Complex discs and their applications

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Several complex variables - one at a time!

A complex (or J-complex) disc is a map $\phi : \overline{\mathbb{D}} \to M$, of the standard unit disc $\mathbb{D} \subset \mathbb{C}$ to a complex (or almost complex) manifold M, $\phi \in \mathcal{O}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$. We will describe methods for constructing complex discs and discuss their applications.

In Part I we will discuss applications to analytic continuation of holomorphic an CR functions.

In Part II we will consider applications to symplectic rigidity and non-squeezing properties of differential equations.

Part I: Complex discs

- Continuity principle
- CR manifolds and CR functions
- Baouendi-Treves approximation theorem
- Bishop's equation
- The edge-of-the-wedge theorem
- Minimality
- Kneser-Lewy extension theorem
- Strip-problems
- Analytic continuation from a family of lines
- Boundary Hartogs theorems

Continuity principle

Theorem (Hartogs, Levi, Behnke, Stein, Cartan, Thullen, etc)

Let ϕ_t be a family of complex discs in a domain $\Omega \subset \mathbb{C}^n$ and let ϕ be a complex disc in Ω. Suppose $\phi_t \to \phi$ as $t \to 0$ uniformly in $\overline{\mathbb{D}}$. Let $f \in \mathcal{O}(\Omega)$. Then f holomorphically extends to a neighborhood of $\phi(\overline{\mathbb{D}})$.

CR manifolds

Let M be a smooth real submanifold in $\mathbb{C}^n$. Recall the *complex tangent space* at $p \in M$

$$T^c_p(M) = T_p(M) \cap JT_p(M), \; p \in M.$$

Here $J : \mathbb{C}^n \to \mathbb{C}^n$ is the operator of multiplication by $i = \sqrt{-1}$. The manifold M is called a *CR manifold* if $\dim T^c_p(M)$ does not depend on $p \in M$. The manifold M is called *generic* if $T_p(M)$ spans $T_p(\mathbb{C}^n) \simeq \mathbb{C}^n$ over $\mathbb{C}$ for all $p \in M$, that is,

$$T_p(M) + JT_p(M) = \mathbb{C}^n.$$

For instance, all real hypersurfaces are generic. If M is generic, then M is a CR manifold and

$$\dim_{\mathbb{C}} T^c_p(M) + \operatorname{cod} M = n,$$

where $\operatorname{cod} M$ is the codimension of M in $\mathbb{C}^n$. The dimension $\dim_{\mathbb{C}} T^c_p(M)$ is called the CR dimension of M and is denoted by $\dim_{CR} M$.

CR functions

A C^1 function f on M is called a *CR function* if df is $\mathbb{C}$*-linear* on $T^c_p(M),\ p \in M$. In other words, f is a CR function if

$$df \wedge dz_1 \wedge \cdots \wedge dz_n|_M = 0.$$

This condition uses only holomorphic differentials $dz_j = dx_j + idy_j$. For a continuous function f on M, we say that f is a CR function if the above condition holds in the sense of distributions.

Baouendi-Treves approximation theorem

Theorem (Baouendi and Treves, 1981)

Let M be a generic manifold in $\mathbb{C}^n$. Then for every point $p \in M$ there is a neighborhood $U \subset M$ of p such that for every continuous CR function f on M there is a sequence of polynomials f_λ such that $f_\lambda|_U$ converge uniformly to $f|_U$ as $\lambda \to \infty$.

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Let f be a CR function on M. Let $\phi : \overline{\mathbb{D}} \to \mathbb{C}^n$ be a small complex disc attached to M, that is $\phi(\partial\mathbb{D}) \subset M$. By the above theorem, locally f is a limit of a sequence of polynomials f_λ. They converge to f on the boundary $\phi(\partial\mathbb{D})$. By the maximum principle, they converge on the set $\phi(\mathbb{D})$.

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Proof

Let $M_0 \subset M$ be a maximally real submanifold through p, that is $p \in M_0$, $T^c(M_0) = 0$, and $\dim M_0 = n$. We introduce coordinates in $\mathbb{C}^n$ in such a way that $p = 0$ and

$$T_p(M_0) = \mathbb{R}^n \subset \mathbb{C}^n.$$

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$$T_p(M_0) = \mathbb{R}^n \subset \mathbb{C}^n.$$

We introduce the entire functions

$$f_\lambda(z) = (\lambda/\pi)^{n/2} \int_{M_0} f(w) e^{-\lambda(z-w)^2} dw_1 \wedge \cdots \wedge dw_n,$$

where $(z-w)^2 := \sum (z_j - w_j)^2$, $\lambda > 0$.

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After shrinking M_0 if necessary

$$(\lambda/\pi)^{N/2} e^{-\lambda(z-w)^2} dw_1 \wedge \cdots \wedge dw_N$$

form a δ-shaped sequence as $\lambda \to \infty$. Thus

$$f_\lambda(z) \to f(z) \quad \text{for} \quad z \in M_0 \quad \text{as} \quad \lambda \to \infty.$$

We now prove that $f_\lambda \to f$ in a neighborhood of $p \in M$. Let us view M_0 as a manifold with boundary, and let $M_1 \subset M$ be a slight perturbation of M_0 with the same boundary.

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We define $\tilde{f}_\lambda = \int_{M_1} \dots$ by integrating the same expression as in f_λ. Then by the same argument, $\tilde{f}_\lambda(z) \to f(z)$ for $z \in M_1$. But actually $\tilde{f}_\lambda(z) = f_\lambda(z)$ for all $z \in \mathbb{C}^n$.

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Indeed, M_0 and M_1 bound a submanifold $M_{01} \subset M$, $\partial M_{01} = M_0 - M_1$. Since $e^{-\lambda(z-w)^2}$ is holomorphic and $df \wedge dw_1 \wedge \cdots \wedge dw_n|_M = 0$, the integrand is a closed form on M. By the Stokes formula $f_\lambda - \tilde{f}_\lambda = \int_{M_0} - \int_{M_1} = \int_{M_{01}} = 0$. Thus f_λ converge to f on every perturbation M_1 of M_0 in M, hence in a neighborhood of p on M. To approximate f by polynomials, one takes the Taylor polynomials of f_λ. QED

Theorem (Bishop, 1965)

Let M be a generic manifold in $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^m$, $\operatorname{cod} M = k$, $\dim_{CR} M = m$. Let $p \in M$. Assume that the projection $w : \mathbb{C}^n \to \mathbb{C}^m$ maps $T_p^c(M)$ isomorphically to $\mathbb{C}^m$. Then for every $q \in M$ close to p there exists a unique analytic disc ϕ attached to M having a given w-component and passing through q, that is, $\phi(1) = q$.

Bishop's equation

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Proof. Let $p = 0$. We choose w to be part of coordinate functions in $\mathbb{C}^n$ and complete it to a system of holomorphic coordinates $(z = x + iy, w) \in \mathbb{C}^k \times \mathbb{C}^m$.
Then we can choose the z coordinates so that $T_p(M)$ has the equation $x = 0$, so $T_p^c(M)$ has the equation $z = 0$. Then M has a local equation

$$x = h(y, w),$$

where h is a smooth function with $h(0) = 0$ and $dh(0) = 0$.

Let $\zeta \mapsto \phi(\zeta) = (x(\zeta) + iy(\zeta), w(\zeta))$ be an analytic disc in $\mathbb{C}^n$. The disc ϕ is attached to M if and only if

$$x(\zeta) = h(y(\zeta), w(\zeta)), \qquad |\zeta| = 1.$$

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$$y = T_1 h(y, w) + y_0, \qquad y_0 = y(1).$$

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The existence and uniqueness of the solution to Bishop's equation follow by the implicit function theorem. The solution defines $\phi(\zeta)$ for $|\zeta| = 1$. For all $\zeta \in \mathbb{D}$, the disc is obtained by harmonic extension. QED

The edge-of-the-wedge theorem

Let M be a generic manifold in $\mathbb{C}^n$. Let $N_p(M) := T_p(\mathbb{C}^n)/T_p(M)$ be the normal space to M in $\mathbb{C}^n$. The spaces $N_p(M)$ form the normal bundle $N(M)$.
Let Γ be an *open cone* in $N_p(M)$. Let U be a neighborhood of p in M. We can identify Γ with a cone in a transverse plane Π through p, $\Pi \oplus T_p(M) = \mathbb{C}^n$. *A wedge W with direction cone Γ* is a set of the form

$$W = ((M \cap U) + \Gamma) \cap U.$$

The edge-of-the-wedge theorem

Theorem (Ayrapetian and Henkin, 1981)

Let $M \subset \mathbb{C}^n$ be generic, and let $p \in M$. Let M_j $1 \leq j \leq k = \operatorname{cod} M$ be manifolds with boundary M. Let $\xi_j \in T_p(M_j)/T_p(M) \subset N_p(M)$ point inside M_j and let ξ_j form a basis of $N_p(M)$. Let Γ be the convex span of $\xi, \ldots, \xi_k$. Then all continuous CR functions on $M \cup \bigcup_{j=1}^k M_j$ extend holomorphically to the same wedge with direction cone Γ', where $\Gamma' \subset \Gamma$ is any finer cone.

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Proof (Ayrapetian, 1989). We show that analytic discs attached to $X = M \cup \bigcup_{j=1}^{k} M_j$ fill up a wedge W with edge M and direction cone Γ'. Then by the Baouendi-Treves approximation theorem (it still holds in this situation), all CR functions on X extend to be holomorphic in W, as desired.

We will write a version of the Bishop equation for X. We first choose coordinates and introduce a parametric equation

$$x = h(y, w, t),$$

where $(z, w) \in \mathbb{C}^k \times \mathbb{C}^m = \mathbb{C}^n$, $t \in \mathbb{R}^k$, and h is a smooth $\mathbb{R}^k$-valued function in a neighborhood of 0. Then the manifold M has the equation $x = h(y, w, 0)$ while M_j is defined as $t_j > 0,\ t_l = 0$ for $l \neq j$. In these coordinates the cone Γ turns into $\mathbb{R}^k_+ := \{t \in \mathbb{R}^k : t_j \geq 0,\ 1 \leq j \leq m\}$.

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Divide the circle $\partial\mathbb{D}$ into disjoint arcs $\partial\mathbb{D} = \bigcup_{j=1}^k \overline{\gamma}_j$ and let $\psi_j \geq 0$, $1 \leq j \leq k$, be nonzero smooth functions on $\partial\mathbb{D}$, with support in γ_j. We define for $\zeta \in b\Delta$, $\lambda \in \mathbb{R}^k_+$

$$t(\zeta) = (\lambda_1\psi_1(\zeta), \ldots, \lambda_k\psi_k(\zeta)).$$

We take $w(\zeta) = w_0 = \mathrm{const}$ and $y(0) = y_0 \in \mathbb{R}^k$. Consider the Bishop equation

$$y = Th(y, w_0, t) + y_0.$$

For small λ, the solutions exist and define complex discs attached to X. They cover the desired wedge. QED

Minimality

Let M be a CR manifold in $\mathbb{C}^n$. We say that M is minimal at $p \in M$ if there is no proper CR submanifold $S \subset M$ through p such that $\dim_{CR} S = \dim_{CR} M$. By "proper" we mean that $\dim S < \dim M$.

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If M has finite type at $p \in M$, then M is minimal at p. Indeed, if $S \subset M$ with the indicated properties exists, then complex tangent vector fields can be restricted to S, their commutators stay in $T_p(S)$, so they cannot span the whole space $T_p(M)$. In case M is real analytic, M is minimal at p if and only if M has finite type at p.

The minimality is necessary and sufficient for wedge extendibility.

Theorem (Tumanov, 1988)

Let M be a generic manifold in $\mathbb{C}^N$. Suppose M is minimal at $p \in M$. Then there is a wedge W with edge M near p such that all CR functions on M extend to be holomorphic in W.

Theorem (Baouendi and Rothschild, 1990)

Let M be a generic manifold in $\mathbb{C}^N$. Suppose M is not minimal at $p \in M$. Then for every neighborhood $U \subset M$ of p there exists a CR function in U that does not extend to any wedge with edge M near p.

Thrépreau (1985) proved the above theorems in the hypersurface case.

We give a short sketch of the proof that minimality implies wedge-extendibility.
Let $\mathcal{A}_p$ denote the set of all complex discs ϕ attached to M *through* $p \in M$, that is $\phi(1) = p$.
We consider the evaluation maps

$$\begin{aligned} &\mathcal{F} : \mathcal{A}_p \to M, && \mathcal{F}(\phi) = \phi(-1), \\ &\mathcal{G} : \mathcal{A}_p \to N_p(M), && \mathcal{G}(\phi) = [\phi_\nu(1)], \end{aligned}$$

where $\phi_\nu(1) = \partial\phi(1)/\partial\nu$ is the derivative of ϕ in the direction of the inner normal ν to $\partial\mathbb{D}$ at 1, and the brackets mean the class in $N_p(M)$. Then $\mathcal{F}$ and $\mathcal{G}$ are smooth maps. We set

$$F_\phi = \text{Range}\,\mathcal{F}'(\phi) \subset T_{\phi(-1)}(M)$$
$$G_\phi = \text{Range}\,\mathcal{G}'(\phi) \subset N_p(M)$$

By lifting ϕ to the cotangent bundle and using duality, we can show that

$$\dim M - \dim F_\phi = \mathrm{cod}\, M - \dim G_\phi, \qquad T^c_{\phi(-1)}(M) \subset F_\phi.$$

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If the main result fails, then we choose a disc $\phi \in \mathcal{A}_p$ with the highest $\dim G_\phi$, which is still less than $\operatorname{cod} M$. Then for this disc $\dim F_\phi < \dim M$.

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Then by the constant rank theorem, the range of $\mathcal{F}$ will contain a manifold S through $\phi(-1)$, $\dim S < \dim M$. Since $T^c_{\psi(-1)}(M) \subset F_\psi$ for every $\psi \in \mathcal{A}_p$, $\dim_{CR} S = \dim_{CR} M$.

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Finally, one can show that we can replace ϕ by $\tilde{\phi}$, $\tilde{\phi}(\zeta) = \phi(\zeta^2)$, for which $\tilde{\phi}(-1) = \tilde{\phi}(1)$. The constructed submanifold S will contradict the minimality. QED

Kneser-Lewy extension theorem

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Probably not. Recall the extension to a convex side of a hypersurface.
Let $M = \{z \in \mathbb{C}^n : r(z) = 0\}$ be a smooth hypersurface. Recall that the Levi form of M is a hermitian form on $T_p^c(M)$ defined by

$$L(p)(\xi, \bar{\xi}) = \sum_{j,k=1}^{n} r_{j\bar{k}}(p)\xi_j\bar{\xi}_k, \qquad \xi \in T_p^c(M),$$

where $r_{j\bar{k}} = \partial^2 r/\partial z_j\, \partial \bar{z}_k$.

Theorem (Kneser, 1936; Lewy, 1956)

Suppose $L(p)(\xi, \bar{\xi}) > 0$. Then all CR functions on M extend holomorphically to the side $r(z) < 0$ near p.

Proof by H. Lewy

For simplicity $n = 2$. Let (z, w) be coordinates in $\mathbb{C}^2$. Let Ω be the convex side of M, and let G be the projection of Ω to the z-coordinate. Put

$$\Gamma_z = \{w \in \mathbb{C} : (z, w) \in M\}.$$

The hypothesis on the Levi form imply that we can choose the coordinates such that $p = (z_0, w_0), z_0 \in M$, and for every $z \in G$ close to z_0, Γ_z is a simple closed curve in M. (By the Baouendi-Treves theorem, we can stop here.)

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For the extension we have only one candidate

$$F(z, w) = \frac{1}{2\pi i} \int_{\Gamma_z} \frac{f(z, \zeta)\, d\zeta}{\zeta - w}.$$

We show that F is a holomorphic extension of f as desired. Obviously, F is holomorphic in w. We will prove F is holomorphic in z.

To show that F is an extension, we need the vanishing of the moments

$$m_k(z) = \int_{\Gamma_z} \zeta^k f(z,\zeta)\, d\zeta, \qquad k \geq 0.$$

We will see that m_k is holomorphic. Note that Γ_z shrinks into a point as $z \to \partial G$. Then $m_k(z) \to 0$ as $z \to 0$. Hence $m_k(z) = 0$ for all $z \in G$.

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To show that F and m_k are holomorphic in z, we put

$$\Phi(z) = \int_{\Gamma_z} H(z,\zeta)\, d\zeta,$$

where $H(z,\zeta)$ is the integrand in the formula for F or m_k.
We apply the Morera theorem to Φ. Let $\gamma \subset G$ be a closed loop. Consider the "torus" $T = \{(z,\zeta) \in M : z \in \gamma, \zeta \in \Gamma_z\}$. Then T bounds a "solid torus" $S \subset M$ obtained by filling the loop γ.
Then by Stokes' formula, since H is a CR function,

$$\int_\gamma \Phi(z)\, dz = \int_T H(z,\zeta)\, d\zeta \wedge dz = \int_S dH(z,\zeta) \wedge d\zeta \wedge dz = 0.$$

QED

21-st century

Strip-problems

Let Ω be a domain in complex plane and let C_t be a continuous one parameter family of Jordan curves such that $\bigcup C_t = \overline{\Omega}$. Let f be a continuous function in $\overline{\Omega}$ such that the restrictions $f|_{C_t}$ extend holomorphically inside C_t. When does this imply that f is holomorphic in Ω?

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This question is still largely open. It is related to inverse problems for PDE and integral geometry. There are partial results by Agranovsky, Ehrenpreis, Globevnik, Tumanov, and others.

Strip-problems

Let Ω be a domain in complex plane and let C_t be a continuous one parameter family of Jordan curves such that $\bigcup C_t = \overline{\Omega}$. Let f be a continuous function in $\overline{\Omega}$ such that the restrictions $f|_{C_t}$ extend holomorphically inside C_t. When does this imply that f is holomorphic in Ω?

This question is still largely open. It is related to inverse problems for PDE and integral geometry. There are partial results by Agranovsky, Ehrenpreis, Globevnik, Tumanov, and others.

Agranovsky answered the question when both the curves and the function are real analytic.

The name “Strip-problems” is due to the following special case.

Theorem (Tumanov, 2004)

Let f be a continuous function in the strip $|\mathrm{Im}\, z| \leq 1$. Suppose for every $t \in \mathbb{R}$ the restriction of f to the circle $C_t = \{z \in \mathbb{C} : |z - t| = 1\}$ extends holomorphically inside the circle. Then f is holomorphic in the strip $|\mathrm{Im}\, z| < 1$.

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We note that if we restrict to the family $\{C_t : |t| < 1 - \epsilon\}$ for some $\epsilon > 0$, then the result fails. The function $f(z) = z^2/\overline{z}$ extends inside the circles but is not holomorphic in Ω.

Testing analyticity on circles

We present a more general result.
Let $\{C_t : \alpha \leq t \leq \beta\}$, be a continuous one parameter family of circles in complex plane $\mathbb{C}$ with centers at $c(t) \in \mathbb{C}$ and radii $r(t) > 0$. Let D_t denote the disc bounded by C_t.
Suppose the following hold.

- (a) $\overline{D}_\alpha \cap \overline{D}_\beta = \emptyset$, that is $|c(\alpha) - c(\beta)| > r(\alpha) + r(\beta)$.
- (b) The functions $c(t)$ and $r(t)$ are piecewise C^3 smooth. The curve $t \mapsto c(t)$ is injective and regular, that is $c'(t) \neq 0$.
- (c) No circle C_t is contained in the closed disc $\overline{D}_s$ for $t \neq s$, that is $|c(t) - c(s)| > |r(t) - r(s)|$.
- (d) $|c'(t)| > |r'(t)|$ whenever defined.

Theorem (Tumanov, 2007)

Let the family $\{C_t : \alpha \le t \le \beta\}$ *satisfy (a)-(d). Let* $\Omega = \bigcup D_t$*. Let* $f : \overline{\Omega} \to \mathbb{C}$ *be a continuous function. Suppose for every* $\alpha \le t \le \beta$ *the restriction* $f|_{C_t}$ *extends holomorphically to* D_t*. Then* f *is holomorphic in* Ω*.*

The condition (a) is crucial, the others being added for simplicity and convenience of the proof. The smoothness in excess of C^1 is used only to deal with triple intersections of the circles. The condition that the circles can't lie inside one another is natural because otherwise the values of f on them are unrelated. We assume the slightly stronger property (c) that they can't even touch. The condition (d) is the infinitesimal version of (c). In fact (c) implies (d) with possible equality, but for simplicity we assume the strict inequality.

Proof for a strip

We will use that f extends into $D_t = \{z \in \mathbb{C} : |z - t| < 1\}$ for $|t| \leq 1 + \epsilon$ for some $\epsilon > 0$ and prove that f is holomorphic in $\Omega = \bigcup_{|t|<1+\epsilon} D_t$. We define

$$X_t = \{(z, w) \in \mathbb{C}^2 : (z - t)(w - t) = 1, |z - t| \leq 1\},$$
$$\Sigma = \{(z, w) \in \mathbb{C}^2 : w = \overline{z}\}.$$

One can see that $X_t \cap X_s \subset \Sigma$ for $t \neq s$.

We define $M = \bigcup_{|t|<1+\epsilon} X_t$. Then $M \setminus \Sigma$ is a piecewise smooth Levi-flat hypersurface in $\mathbb{C}^2$. Let f_t denote the holomorphic extension of f inside C_t for $|t| < 1 + \epsilon$. For $(z, w) \in X_t$ we define

$$F(z, w) = f_t(z).$$

Then F is a continuous CR function on M because F is holomorphic on the fibers X_t.
We plan to prove that F actually is independent of w. That would mean that all the extensions $f_t(z)$ match at z, and f is holomorphic.

Let $\Omega' = \Omega \setminus (\overline{D}_{-1-\epsilon} \cup \overline{D}_{1+\epsilon})$. For $z \in \Omega'$ put

$$\Gamma_z = \{w \in \mathbb{C} : (z, w) \in M\}.$$

If $z \notin \mathbb{R}$, then Γ_z is a simple closed curve in $\mathbb{C}$ parametrized by

$$t \mapsto w(t) = t + \frac{1}{z - t}, \qquad t \in I_z.$$

Here I_z is the interval $I_z = \{t : |z - t| \leq 1\} \subset \mathbb{R}$. The curve Γ_z is closed because for both endpoints of I_z we have $|z - t| = 1$, which implies $w = \overline{z}$. If $z \in \mathbb{R}$, then $\Gamma_z = \mathbb{R}$.

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We note that Γ_z shrinks into a point as $\mathrm{Im}\, z \to 1$. By the argument of H. Lewy described above, the CR function F holomorphically extends inside Γ_z for $z \in \Omega' \setminus \mathbb{R}$. When $z \in \Omega'$ crosses the real line $\mathbb{R}$, the interior and exterior of the loop Γ_z interchange. This implies that $F(z, w)$ extends to the whole Riemann sphere for every $z \in \mathbb{R}$, hence $F(z, w)$ is independent of z, and f is holomorphic. QED

Analytic continuation from a family of lines

Theorem (Tumanov, 2008)

Let $\Lambda \subset \mathbb{R}^2$ be a C^2-smooth convex curve with strictly positive curvature. Let f be a complex function in the exterior of Λ. Denote by l_λ the tangent line to Λ at $\lambda \in \Lambda$. Suppose that for every $\lambda \in \Lambda$, the restriction $f|_{l_\lambda}$ extends to entire function f_λ on $\mathbb{C} = \mathbb{R}^2$. Suppose that the map $(z, \lambda) \mapsto f_\lambda(z)$ is continuous. Then f extends as an entire function on $\mathbb{C}^2$.

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In the case Λ is the unit circle, the result was obtained by Aguilar, Ehrenpreis and Kuchment (1996) as the characterization of the range of a version of the Radon transform. Öktem (1998) gave a proof using a separate analyticity result by Siciak (1969). For a general convex curve Λ, the proof is based on the extendability of CR functions, specifically, Lewy's (1956) proof of the classical extension theorem.

Boundary Hartogs theorems

Here we refer to Hartogs's name for both separate analyticity and extension from the boundary!

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Let $\Omega \subset \mathbb{C}^n$ be a convex domain with smooth boundary $\partial\Omega$ and let f be a continuous function on ∂D. Suppose for every complex line L the restriction $f|_{L\cap\partial\Omega}$ holomorphically extends into $L \cap \Omega$. Then f extends to Ω as a holomorphic function of n variables (Stout, 1977).

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The condition of holomorphic extendibility into sections $L \cap \Omega$ by all complex lines L seems excessive, because it suffices to use only the lines close to the tangent lines to $\partial\Omega$.

Indeed, for simplicity assume $f \in C^1(\partial\Omega)$. Then the Morera condition for L as L approaches a tangent line L_0 at $z_0 \in \partial\Omega$ implies that the $\overline{\partial}$ derivative of f at z_0 along L_0 equals zero. Then f holomorphically extends to Ω by the classical Hartogs-Bochner theorem.

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This question has been the subject of work by Agranovsky, Dinh, Globevnik, Stout, Rudin, etc. Glovevnik and Stout (1991) conjectured that the result would hold for a family of all lines tangent to a smooth subdomain $\Omega_1 \subset \Omega$. The conjecture is still open.

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Agranovsky proved a general result implying the conjecture in case all data are real analytic.

We would like to mention the following partial result.

Theorem (Baracco, Tumanov, Zampieri, 2007)

Let $\Omega_1 \subset \Omega$ be smooth bounded strongly convex domains in $\mathbb{C}^n$. Suppose a function $f \in C(\partial\Omega)$ extends holomorphically along every complex geodesics for Ω which is tangent to $\partial\Omega_1$. Then f extends holomorphically to Ω.

For a ball, complex geodesics (also called extremal or stationary discs) turn into the usual complex lines. Arguably, for a general convex domain they are more appropriate than the lines. If Ω is a ball, and Ω_1 is a general convex subdomain, the above theorem proves the conjecture of Globevnik and Stout.

The proof employs the method described above. We add an extra variable, the fiber coordinate in the projectivized cotangent bundle. Then using the lifts of the extremal discs we lift the given function f to a CR function on the boundary of a wedge W whose edge is the projectivized conormal bundle of $\partial\Omega_1$. Then using the theory of CR functions we extend it to a bounded holomorphic function in W. Finally since W contains "large" discs, we prove that the lifted function actually does not depend on the extra variable, which proves the result.

Extension from bunches of lines

We now consider families of lines through finitely many points. There are interesting results for holomorphic extendibility of continuous, smooth, or real-analytic functions on the boundary (Agranovsky, Baracco, Globevnik, and others). Restricting to continuous functions, we mention the following result proving a conjecture by Agranovsky.

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Theorem (Baracco, 2013; Globevnik, 2013)

Let f be a continuous function on the sphere $\partial\mathbb{B}^2$ that extends holomorphically into sections by complex lines intersecting the set of 3 non-collinear points. Then, f extends holomorphically into $\mathbb{B}^2$.

Globevnik's version is more general allowing points outside $\mathbb{B}^2$. For higher dimension, the result is nowhere published. For domains other than a ball, the corresponding result is unknown.

Part II: J-complex discs

- Almost complex structures
- J-complex discs in a cylinder
- Gromov's Non-Squeezing Theorem
- Real bidisk
- Modified Cauchy-Green operators
- Non-squeezing in Hilbert space
- Discrete non-linear Schrdinger equation

Almost complex structures

M is a smooth real manifold M, $\dim M = 2n$.

(M, J) is an almost complex manifold

$J_p : T_pM \to T_pM$, $J_p^2 = -I$, I the identity map.

J_{st} is the standard almost complex structure in $\mathbb{C}^n$.

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A smooth map $f : (M', J') \to (M, J)$ is called (J', J)-complex if it satisfies the Cauchy-Riemann equations.

$$df \circ J' = J \circ df.$$

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For $M' = \mathbb{D}$ and $J' = J_{st}$, we call the map f a J-complex disc.

$z = (z_1, \ldots, z_n)$ local coordinates on (M, J).

J is represented by a complex $n \times n$ matrix $A(z)$ such that

$$Av = (J_{\mathrm{st}} + J)^{-1}(J_{\mathrm{st}} - J)\overline{v}, \quad v \in \mathbb{C}^n.$$

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Let $f : \mathbb{D} \to (M, J)$ be a smooth map. The Cauchy-Riemann equations in term of the matrix A take the form

$$f_{\overline{\zeta}}(\zeta) = A(f(\zeta))\overline{f}_{\overline{\zeta}}(\zeta)$$

This equation is a vector version of the Beltrami equation.

Let (M, ω) be a symplectic manifold. Let ω be a nondegenerate 2-form, $d\omega = 0$. (We only need the standard symplectic form on $\mathbb{C}^n$.) Let J be an almost complex structure on M.

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Let J be an almost complex structure on a domain $\Omega \subset \mathbb{C}^n$. Suppose $J \leftrightarrow A$. Then J is tamed by the standard symplectic form if and only if

$$\|A\| < 1.$$

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If f is a usual complex disc in $\mathbb{C}^n$, then $\mathrm{Area}(f)$ with respect to the standard symplectic form is the usual area.

Theorem (Sukhov and Tumanov, 2014)

Let $\Sigma = \mathbb{D} \times \mathbb{C}^{n-1}$, $n \geq 2$. Let A be a continuous (resp. smooth) $n \times n$ matrix function on $\mathbb{C}^n$ such that $A|_{\mathbb{C}^n \setminus \Sigma} = 0$.

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and $\mathrm{Area}(f) = \pi$.

Gromov's Non-Squeezing Theorem

- Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$; then $\mathbb{D} = \mathbb{B}^1 \subset \mathbb{C}$ is the unit disc. $\mathbb{B}^n(r)$ is the ball of radius r.

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Let $r, R > 0$. Suppose there is a symplectic embedding $\psi : \mathbb{B}^n(r) \to \mathbb{D}(R) \times \mathbb{C}^{n-1}$. Then $r \leq R$.

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- Gromov's proof is based on complex analysis, namely on J-complex (pseudoholomorphic) curves.

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- We give a simple direct proof of Gromov's theorem using a new method for constructing J-complex discs.

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WLOG, shrinking r if necessary, assume ψ extends to a neighborhood of $\overline{\mathbb{B}^n(r)}$.

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WLOG, shrinking r if necessary, assume ψ extends to a neighborhood of $\overline{\mathbb{B}^n(r)}$.

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Proof of Non-Squeezing

We prove Gromov's Non-Squeezing Theorem assuming the above result (Sukhov-Tumanov, 2014).

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Then $\|A\| < 1$. Extend A to $\mathbb{C}^n$ satisfying the hypotheses of Main Theorem.

Then there exist a J-complex disc satisfying the conclusions of Main Theorem, in particular $f(0) = \psi(0)$ and $\mathrm{Area}(f) = \pi$.

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Hence $r \leq 1$ contrary to the assumption. The proof is complete.

Real bidisk

We present a few results on symplectic rigidity obtained by using J-complex discs. We consider domains in the space $\mathbb{C}^n$ with the standard symplectic structure.
We introduce the real bidisc

$$\mathbb{D}^2_{\mathbb{R}} = \{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : x_1^2 + x_2^2 < 1, y_1^2 + y_2^2 < 1\}.$$

The domains $\mathbb{D}^2_{\mathbb{R}}$ and $\mathbb{D}^2$ have the same volume.

Theorem (Wong, 2014)

The domains $\mathbb{D}^2_{\mathbb{R}}$ and $\mathbb{D}^2$ are not symplectomorphic. Furthermore, $\mathbb{D}^2_{\mathbb{R}}$ admits a symplectic embedding in $\mathbb{D}(r) \times \mathbb{C}$ if and only if $r \geq 2/\sqrt{\pi}$.

Real bidisk

Theorem (Wong, 2014)

For $n \geq 2, r \geq 1$ the domains $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ are not symplectomorphic.

It is unknown whether this result holds for $r < 1$.

Attempting to prove Main Theorem

Notation:
$\zeta \in \overline{\mathbb{D}}$, $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1} = \mathbb{C}^n$,
$f(\zeta) = (z(\zeta), w(\zeta))$.

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Cauchy-Riemann equations $f_{\overline{\zeta}} = A\overline{f}_{\overline{\zeta}}$, that is:

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The boundary condition is non-linear. Most if not all general results assume linear boundary conditions.

Let Δ be a triangle.
Let $\mathbb{D} \to \Delta$ be an area preserving map.
Then it gives rise to a sympectomorphism
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Introduce the triangle

$$\Delta = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1 - |\operatorname{Re} z|\}.$$

Note $\operatorname{Area}(\Delta) = 1$, so we will be looking for a disc of area 1.

Reduction to modified Cauchy-Green operators

Recall the Cauchy-Green operator

$$Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(t)\, dt \wedge d\bar{t}}{t - \zeta}.$$

$T : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{D})$ is bounded for $p > 1$.
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We look for a solution of Cauchy-Riemann equations in the form

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T_1 satisfies $\mathrm{Re}\,(T_1 u)|_{b\mathbb{D}} = 0$.
$T_2 u|_{b\mathbb{D}}$ takes values in the lines L_j parallel to the sides of Δ.

Modified Cauchy-Green operators

Let Q be a non-vanishing holomorphic function in $\mathbb{D}$. We define

$$\begin{aligned} T_Q u(\zeta) &= Q(\zeta)\left(T(u/Q)(\zeta) + \zeta^{-1}\overline{T(u/Q)(1/\bar{\zeta})}\right) \\ &= Q(\zeta)\int_{\mathbb{D}}\left(\frac{u(t)}{Q(t)(t-\zeta)} + \frac{\overline{u(t)}}{\overline{Q(t)}(\bar{t}\zeta - 1)}\right)\frac{dt\wedge d\bar{t}}{2\pi i}. \end{aligned}$$

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$T_1 f = T_Q f + 2i\mathrm{Im}\, Tf(1)$ with $Q(\zeta) = \zeta - 1$. Then
$\mathrm{Re}\,(T_1 u)|_{b\mathbb{D}} = 0$ (Vekua).

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$T_2 = T_Q$ with $Q(\zeta) = \sigma(\zeta-1)^{1/4}(\zeta+1)^{1/4}(\zeta-i)^{1/2}$, $\sigma = \mathrm{const}$. Then $T_2 u(\gamma_j) \subset L_j$. Here γ_j, $j = 0, 1, 2$, denote the arcs $[-1, 1]$, $[1, i]$, $[i, -1]$ respectively.

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Operators similar to T_2 were introduced by Antoncev and Monakhov for application to problems of gas dynamics.

Operators S_j

Recall that the operator $S = \partial T$ for the whole plane is an isometry of $L^2(\mathbb{C})$. It turns out the operators $S_j = \partial T_j$, $j = 1, 2$, have similar properties.

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The first assertion follows by the corresponding property the operator S because the difference is a smoothing operator.

The second assertion looks like a fluke. It follows because the boundary values of $T_j u$ do not bound a positive area.

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The Cauchy-Riemann equation $f_{\bar\zeta} = A\bar f_{\bar\zeta}$ turns into the integral equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = A(z, w) \begin{pmatrix} \overline{S_2 u} + \overline{\Phi'} \\ \overline{S_1 v} \end{pmatrix}.$$

How to satisfy $z(\tau) = z_0$

Using the equation

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Here $\Psi : \mathbb{C} \to \overline{\mathbb{D}}$ is a continuous map defined as follows.

$$\Psi(z) = \begin{cases} \Phi^{-1}(z) & \text{if } \ z \in \overline{\Delta}, \\ \Phi^{-1}(b\Delta \cap [z_0, z]) & \text{if } \ z \notin \overline{\Delta}. \end{cases}$$

Existence of solution

We now have the system

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\begin{aligned}
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By a priori estimates in L^p for some $p > 2$, we show that the system defines a compact operator. By Schauder principle the system has a solution.

Properties of the solution

Now that all the quantities (z, w, u, v, τ) are defined, we claim they have all the desired properties.

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- $\mathrm{Area}(f) = 1$ by the boundary conditions of T_1 and T_2. Indeed, $\mathrm{Area}(f) = \mathrm{Area}(z) + \mathrm{Area}(w)$. $\mathrm{Area}(z) = 1$ by the previous item. $\mathrm{Area}(w) = 0$ because every component of w takes values on a real line.

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The proof is complete.

Infinite-dimensional versions

Flows of Hamiltonian PDEs are symplectic transformations. Non-squeezing property is of great interest. There are many results for specific PDEs.

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- Kuksin (1994-95) proved a general non-squeezing result for symplectomorphisms of the form $F = I +$ compact.
- Bourgain (1994-95) proved the result for cubic NLS. Consider time t flow $F : u(0) \mapsto u(t)$ of the equation

$$iu_t + u_{xx} + |u|^p u = 0, \quad x \in \mathbb{R}/\mathbb{Z}, t > 0.$$

  Then F is a symplectic transformation of $L^2(0,1)$, $0 < p \leq 2$. Bourgain proved the non-squeezing property for $p = 2$.

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- Finally, Fabert (2015) proposes a proof of the general result using non-standard analysis.

We prove a non-squeezing result for a symplectic transformation F of the Hilbert space assuming that the derivative F' is bounded in Hilbert scales. We apply our result to discrete nonlinear Schrödinger equations.

Hilbert scales

Let $\mathbb{H}$ be a complex Hilbert space with fixed orthonormal basis $(e_n)_{n=1}^{\infty}$. Let $(\theta_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\theta_n \to \infty$ as $n \to \infty$, for example, $\theta_n = n$.

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For $s \in \mathbb{R}$ we define $\mathbb{H}_s$ as a Hilbert space with the following norm:

$$\|x\|_s^2 = \sum |x_n|^2 \theta_n^{2s}, \qquad x = \sum x_n e_n.$$

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The family $(\mathbb{H}_s)$ is called the Hilbert scale corresponding to the basis (e_n) and sequence (θ_n). We have $\mathbb{H}_0 = \mathbb{H}$. For $s > r$, the space $\mathbb{H}_s$ is dense in $\mathbb{H}_r$, and the inclusion $\mathbb{H}_s \subset \mathbb{H}_r$ is compact.

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Let $\mathbb{H}$ be a complex Hilbert space with fixed orthonormal basis $(e_n)_{n=1}^{\infty}$. Let $(\theta_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\theta_n \to \infty$ as $n \to \infty$, for example, $\theta_n = n$.
For $s \in \mathbb{R}$ we define $\mathbb{H}_s$ as a Hilbert space with the following norm:

$$\|x\|_s^2 = \sum |x_n|^2 \theta_n^{2s}, \qquad x = \sum x_n e_n.$$

The family $(\mathbb{H}_s)$ is called the Hilbert scale corresponding to the basis (e_n) and sequence (θ_n). We have $\mathbb{H}_0 = \mathbb{H}$. For $s > r$, the space $\mathbb{H}_s$ is dense in $\mathbb{H}_r$, and the inclusion $\mathbb{H}_s \subset \mathbb{H}_r$ is compact.

Example. $\mathbb{H} = L^2(0,1)$ with the standard Fourier basis, $\theta_n = (1 + n^2)^{1/2}$, $n \in \mathbb{Z}$. Then $\mathbb{H}_s$ is the standard Sobolev space.

Non-squeezing in Hilbert space

Let $\mathbb{B}(r) = \mathbb{B}^\infty(r)$ be the ball of radius r in $\mathbb{H}$.

Theorem (Sukhov and Tumanov, 2016)

Let $r, R > 0$. Let $F : \mathbb{B}(r) \to \mathbb{D}(R) \times \mathbb{H}$ be a symplectic embedding of class C^1. Suppose there is $s_0 > 0$ such that for every $|s| < s_0$ the derivative $F'(z)$ is bounded in $\mathbb{H}_s$ uniformly in $z \in \mathbb{B}(r)$. Then $r \le R$.

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Under the hypotheses of the theorem, our finite dimensional proof goes through by estimates in $L^p(\mathbb{D}, \mathbb{H}_s)$.

Discrete non-linear Schrödinger equation

Consider the following system of equations

$$iu_n' + f(|u_n|^2)u_n + \sum_k a_{nk}u_k = 0. \tag{1}$$

Here $u(t) = (u_n(t))_{n\in\mathbb{Z}}$, $u_n(t) \in \mathbb{C}$, $t \geq 0$.

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We assume that $f : \mathbb{R}_+ \to \mathbb{R}$ and its derivative are continuous on the positive reals, furthermore, $\lim_{x\to 0} f(x) = \lim_{x\to 0}[xf'(x)] = 0$. For example, one can take $f(x) = x^p$ with real $p > 0$. The hypotheses on the function f are imposed in order for the flow of (1) to be C^1 smooth.

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Here $A = (a_{nk})$ is an infinite matrix independent of t. Furthermore, A is a hermitian matrix, that is, $a_{nk} = \overline{a_{kn}}$. For simplicity we also assume that the entries a_{nk} are uniformly bounded and there exists $m > 0$ such that $a_{nk} = 0$ if $|n-k| > m$.

Discrete non-linear Schrödinger equation

The equation (1) with $f(x) = x$ is called the discrete self-trapping equation. The special case with $a_{nk} = 1$ if $|n-k| = 1$ and $a_{nk} = 0$ otherwise, is the discrete nonlinear (cubic) Schrödinger equation:

$$iu_n' + |u_n|^2 u_n + u_{n-1} + u_{n+1} = 0.$$

There are other discretizations of the Schrödinger equation, in particular, the Ablowitz-Ladik model that can be treated in a similar way.

Discrete non-linear Schrödinger equation

The equation (1) can be written in the Hamiltonian form:

$$u_n' = i\frac{\partial H}{\partial \overline{u_n}}.$$

The Hamiltonian H is given by

$$H = \sum_n F(|u_n|^2) + \sum_{n,k} a_{nk}\overline{u_n}u_k,$$

here $F' = f$ and $F(0) = 0$.

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One can verify that our main result applies to (1), hence, the non-squeezing property holds for the flow of (1).

Thank you!