{0\}$. So π_1 is a 2^p-to-1 branched covering.

Moser-Webster's surface case ($p = 1$):

- Condition B is satisfied, if $\gamma \neq 0$.
- $\pi_1 \colon \mathcal{M} \to \mathbf{C}^2$ admits a non-trivial deck transformation τ_1, and $\tau_2 = \overline{\tau_1 \rho}$ is a deck transformation of π_2.

$$(\tilde{z}', \tilde{w}') := \tau_1(z', w')$$
$$\pi_1(\tau_1(z', w')) = \pi_1(z', w') = (z', E(z', w')),$$
$$E(z', \tilde{w}') = E(z', w').$$

However, τ_1, τ_2 do not commute and $\sigma = \tau_1 \tau_2$ is reversible:

$$\sigma^{-1} = \tau_i \sigma \tau_i, \quad \tau_i^2 = I$$

which turns out to be essential in deriving the Moser-Webster normal form.

Main issue for $p > 1$: Deck transformation can be destroyed by perturbation.

Example:

$$z_3 = z_1\overline{z}_1 + \gamma_1\overline{z}_1^2 + \epsilon_2\overline{z}_2^3,$$
$$z_4 = z_2\overline{z}_2 + \gamma_2\overline{z}_2^2 + \epsilon_1\overline{z}_1^3.$$

For generic ϵ_j, the only deck transformation of π_1 is the identity.

Basic conditions ($p > 1$)

Lemma. *All deck transformations of π_1 generate an abelian group of involutions of order 2^l, $l \leq p$.*

Condition D. *The branched covering π_1 of $\mathcal{M}$ admits the maximum 2^p deck transformations.*

Example $z_{p+j} = \left(\sum_k b_{j,k}\bar{z}_k + R_j(z', \bar{z}')\right)^2, \quad \mathrm{ord}_0 R_j(0, \bar{z}') \geq 2.$

Then the deck transformations of π_1 admits a unique set of generators

$$\tau_{11}, \ldots, \tau_{1p}, \quad \operatorname{codim} \operatorname{Fix}(\tau_{1j}) = 1.$$

$\tau_1 = \tau_{11} \cdots \tau_{1p}$ is the unique deck transformation with

$$\operatorname{codim} \operatorname{Fix}(\tau_1) = p.$$

Roles of generalized Moser-Webster involutions

We call

$$\{\tau_{11}, \dots, \tau_{1p}, \rho\}$$

the Moser-Webster involutions.
Set

$$\tau_1 = \tau_{11} \cdots \tau_{1p}, \quad \tau_2 = \rho\tau_1\rho, \quad \sigma = \tau_1\tau_2.$$

Basic relation:

$$\{M\}/\sim \; \equiv \; \{\{\tau_{11}, \dots, \tau_{1p}, \rho\}\}/\sim,$$

provided the involutions τ_{1j}, ρ have the correct linear parts.

Condition E: $\sigma = \tau_1\tau_2$ has $2p$ distinct eigenvalues. Thus

$$\sigma \colon \xi_j' = \mu_j\xi_j + O(2), \quad \eta_j' = \mu_j^{-1}\eta_j + O(2).$$

Model: Product quadrics

We consider a product quadric that has 3 types of components

$$Q_e \subset \mathbf{C}^2\colon z_2 = (z_1 + 2\gamma_e \overline{z}_1)^2;$$
$$Q_h \subset \mathbf{C}^2\colon z_2 = (z_1 + 2\gamma_h \overline{z}_1)^2;$$

$$\boxed{\mathrm{Q}_s \subset \mathbf{C}^4\colon z_3 = (z_1 + 2\gamma_s \overline{z}_2)^2, \quad z_4 = (z_2 + 2(1-\overline{\gamma}_s)\overline{z}_1)^2.}$$

Here

$$0 < \gamma_e < 1/2, \quad 1/2 < \gamma_h < \infty, \quad \operatorname{Re}\gamma_s < 1/2, \quad \operatorname{Im}\gamma_s > 0.$$

Note Q_s is a new type of complex tangent, called complex type.

There are other quadratic invariants $\Longrightarrow$ holomorphic classification “à la Bishop”. The σ has eigenvalues

$$\mu_e, \quad \mu_e^{-1}, \quad \mu_h, \quad \mu_h^{-1}, \quad \mu_s, \quad \mu_s^{-1}, \quad \overline{\mu}_s, \quad \overline{\mu}_s^{-1},$$
$$\mu_e > 1, \quad |\mu_h| = 1, \quad |\mu_s| > 1.$$

Problem and setting

Our ultimate goal is to classify, up to conjugacy by biholomorphism of $(\mathbb{C}^n, 0)$, higher order perturbations of products of quadrics :

$$M : \quad z'' = Q_2(z', \bar{z}') + R_{\geq 3}(z', \bar{z}').$$

Our strategies to transform to a normal form $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$:

- First step: Normalize $\{\sigma, \tau_1, \tau_2, \rho\}$.
- Second step: Normalize $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$.

Negative result: Divergence of ALL normal forms

Theorem. *There exists a real analytic M^6 with pure elliptic type in $\mathbf{C}^6$ for which all Poincaré-Dulac normal forms of σ are divergent.*

Main ingredients :

- Under a nondegeneracy condition, there is a **unique** formal normal form
- Uses small divisors by Siegel.

X. Gong: There exists a divergent Birkhoff normal form for classical analytic Hamiltonian functions (2013).

Positive results.

Let M be a higher order perturbation a product quadric Q. We say that M has an abelian CR singularity, if there exist parings

$$\sigma_j = \tau_{1j}\tau_{2j'}, \quad 1 \leq j \leq p$$

satisfying

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad i, j = 1, \ldots, p.$$

Theorem. *Let M be real analytic and of an abelian CR singularity. Suppose that M has distinct eigenvalues and has no hyperbolic component. Then M is holomorphically equivalent to a normal form, which has infinitely many invariants when $p > 1$.*

Example :

$$M\colon z_3 = |z_1|^2 + \gamma_1(z_1^2 + \overline{z}_1^2) + O(3), \quad z_4 = (z_2 + 2\gamma_2\overline{z}_2 + z_2 z_3)^2.$$

Normal form :

$$\widehat{M}\colon z_{p+j} = \Lambda_{1j}(\zeta)\zeta_j, \quad 1 \leq j \leq p,$$

where $\zeta = (\zeta_1, \ldots, \zeta_p)$ are the convergent solutions to

$$\zeta_s = \frac{\Lambda_{1,s}(\zeta) + \Lambda_{1,s}^3(\zeta)}{(1 - \Lambda_{1,s}^2(\zeta))^2} z_s \overline{z}_{s+s_*} - \frac{\Lambda_{1,s}(\zeta)}{(1 - \Lambda_{1,s}^2(\zeta))^2}(z_s^2 + \Lambda_{1,s}^2(\zeta)\overline{z}_{s+s_*}^2),$$

- ...

Consequence : "Flatening" : $M \subset \{z_{p+e} = \bar{z}_{p+e}, \quad z_{p+s} = \bar{z}_{p+s_*+s}\}$

Hull of holomorphy of abelian CR singularity of elliptic type

In elliptic case, the normal form has a special form:

$$A_j(x'')|z_j|^2 - B_j(x'')(z_j^2 + \overline{z}_j^2) = x_{p+j}, \quad 1 \leq j \leq p; \tag{1}$$
$$y'' = 0. \tag{2}$$

Theorem (local hull of holomorphy). *Assume : "distinct eigenvalues". Then local hull of holomorphy of M near 0 is foliated by non-linear real analytic polydiscs with boundary in M.*
The hull is the intersection of p half domains with boundary in $\mathbf{C}^p \times \mathbf{R}^p$.

Results for M with minimum ($p = 1$) complex tangent space:

Kenig-Webster, Huang-Krantz, Huang (local hull of holomorphy, smooth and real analytic).

Rigidity of product quadrics

Theorem. *If M is formally equivalent to a product quadric*

$$Q_e \subset \mathbf{C}^2\colon z_{p+e} = (z_e + 2\gamma_e \overline{z}_e)^2;$$
$$Q_h \subset \mathbf{C}^2\colon z_{p+h} = (z_h + 2\gamma_h \overline{z}_h)^2;$$
$$Q_s \subset \mathbf{C}^4\colon z_{p+s} = (z_s + 2\gamma_s \overline{z}_{s_*+s})^2, \quad z_{p+s_*+s} = (z_{s_*+s} + 2(1-\overline{\gamma}_s)\overline{z}_s)^2.$$

Assume that each γ_h is a Bruno number and all γ_j's are distinct. Then M is holomorphically equivalent to the product quadric.

Earlier works:

- X. Gong, for $p = 1$ and under Siegel small divisors condition.

Strategy:

- Fact : σ_j's commute pairwise and are simultaneously formally linearizable.
- S. $\implies \exists \Phi$ biholom. of $(\mathbb{C}^n, 0)$ s.t.

$$\begin{aligned} \sigma_j \circ \Phi &= \Phi \circ D\sigma_j(0), \\ \rho \circ \Phi &= \Phi \circ \rho. \end{aligned}$$

- τ_1, τ_2 are holomorphically linearizable by a biholom. that commutes with $D\sigma_j(0)$'s and ρ,
- $\tau_{1,1}, \ldots, \tau_{1,p}$ are simultaneously holomorphically linearizable by a biholom. that commutes $D\tau_1(0), D\tau_2(0)$, ρ.

A local analytic geometric problem

A complex submanifold $K^p \subset \mathbf{C}^{2p}$ is said to be **attached** to M, if

$$K^p \cap M^{2p} \supset K_1^p \cup K_2^p$$

where K_1^p, K_2^p are totally real and intersect transversally.

We prove that the complexifications of K_1, K_2 in $\mathcal{M}$ are invariant complex submanifolds of σ. (However, the projections of the complexifications in z-space might be the same.)

Pöschel's theorem: if

$$\sigma\colon \xi_j = \mu_j \xi_j + O(2), \quad \eta_j = \mu_j^{-1} \eta_j + O(2),$$
$$|\mu_j| > 1$$

is nonresonant: i.e. $\mu^P \neq 1$ for $P \neq 0$, then σ has a holomorphic invariant submanifold tangent to $\eta = 0$.

Existence of attached complex submanifolds for possible non-abelian CR singularity

By theorem of S. that gives the existence of invariant analytic set of commuting biholomorphisms, we also obtain:

Theorem. *Suppose M has no elliptic component and its eigenvalues satisfy a Bruno type condition. Then ALL attached formal submanifolds are convergent.*

Klingenberg's theorem: M is in $\mathbf{C}^2$, $(p = 1)$, hyperbolic, and its eigenvalue satisfies a Siegel type condition.

Convergence of normalization

Let $\{F_i\}_{i=1}^{\ell}$ be a family of biholomorphisms whose linear parts at the origin are given by $\{D_i\}_{i=1}^{\ell}$ with

$$D_i\colon x_j' = \mu_{ij} x_j, \quad 1 \leq j \leq n.$$

A formal normal form $\{\hat{F}_i\}$ of $\{F_i\}$ is **completely integrable** if

- each $\hat{F}_i$ has the form

$$x_j' = \hat{\mu}_{ij}(x) x_j, \quad j = 1, \dots, n$$

  where $\hat{\mu}_{ij} \circ D_k = \hat{\mu}_{ij}$;
- for each $(j, Q) \in \{1, \dots, n\} \times \mathbf{N}^n$ with $|Q| \geq 2$,

$$\hat{\mu}_i(x)^Q \equiv \hat{\mu}_{ij}(x) \text{ for all } i, \quad \text{if} \quad \hat{\mu}_i^Q(0) = \hat{\mu}_{ij}(0) \text{ for all } i.$$

Example $n = 2$, $l = 1$, $\mu_2 = \mu_1^{-1}$, $\hat{\mu}_2(x) = \hat{\mu}_1^{-1}(x)$.

Poincaré type condition.

The family D of linear maps is of Poincaré type, if eigenvalues $\hat{\mu}_m(0) = \mu_m$ satisfy: If $(j, Q) \in \{1, \ldots, n\} \times \mathbf{N}^n$ and $\mu_m^Q - \mu_{mj} \neq 0$ for some m, there exists

$$(m', Q') \in \{1, \ldots, \ell\} \times \mathbf{N}^n$$

such that:

- $\mu_k^{Q'} = \mu_k^Q$ for all $1 \leq k \leq \ell$ and $Q' - Q \in \mathbf{N}^p \cup (-\mathbf{N}^p)$,
- $\mu_{m'}^{Q'} - \mu_{m'j} \neq 0$, and
- $\max\left(|\mu_{m'}^{Q'}|, |\mu_{m'}^{-Q'}|\right) > c^{-1} d^{|Q'|}$. Here $d > 1$.

Example:

$$\mu_{ij} = 1, \quad i \neq j; \qquad |\mu_{ii}| \neq 0, 1.$$

Theorem (Complete integrability). *If a family of commutative biholomorphic mappings is formally integrable and of Poincaré type, the family is holomorphically integrable.*

Earlier works:

- Moser for hyperbolic area-preserving mappings
- Moser-Webster: reversible holomorphic mappings: $\sigma = \tau_1\tau_2$ with τ_i being involutions.
- S.: Family of commuting vector fields whose eigenvalues satisfying a Bruno type small divisors condition.

We can apply the theorem to abelian CR singularity that does not have any hyperbolic components.