Characterizing strongly pseudoconvex domains using their intrinsic geometry

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1 Introduction

Recall: A bounded domain $\Omega \subset \mathbb{C}^d$ with C^2 boundary is called strongly pseudoconvex if its Levi form is positive definite

Note: This definition is extrinsic - it depends on how Ω, viewed as a complex manifold, is embedded into $\mathbb{C}^d$.

Question: Is their an intrinsic definition of strong pseudoconvexity, i.e. a definition that only depends on Ω as a complex manifold?

One Motivation

Question Suppose $\Omega_1, \Omega_2 \subset \mathbb{C}^d$ are bounded, have C^2 boundary, and are biholomorphic. If Ω_1 is strongly pseudoconvex, is Ω_2 strongly pseudoconvex?

<u>Remark</u> If Ω_1, Ω_2 have C^∞ boundaries the answer is yes (Bell, Annals of Math, 1981), but the proof uses deep analytic methods and doesn't work in C^2 regularity.

Properties of Strongly pseudoconvex domains

<u>Claim:</u> The asymptotic complex geometry of a strongly pseudoconvex domain coincides with the complex geometry of $\mathbb{B}_d$ – the unit ball.

Holomorphic Sectional Curvature

Suppose X is a complex manifold with a complete Kähler metric g. Given $v \in TX$ non-zero the <u>holomorphic sectional curvature</u> of v is the sectional curvature of the 2-plane spanned by v, Jv.

<u>Thm</u> [Hawley 1953, Igusa 1954] (X,g) simply connected with constant negative sectional curvature $\Rightarrow X \cong_{\text{biholo.}}$ unit ball.

Every bounded domain $\Omega \subset \mathbb{C}^d$ has a Kähler metric – the Bergman metric

- This is intrinsic [Kobayashi, TAMS 1959]
- Let $H_\Omega(v)$ be the holomorphic sectional curvature at $v \in T\Omega$
- On unit ball $H_{\mathbb{B}_d} \equiv \frac{-4}{d+1}$

<u>Thm</u> [Klembeck 1978] $\Omega \subset \mathbb{C}^d$ bounded strongly pseudoconvex with C^∞ boundary, then

$$\lim_{z \to \partial\Omega} \max_{v \in T_z\Omega \setminus \{0\}} \left| H_\Omega(v) - \left(-\frac{4}{d+1}\right) \right| = 0.$$

<u>Rmk</u> Actually C^2 boundary is enough (K.T. Kim-Yu 1996)

<u>The squeezing function</u>

Given a bounded domain $\Omega \subset \mathbb{C}^d$ define $s_\Omega : \Omega \to (0,1]$ to be

$$s_\Omega(z) = \sup\left\{ r > 0 : \exists f : \Omega \to \mathbb{B}_d \text{ 1-1, holo., such that } f(\Omega) \supset r\mathbb{B}_d \right\}$$

Prop [Deng-Guan-Zhang 2012] "sup" = "max" in definition of S_Ω, so if $S_\Omega(z)=1$ for some $z \in \Omega$, then $\Omega \cong_{biholo} \mathbb{B}_d$.

Thm [Diederich-Fornaess-Wold 2014, Deng-Guan-Zhang 2016]
$\Omega \subset \mathbb{C}^d$ bounded strongly pseudoconvex with C^2 boundary, then

$$\lim_{z \to \partial\Omega} S_\Omega(z) = 1.$$

Rmk This Thm implies Klembeck's Result.

② RESULTS

For convex domains with $C^{2,\alpha}$ boundary we can prove strong converses to the previous theorems.

Thm A (Z.) For any $d \geq 2$, $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha, d) > 0$ such that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $C^{2,\alpha}$ boundary and there exists a compact set $K \subset \Omega$ such that

$$\max_{v,w \in T_z\Omega \setminus \{0\}} \left| H_\Omega(v) - H_\Omega(w) \right| \leq \varepsilon$$

for all $z \in \Omega \setminus K$, then Ω is strongly pseudoconvex.

Thm B (Z.) For any $d \geq 0$, $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha, d) > 0$ such that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $C^{2,\alpha}$ boundary and there exists a compact set $K \subset \Omega$ such that

$$S_\Omega(z) \geq 1 - \varepsilon$$

for all $z \in \Omega \setminus K$, then Ω is strongly pseudoconvex.

Both Thm A and Thm B are consequences of a more general result, but first some definitions:

Defn Let $\mathbb{X}_d$ be the collection of all convex domains in $\mathbb{C}^d$ which do not contain an affine complex line. Then let

$$\mathbb{X}_{d,0} = \{ (\Omega, z) : \Omega \in \mathbb{X}_d \,,\ z \in \Omega \}.$$

<u>Remark</u> (1) If $\Omega \subset \mathbb{C}^d$ is a convex domain, then

$$\Omega = T(\Omega' \times \mathbb{C}^k) \quad \text{for some}$$

$0 \leq k \leq d$, $T \in GL_d(\mathbb{C})$, $\Omega' \in \mathbb{X}_{d-k}$.

(2) If $\Omega \subset \mathbb{C}^d$ is a convex domain, then

$\Omega \in \mathbb{X}_d \iff \Omega$ is biholomorphic to a bounded domain

(3) The sets $\mathbb{X}_d$ and $\mathbb{X}_{d,0}$ have a natural topology (described later)

<u>Defn</u> A function $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is called <u>intrinsic</u> if $f(\Omega_1, z_1) = f(\Omega_2, z_2)$ whenever there exists a biholomorphism $\Psi : \Omega_1 \to \Omega_2$ with $\Psi(z_1) = z_2$.

<u>Ex</u> The functions

$$(\Omega, z) \longrightarrow s_\Omega(z)$$

and

$$(\Omega, z) \longrightarrow \max_{v,w \in T_z\Omega \setminus \{0\}} |H_\Omega(v) - H_\Omega(w)|$$

are intrinsic.

Fact $\mathbb{B}_d$ is homogeneous so if $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is intrinsic, then

$$f(\mathbb{B}_d, z) = f(\mathbb{B}_d, 0)$$

for all $z \in \mathbb{B}_d$.

We have the following generalization of Klembeck's Theorem:

Prop (Z) Suppose $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is a continuous intrinsic function. If $\Omega \in \mathbb{X}_d$ and $\xi \in \partial\Omega$ is a strongly pseudoconvex point of $\partial\Omega$, then

$$\lim_{z \to \xi} f(\Omega, z) = f(\mathbb{B}_d, 0).$$

And the following converse

Thm (Z) Suppose $d \geq 2$, $\alpha > 0$, and $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is a continuous intrinsic function with the following property:

if $\Omega \in \mathbb{X}_d$ and $f(\Omega, z) = f(\mathbb{B}_d, 0)$ for all $z \in \Omega$, then $\Omega \cong_{\text{biholo}} \mathbb{B}_d$. Then

there exists $\varepsilon = \varepsilon(\alpha, d, f) > 0$ such that:
if Ω is a bounded convex domain with $C^{2,\alpha}$ boundary and there exists a compact subset $K \subset \Omega$ such that

$$|f(\Omega, z) - f(\mathbb{B}_d, 0)| \leq \varepsilon$$

for all $z \in \Omega \setminus K$, then Ω is strongly pseudoconvex.

Rmk ① A similar result holds in the case when f is upper semicontinuous

② Even in the convex case the $C^{2,\alpha}$ condition seems necessary:

Thm (Fornaess-Wold 2016) For any $d \geq 2$ there exists a bounded convex domain $\Omega \subset \mathbb{C}^d$ with C^2 boundary which is not strongly pseudoconvex but

$$\lim_{z \to \partial\Omega} f(\Omega, z) = f(\mathbb{B}_d, 0)$$

for every continuous intrinsic function $f : \mathbb{X}_{d,0} \to \mathbb{R}$.

Rmk Not exactly what they proved, but follows easily from their work.

3 Outline of Proofs

The proofs of Thms A, B, C have two main steps:

1. Use the rescaling method to construct domains biholomorphic to $\mathbb{B}_d$
2. Use the behavior of the Kobayashi metric to restrict the shape of domains biholomorphic to $\mathbb{B}_d$.

4 The Space of Convex Domains and Rescaling

(following S. Frankel 1991)

Again let

$\mathbb{X}_d$ = the set of convex domains in $\mathbb{C}^d$ which do not contain a complex affine line

$\mathbb{X}_{d,0} = \{(\Omega, z) : \Omega \in \mathbb{X}_d, z \in \Omega\}$

A topology on $\mathbb{X}_{d,0}$:

Given $p \in \mathbb{C}^d$ and $A \subset \mathbb{C}^d$ define

$$d_{Euc}(p, A) = \inf\{ d_{Euc}(p, a) : a \in A \}$$

Given two compact sets $A, B \subset \mathbb{C}^d$ define the <u>Hausdorff distance</u>

$$\mathrm{Haus}(A,B) = \max\left\{ \max_{a \in A} d_{Euc}(a, B), \max_{b \in B} d_{Euc}(b, B) \right\}$$

<u>Defn</u> (Local Hausdorff Topology) We say $\Omega_n \in \mathbb{X}_d$ converges to $\Omega \in \mathbb{X}_d$ if there exists $R_0 \geq 0$ such that

$$\lim_{n \to \infty} \mathrm{Haus}\left(\overline{\Omega_n \cap \mathbb{B}(0;R)}, \overline{\Omega \cap \mathbb{B}(0;R)} \right) = 0$$

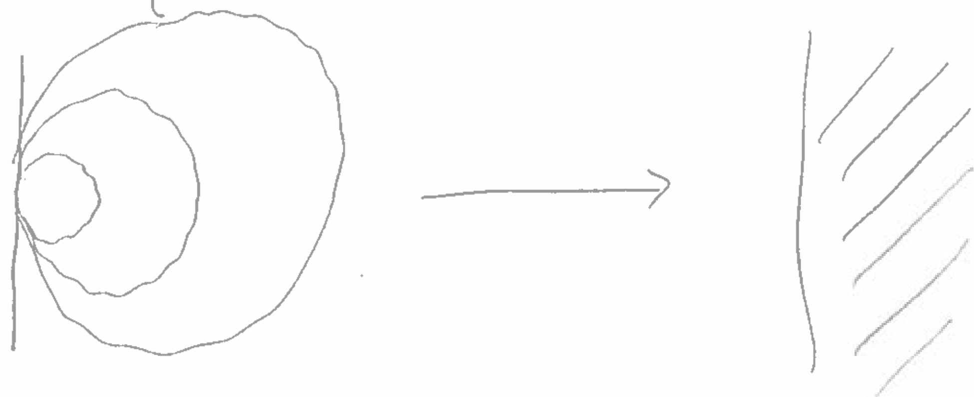
for all $R \geq R_0$ where

$$\mathbb{B}(0;R) = \{ z \in \mathbb{C}^d : \|z\| < R \}.$$

Next, We say $(\Omega_n, z_n) \in \mathbb{X}_{d,0}$ converges to $(\Omega, z) \in \mathbb{X}_{d,0}$ if $\Omega_n \to \Omega$ in $\mathbb{X}_d$ and $z_n \to z$ in $\mathbb{C}^d$.

Example

$$\Omega_n = \{ z \in \mathbb{C} : |z-n| < n \} \longrightarrow \mathcal{H} = \{ z \in \mathbb{C} : \mathrm{Re}(z) > 0 \}$$



Note: $\mathrm{Haus}(\Omega_n, \mathcal{H}) = \infty$ for all n thus motivating the introduction of the local Hausdorff topology.

Action of the Affine Group

Let $\mathrm{Aff}(\mathbb{C}^d)$ be the group of affine automosphisms $\mathbb{C}^d \to \mathbb{C}^d$.

Then $\mathrm{Aff}(\mathbb{C}^d)$ acts on $\mathbb{X}_{d,0}$:

$$T \cdot (\Omega, z) = (T(\Omega), T(z)).$$

<u>Thm</u> [Frankel 1991] The group $\mathrm{Aff}(\mathbb{C}^d)$ acts co-compactly on $\mathbb{X}_{d,0}$, that is there exists a compact set $K \subset \mathbb{X}_{d,0}$ such that $\mathrm{Aff}(\mathbb{C}^d) \cdot K = \mathbb{X}_{d,0}$.

Rmk K is explicit: let

$$\Omega_1 = \text{Convex Hull}\left(\bigcup_{i=1}^{d} \mathbb{D} e_i \right) \qquad e_1, \dots, e_d \text{ standard basis of } \mathbb{C}^d$$

and

$$\Omega_2 = \{ (z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_i) > 1 \text{ for all } i \}$$

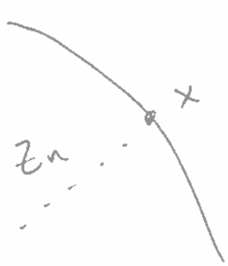
Then let

$$K = \{ (\Omega, 0) : \Omega_1 \subset \Omega \subset \Omega_2 \}.$$

A simple consequence: For any $d > 0$ there exists $C = C(d) > 0$ such that the Bergman and Kobayashi metrics are C-bi-Lipschitz on any $\Omega \in \mathbb{X}_d$.

Pf Enough to verify at $z = 0$ for $\Omega \in K$. $\blacksquare$

Rescaling: Suppose $\Omega \in \mathbb{X}_d$ and $z_n \in \Omega$ is a sequence with $z_n \to x \in \partial\Omega$. Then there exists $A_n \in \operatorname{Aff}(\mathbb{C}^d)$ such that $A_n(\Omega_n, z_n) \in K$. So, by compactness, there exists $n_j \to \infty$ such that $A_{n_j}(\Omega_{n_j}, z_{n_j}) \to (\Omega_\infty, z_\infty)$



This is called "Rescaling".

Defn Given a bounded convex domain define

$$\mathrm{BlowUp}(\Omega) = \overline{\mathrm{Aff}(\mathbb{C}^d)\cdot\Omega} \cap \mathbb{X}_d \setminus \mathrm{Aff}(\mathbb{C}^d)\cdot\Omega$$

note: $\Omega_\infty \in \mathrm{BlowUp}(\Omega) \iff \exists\, z_n \in \Omega$ s.t. $z_n \to x \in \partial\Omega$ $\exists\, z_\infty \in \Omega_\infty$ and $\exists\, A_n \in \mathrm{Aff}(\mathbb{C}^d)$ s.t.

$$A_n(\Omega, z_n) \longrightarrow (\Omega_\infty, z_\infty)$$

Claim: There is a close connection between the the asymptotic geometry of Ω and the sets in $\mathrm{BlowUp}(\Omega)$.

Prop If $\Omega \subset \mathbb{C}^d$ is a bounded convex domain, then

$$\lim_{z\to\partial\Omega} s_\Omega(z) = 1 \iff \text{every } \Omega_\infty \in \mathrm{BlowUp}(\Omega) \text{ is biholomorphic to } \mathbb{B}_d.$$

Pf ($\Rightarrow$) Pick $z_n \in \Omega$ w/ $z_n \to x$, $z_\infty \in \Omega_\infty$, and $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ s.t. $A_n(\Omega, z_n) \to (\Omega_\infty, z_\infty)$

Now $\exists\, r_n \to 1$ and $f_n : \Omega \to \mathbb{B}_d$ such that f_n is 1-1, $f_n(z_n) = 0$, and $r_n \mathbb{B}_d \subset f_n(\Omega)$.

Now with some work one can show that $g_n = f_n \circ A_n^{-1} : A_n\Omega \to \mathbb{B}_d$ converges to a biholomorphism $g : \Omega_\infty \to \mathbb{B}_d$.

($\Leftarrow$) Similar $\blacksquare$

<u>Claim:</u> If Ω has some property, then we can sometimes find $\Omega_\infty \in \text{BlowUp}(\Omega)$ with a stronger property.

<u>Prop</u> [Z 2016] Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. If $\partial\Omega$ has a point of infinite type, then there exists $\Omega_\infty \in \text{BlowUp}(\Omega)$ such that $\partial\Omega_\infty$ contains a non-trivial complex affine disk.

<u>Rmk</u> This Proposition was an important step in proving:

<u>Thm</u> [Z 2016] Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. Let K_Ω be the kobayashi distance on Ω. Then:

(Ω, K_Ω) is Gromov hyperbolic

$\Longleftrightarrow$

Ω has finite type.

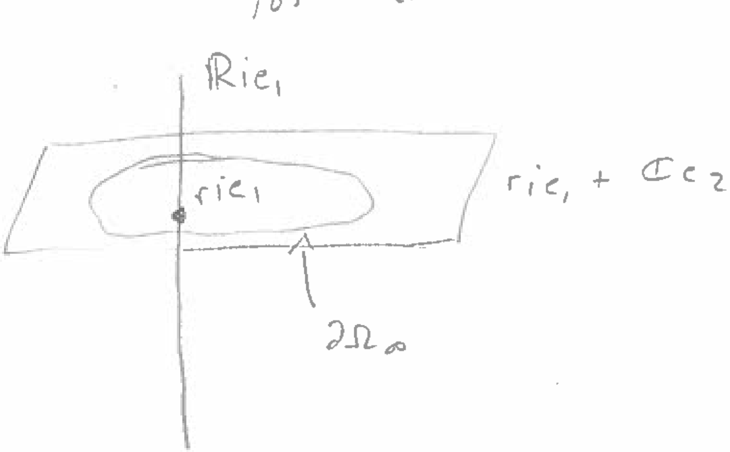
In the proof of Thm A, B, and C the following rescaling result is useful:

Prop Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $C^{2,\alpha}$ boundary. If Ω is not strongly pseudoconvex, then there exists $\Omega_\infty \in \mathrm{BlowUp}(\Omega)$ such that

① $\{(z,0,\dots,0) : \mathrm{Im}(z) > 0\} \subset \Omega_\infty \subset \{(z_1,\dots,z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > 0\}$

② $\mathrm{dist}_{Euc}\left(rie_1 , \partial\Omega_\infty \cap (rie_1 + \mathbb{C}e_2)\right) \leq r^{1/(2+\alpha)}$

for all $r \geq 1$



Rmk When $d = 2$, ① and ② says that Ω_∞ looks sort of like:

$$\{(z_1,z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_1) > |z_2|^{2+\alpha}\}$$

Rmk The domains $\Omega_\infty \in \mathrm{BlowUp}(\Omega)$ often have stronger properties than Ω, but there is some cost.

① Ω_∞ will be unbounded

② $\partial\Omega_\infty$ will, in general, not be even C^1 smooth.

5 Obstructions to biholomorphisms

Q Given two domains $\Omega_1, \Omega_2 \subset \mathbb{C}^d$ how can you show they are non-biholomorphic?

Classical Approach: Show that any biholomorphism $F: \Omega_1 \to \Omega_2$ extends to a CR-automorphism $\partial\Omega_1 \to \partial\Omega_2$. Then develop invariants on the boundary using differential geometry.

Problem Doesn't work if $\partial\Omega_1, \partial\Omega_2$ have low regularity.

Using the asymptotic behavior of geodesics in the Kobayashi metric we can prove:

Prop Suppose $d \geq 2$ and $\Omega \subset \mathbb{C}^d$ is a convex domain such that

1. $\{(z,0,\ldots,0) : \operatorname{Im}(z)>0\} \subset \Omega \subset \{(z_1,\ldots,z_d) : \operatorname{Im}(z_1)>0\}$
2. Ω is biholomorphic to $\mathbb{B}_d$

Then

$$\lim_{r\to\infty} \frac{1}{r} \log d_{Euc}\left(e^r i e_1, (e^r i e_1 + \mathbb{C} w) \cap \partial\Omega\right) = 1/2$$

for all $w \in \operatorname{Span}_{\mathbb{C}}\{e_2,\ldots,e_d\}$ nonzero.

<u>Rmk</u> ① Here $e_1, e_2, \dots, e_d$ is the standard basis

② If

$$P = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) > \sum_{i=2}^{d} |z_i|^2 \right\}$$

Then

(a) $P \cong_{\text{biholo}} \mathbb{B}_d$

(b) $\{ (z, 0, \dots, 0) : \operatorname{Im}(z) > 0 \} \subset P \subset \{ (z_1, \dots, z_d) : \operatorname{Im}(z_1) > 0 \}$

(c) $d_{Euc}\left(e^r i e_1, (e^r i e_1 + \mathbb{C} w) \cap \partial\Omega \right) = e^{1/2\, r}$

for any $w \in \operatorname{Span}_{\mathbb{C}} \{ e_2, \dots, e_d \}$ nonzero and $r \in \mathbb{R}$

So Proposition says that any domain satisfying (a) and (b) asymptotically satisfies (c).

<u>The Kobayashi Distance</u>

A key tool in the proof of the Proposition is the Kobayashi distance.

Let ρ be the Poincaré distance on $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. Given a domain $\Omega \subset \mathbb{C}^d$ and $z, w \in \Omega$ define

$$L_\Omega(z,w) = \inf \left\{ \rho(u,v) : \begin{array}{l} \exists f : \mathbb{D} \to \Omega \text{ holomorphic} \\ \text{with } f(u) = z \text{ and } f(v) = w \end{array} \right\}$$

= "distance" between z and w measured via holomorphic disks

$L_\Omega(z,w)$ may not be a distance:

- may not satisfy the triangle inequality
- may have $L_\Omega(z,w) = \infty$.

To get a distance define

$$k_\Omega(z,w) = \inf \left\{ \sum_{i=0}^{N} L_\Omega(a_i, a_{i+1}) : \begin{array}{l} N > 0;\ a_0, \dots, a_{N+1} \in \Omega; \\ a_0 = z,\ a_{N+1} = w \end{array} \right\}$$

<u>Thm</u> (Kobayashi)

1. If $\Omega \subset \mathbb{C}^d$ is a bounded domain, then (Ω, k_Ω) is a metric space
2. If $\Omega_1, \Omega_2 \subset \mathbb{C}^d$ are bounded domains and $f : \Omega_1 \to \Omega_2$ is a biholomorphism, then

$$k_{\Omega_2}(f(z), f(w)) = k_{\Omega_1}(z,w)$$

for all $z, w \in \Omega_1$.

For convex domains we have the following:

<u>Thm</u> (Barth, 1980) If $\Omega \in \mathbb{X}_d$, then k_Ω is a complete metric on Ω.

The Kobayashi Distance on the unit ball

$(\mathbb{B}_d, k_{\mathbb{B}_d})$ is a standard model of complex hyperbolic d-space - a classical example of a simply connected complete negatively curved Riemannian mfld.

Almost everything is known about the geometry of $(\mathbb{B}_d, k_{\mathbb{B}_d})$:

<u>Defn</u> If (X, d) is a metric space and $I \subset \mathbb{R}$ is an interval, then a map $\gamma : I \to X$ is called a <u>geodesic</u> if $d(\gamma(s), \gamma(t)) = |t - s| \quad \forall s, t \in I$.

<u>Fact</u> If $\gamma_1, \gamma_2 : [0,\infty) \to (\mathbb{B}_d, K_{\mathbb{B}_d})$ are geodesic rays and

$$\liminf_{s,t\to\infty} K_{\mathbb{B}_d}(\gamma_1(s), \gamma_2(t)) < +\infty,$$

then there exists some $T \in \mathbb{R}$ such that

$$\lim_{t\to\infty} K_{\mathbb{B}_d}(\gamma_1(t), \gamma_2(t+T)) = 0.$$

Moreover,

$$\lim_{t\to\infty} \frac{1}{t} \log K_{\mathbb{B}_d}(\gamma_1(t), \gamma_2(t+T)) = \begin{cases} -2 & \text{if } \star \\ -1 & \text{otherwise} \end{cases}$$

$\star$ = there exists a complex affine line $L \subset \mathbb{C}$ such that

$$\gamma_1, \gamma_2 \subset L \cap \mathbb{B}_d.$$

<u>Proof of Proposition</u>

Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain such that

① $\Omega \cong_{\text{biholo}} \mathbb{B}_d$

② $\{(z,0,\ldots,0) : \mathrm{Im}(z) > 0\} \subset \Omega \subset \{(z_1,\ldots,z_d) : \mathrm{Im}(z_1) > 0\}$

<u>Step 1</u>: Construct geodesics

Fix some $w \in \mathrm{Span}_{\mathbb{C}}\{e_2,\ldots,e_d\}$

Since Ω is convex, open, and

$$\{(z,0,\ldots,0) : \operatorname{Im}(z)>0\} \subset \Omega$$

then

$$(w + \mathbb{C}e_1) \cap \Omega = \{ze_1 + w : \operatorname{Im}(z) > \alpha_w\}$$

for some $\alpha_w > 0$.

Basic estimates on the kobayashi distance then imply that

$$\gamma_w(t) = w + (\alpha_w + e^{2t}) i e_1$$

is a geodesic in (Ω, K_Ω).

<u>Step 2:</u> Distance Estimates

<u>Fact:</u> If $w \in \operatorname{Span}_{\mathbb{C}}\{e_2, \ldots, e_d\}$, then

① $\lim_{t\to\infty} K_\Omega(e^{2t} i, \gamma_w(t)) = 0$

② $\exists\, \theta \in [0, 2\pi]$ such that if $w' = e^{i\theta}$

then

$$-1 = \lim_{t\to\infty} \frac{1}{t} \log K_\Omega(e^{2t} i e_1, \gamma_{w'}(t))$$

$$= (-2) \lim_{t\to\infty} \frac{1}{t} \log d_{Euc}\left(e^t i e_1, (e^t i e_1 + \mathbb{C}w) \cap \partial\Omega\right)$$

<u>Proof</u> Uses basic estimates on Kobayashi

distance + Fact about geodesics in $(\mathbb{B}_d, k_{\mathbb{B}_d})$ ∎

Conclusion: If $w \in \operatorname{Span}_{\mathbb{C}}\{e_2, \dots, e_d\}$, then

$$\lim_{t \to \infty} \frac{1}{t} \log d_{Euc}\left(e^t i e_1, (e^t i e_1 + \mathbb{C}w) \cap \partial\Omega\right) = 1/2 .$$

6 Proofs of Main Theorems

We will sketch the proof of:

Thm B For any $d \geq 2$, $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha, d) > 0$ such that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $C^{2,\alpha}$ boundary and there exists a compact set $K \subset \Omega$ such that

$$s_\Omega(z) \geq 1 - \varepsilon$$

for all $z \in \Omega \setminus K$, then Ω is strongly pseudoconvex.

<u>Proof by Contradiction:</u> suppose not, then for all n there exists Ω_n a bounded convex domain with $C^{2,\alpha}$ boundary such that:

1. Ω_n is not strongly pseudoconvex
2. $s_{\Omega_n}(z) \geq 1 - \frac{1}{n}$ outside a compact set of Ω_n.

By rescaling result: we can find $\hat{\Omega}_n \in \mathrm{BlowUp}(\Omega_n)$ such that

1. $\{(z,0,\dots,0) : \mathrm{Im}(z) > 0\} \subset \hat{\Omega}_n \subset \{(z_1,\dots,z_d) : \mathrm{Im}(z_1) > 0\}$
2. $\mathrm{dist}_{Euc}\left(rie_1 ; \partial\hat{\Omega}_n \cap (rie_1 + \mathbb{C}e_2)\right) \lesssim r^{\frac{1}{2+\alpha}}$

for all $r \geq 1$.

With more work one can do this so that
$$\{\hat{\Omega}_n : n = 1,2,3,\dots\}$$
is relatively compact in $\mathbb{X}_d$. One can also show that $s_{\hat{\Omega}_n}(z) \geq 1 - \frac{1}{n}$ for all $z \in \hat{\Omega}_n$.

Then we can pass to a subsequence such that $\hat{\Omega}_{n_k} \to \hat{\Omega}_\infty$ in $\mathbb{X}_d$. Using properties of local Hausdorff topology:

① $\{(z,0,\dots,0) : \operatorname{Im}(z) > 0\} \subset \hat{\Omega}_\infty \subset \{(z_1,\dots,z_d) : \operatorname{Im}(z_1) > 0\}$

② $\operatorname{dist}_{Euc}(rie_1, \partial\hat{\Omega}_\infty \cap (rie_1 + \mathbb{C}e_2)) \leq r^{\frac{1}{2+\alpha}}$ for all $r \geq 1$.

③ $s_{\hat{\Omega}_\infty} \equiv 1$.

Then ③ $\Rightarrow$ $\hat{\Omega}_\infty \cong_{\text{biholo}} \mathbb{B}_d$. Then Proposition in Section ⑤ implies that

$$\frac{1}{2} = \lim_{r\to\infty} \frac{1}{r} \log \operatorname{dist}_{Euc}\left(e^r ie_1 ; \partial\hat{\Omega}_\infty \cap (rie_1 + \mathbb{C}e_2)\right)$$

But by ② we have

$$\frac{1}{2+\alpha} \geq \lim_{r\to\infty} \frac{1}{r} \log \operatorname{dist}_{Euc}\left(e^r ie_1 ; \partial\hat{\Omega}_\infty \cap (rie_1 + \mathbb{C}e_2)\right)$$

So contradiction. $\blacksquare$