Some aspects of Ohsawa-Takegoshi type theorems

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Prologue

We first recall the fundamental

Theorem 1.1 (Hörmander 1965)

$\Omega \subset \mathbb{C}^n$: pseudoconvex, $\varphi \in PSH(\Omega)$ s.t.

$$i\partial\bar{\partial}\varphi \geq \Theta,$$

Θ : Kähler form on Ω. $\Rightarrow \forall\ \bar{\partial}-$closed $(0,1)$ form v with $\int_\Omega |v|_\Theta^2 e^{-\varphi} < \infty$, $\exists$ a solution u of $\bar{\partial}u = v$ s.t.

$$\int_\Omega |u|^2 e^{-\varphi} \leq \int_\Omega |v|_\Theta^2 e^{-\varphi}.$$

Notion: $|v|_\Theta^2 = \sum \Theta^{j\bar{k}} v_j \bar{v}_k$ where $v = \sum v_j d\bar{z}_j$, $\Theta = i \sum \Theta_{j\bar{k}} dz_j \wedge d\bar{z}_k$ and $(\Theta^{j\bar{k}}) = (\Theta_{j\bar{k}})^{-1}$.

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In case when Θ is only **semipositive**, we need to define $|v|_\Theta$ via duality

$$|v|_\Theta = \sup\left\{|\langle v, X\rangle| : X \in T^{0,1}(\Omega), |X|_\Theta \le 1\right\}$$

where for $X = \sum_j X_j \partial/\partial \bar{z}_j$,

$$|X|^2_\Theta := \sum_{j,k} \Theta_{j\bar{k}} X_j \overline{X}_k.$$

For general $\varphi \in PSH(\Omega)$, Blocki suggested to view $|v|^2_{i\partial\bar\partial\varphi}$ as the infimum of all $0 \le H \in L^\infty_{\mathrm{loc}}$ s.t.

$$i\bar{v} \wedge v \le H i\partial\bar\partial\varphi.$$

Prologue

Among numerous applications of Theorem 1.1, the following one is truly deep.

Theorem 1.2 (Bombieri 1970)

$\varphi \in PSH(\mathbb{B}^n)$, not identically $-\infty \Rightarrow$ the set E of points in every neighborhood of which $e^{-\varphi}$ is not integrable is analytic subset of $\mathbb{B}^n$.

Proof. Set

$$A^2_\varphi = \left\{ f \in \mathcal{O}(\mathbb{B}^n) : \int_{\mathbb{B}^n} |f|^2 e^{-\varphi} < \infty \right\}$$

and $S = \bigcap_{f \in A^2_\varphi} f^{-1}(0)$. It suffices to verify $E = S$.

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$E \subset S$: $a \in E \Rightarrow f(a) = 0, \forall f \in A^2_\varphi$.

$S \subset E$: for $a \in \mathbb{B}^n \backslash E$, choose $0 < \varepsilon \ll 1$ s.t. $e^{-\varphi} \in L^1(B_\varepsilon(a))$. Set $\hat{\varphi} = \varphi + |z|^2 + 2n \log |z - a|$,

$$\chi \in C_0^\infty(B_\varepsilon(a)),\ \chi|_{B_{\frac{\varepsilon}{2}}(a)} = 1,\ |\bar{\partial}\chi| \leq 3\varepsilon^{-1}.$$

By Theorem 1.1, $\exists$ solution of $\bar{\partial} u = \bar{\partial}\chi$ s.t.

$$\int_{\mathbb{B}^n} |u|^2 e^{-\hat{\varphi}} \leq \int_{\mathbb{B}^n} |\bar{\partial}\chi|^2 e^{-\hat{\varphi}} \leq C_n \varepsilon^{-2} \fint_{B_\varepsilon(a)} e^{-\varphi}.$$

Thus $f := \chi - u \in \mathcal{O}(\mathbb{B}^n), f(a) = 1,$

$$\int_{\mathbb{B}^n} |f|^2 e^{-\varphi} \leq C_n \varepsilon^{-2} \fint_{B_\varepsilon(a)} e^{-\varphi}. \qquad (0.1)$$

$\Rightarrow a \notin S$. Q.E.D.

Prologue

A natural question is whether the term ε^{-2} in (0.1) can be removed?

A positive answer is essentially the core of the proof of Ohsawa-Takegoshi L^2 extension theorem, which will be given later. We first recall

Theorem 1.3 (Demailly 1992)

$K_m(z) := \sup\{|f(z)|^2/\|f\|^2_{m\varphi} : f \in A^2_{m\varphi}\}$, *i.e. Bergman kernel of* $A^2_{m\varphi}$, $\varphi \in PSH(\mathbb{B}^n)$. *Then for* $\varepsilon < d(z, \partial\mathbb{B}^n)$,

$$C_n^{-1}\left[\fint_{B_\varepsilon(z)} e^{-m\varphi}\right]^{-1} \le K_m(z) \le \frac{1}{|B_\varepsilon(z)|}\fint_{B_\varepsilon(z)} e^{m\varphi}. \tag{0.2}$$

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Proof. Upper bound: $f \in A^2_{m\varphi} \Rightarrow$

$$|f(z)| \leq \fint_{B_\varepsilon(z)} |f| \leq \left[\fint_{B_\varepsilon(z)} e^{m\varphi}\right]^{1/2} \left[\fint_{B_\varepsilon(z)} |f|^2 e^{-m\varphi}\right]^{1/2}.$$

Lower bound: for fixed z, take $f \in A^2_{m\varphi}$ s.t. $f(z) = 1$,

$$\int_{\mathbb{B}^n} |f|^2 e^{-m\varphi} \leq C_n \fint_{B_\varepsilon(z)} e^{-m\varphi}.$$

As $K_m(z) \geq |f(z)|^2 / \|f\|^2_{m\varphi}$, we are done. Q.E.D.

Corollary 1.4

$$\lim_{m\to\infty} \frac{1}{m} \log K_m(z) = \varphi(z). \tag{0.3}$$

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Proof. Note that $\zeta \in B_\varepsilon(z) \Rightarrow \varphi(\zeta) \le \varphi(z) + o(1)$ as $\varepsilon \to 0$. Letting $m \to \infty$ and $\varepsilon \to 0$ in (0.2)

$$\limsup_{m\to\infty} \frac{1}{m} \log K_m(z) \le \varphi(z).$$

Choose smooth psh functions $\varphi_j \downarrow \varphi$. Using (0.2) with φ replaced by φ_j and let $\varepsilon \to 0$, we have

$$\frac{1}{m} \log K_{m,j}(z) \ge \varphi_j(z) - \frac{\log C_n}{m} \ge \varphi(z) - \frac{\log C_n}{m}.$$

A normal family argument implies $K_{m,j}(z) \downarrow K_m(z)$ as $j \to \infty$. Q.E.D.

Theorem 1.3 also implies a surprisingly short proof of a very deep result of Siu.

Prologue

Theorem 1.5 (Siu 1974)

$\varphi \in PSH(\mathbb{B}^n)$, $\nu(\varphi, z)$: Lelong number of φ at a.

$$\Rightarrow \quad E_c(\varphi) = \{z \in \mathbb{B}^n : \nu(\varphi, z) \geq c\}$$

is an analytic subset of $\mathbb{B}^n$, $\forall\, c > 0$.

Proof. Set $\varphi_m = \frac{1}{m} \log K_m$. We have

$$\sup_{B_\varepsilon(z)} \varphi - C_1 m^{-1} \leq \sup_{B_\varepsilon(z)} \varphi_m \leq \sup_{B_{2\varepsilon}(z)} \varphi - m^{-1} \log(C_2 \varepsilon^{2n}).$$

Since $\nu(\varphi, z) = \lim_{\varepsilon \to 0} \frac{\sup_{B_\varepsilon(z)} \varphi}{\log \varepsilon}$, we have

$$\nu(\varphi, z) - 2n/m \leq \nu(\varphi_m, z) \leq \nu(\varphi, z).$$

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It follows that

$$E_c(\varphi) = \cap_{m \geq m_0} E_{c-2n/m}(\varphi_m).$$

One can show that $E_{c-2n/m}(\varphi_m)$ is an analytic set, so is $E_c(\varphi)$. Q.E.D.

The above results showed the power of the Ohsawa-Takegoshi extension theorem.

Nevertheless, we shall see that techniques of proving the OT extension theorem is even more impressive.

Ohsawa-Takegoshi extension theorem

Theorem 2.1 (Ohsawa-Takegoshi 1987)

$\Omega \subset \mathbb{C}^n$: pseudoconvex, H : complex hyperplane s.t. $\sup_{z\in\Omega} d(z,H) < \infty$. Then for every $\varphi \in PSH(\Omega)$, every $f \in \mathcal{O}(\Omega \cap H)$ with $\int_{\Omega\cap H} |f|^2 e^{-\varphi} < \infty$, there exists $F \in \mathcal{O}(\Omega)$ s.t. $F|_{\Omega\cap H} = f$ and

$$\int_{\Omega} |F|^2 e^{-\varphi} \le C \int_{\Omega\cap H} |f|^2 e^{-\varphi}$$

where C depends only on $\sup_{z\in\Omega} d(z,H)$.

Remark. By induction, one gets an analogous extension theorem when Ω is bounded and H is a complex affine subspace.

Ohsawa-Takegoshi extension theorem

The original approach of OT is built on the general framework of Kähler geometry. The main idea is to use a twisted Bochner-Kodaira-Nakano inequality, inspired by Donnelly-Fefferman (1983) and Donnelly-Xavier (1984).

By using a twisted Morrey-Kohn-Hörmander inequality, Siu (1996) and Berndtsson (1996) are able to give simplified and more accessible proofs.

Chen (2011) observed that OT can be derived directly from Theorem 1.1. We shall explain this approach in detail here.

Ohsawa-Takegoshi extension theorem

One may assume $H = \{z_n = 0\}$, $\sup_\Omega |z_n|^2 < e^{-1}$.

Lemma 2.2

Suppose furthermore $\Omega \subset\subset \mathbb{C}^n$, $\partial\Omega \in C^\infty$, φ : C^∞ strictly psh on $\overline{\Omega}$, $f \in \mathcal{O}(V)$, V : a neighborhood of $\overline{\Omega \cap H}$ in H. Then $\exists\, F \in \mathcal{O}(\Omega)$ s.t. $F|_{\Omega\cap H} = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \leq C_0 \int_V |f|^2 e^{-\varphi} \tag{0.4}$$

where C_0 is a universal constant.

Lemma 2.2 $\Rightarrow$ OT: Choose smooth bounded pseudoconvex domain $\Omega_j \uparrow \Omega$, smooth strictly psh function φ_j on Ω_{j+1} s.t. $\varphi_j \downarrow \varphi$.

Ohsawa-Takegoshi extension theorem

We have $F_j \in \mathcal{O}(\Omega_j)$ s.t. $F_j|_{\Omega_j \cap H} = f$ and

$$\int_{\Omega_j} |F_j|^2 e^{-\varphi_j} \leq C_0 \int_{\Omega_{j+1} \cap H} |f|^2 e^{-\varphi_j} \leq C_0 \int_{\Omega \cap H} |f|^2 e^{-\varphi}.$$

Thus $\{F_j\}$ forms a normal family and it suffices to choose a weak limit of $\{F_j\}$. Q.E.D.

Since for $\varepsilon \ll 1$

$$(\partial V \times \{|z_n| < \varepsilon\}) \cap \Omega = \emptyset,$$

so $\chi(|z_n|^2/\varepsilon^2)f$ gives a smooth extension of f on Ω, where $\chi : \mathbb{R} \to [0,1]$ s.t. $\chi|_{[1,\infty)} = 0$, $\chi|_{(-\infty,1/2]} = 1$. In order that $F = \chi(|z_n|^2/\varepsilon^2)f - u$ is a holomorphic L^2_φ extension of f, it suffices to solve

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$$\bar\partial u = \bar\partial[\chi(|z_n|^2/\varepsilon^2)f] =: v_\varepsilon, \quad u|_{\Omega\cap H} = 0$$

$$\int_\Omega |u|^2 e^{-\varphi} \le C_0 \int_V |f|^2 e^{-\varphi}.$$

Similar as the proof of the Bombieri theorem, one needs to estimate integral of type

$$\int_\Omega |v_\varepsilon|^2_\Theta e^{-\hat\varphi}, \quad \text{provided} \quad i\partial\bar\partial\hat\varphi \ge \Theta.$$

If $\hat\varphi = \varphi + |z_n|^2 + 2\log|z_n|$ and $\Theta = idz_n \wedge d\bar z_n$, then

$$\int_\Omega |v_\varepsilon|^2_\Theta e^{-\hat\varphi} \le C_0\varepsilon^{-2} \int_V |f|^2 e^{-\varphi}.$$

To get rid of the term ε^{-2}, one needs to replace $|z_n|^2$ by some $\psi(z_n)$ with slow growth and large Hessian

Ohsawa-Takegoshi extension theorem

near $z_n = 0$, e.g. Poincaré type potentials:

$$\psi(z_n) = -r \log[-\log(\varepsilon^2 + |z_n|^2)], \; r > 0$$

$$i\partial\bar{\partial}\psi = \frac{\varepsilon^2 |\log(\varepsilon^2 + |z_n|^2)| + |z_n|^2}{(\varepsilon^2 + |z_n|^2)^2 |\log(\varepsilon^2 + |z_n|^2)|^2} r i dz_n \wedge d\bar{z}_n \quad (0.5)$$

In this case, we have for $\varepsilon \ll 1$

$$\int_\Omega |v_\varepsilon|^2_{i\partial\bar{\partial}\psi} e^{-\hat{\varphi}} \leq C_r |\log \varepsilon|^{1+r} \int_V |f|^2 e^{-\varphi},$$

which is useless unless $r = -1$! Thus we have to deal with L^2–estimates with weight $\phi - \psi$, where ϕ, ψ are psh. Such estimates originated from the work of Donnelly-Fefferman (1983).

Ohsawa-Takegoshi extension theorem

Lemma 2.3

$\phi, \psi \in C^2(\overline{\Omega})$ s.t. $\phi + \psi$ is psh on Ω, $\varphi \in PSH(\Omega) \Rightarrow$ the $L^2_{\phi+\varphi}(\Omega)$ minimal solution of $\bar{\partial} u = v$ satisfies

$$\int_\Omega |u|^2 e^{\psi-\phi-\varphi}\left[1 - |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\phi+\psi)} - \kappa\chi_{\mathrm{supp}\, v}|\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\phi+\psi)}\right]$$
$$\leq \int_\Omega (1+\kappa^{-1})|v|^2_{i\partial\bar{\partial}(\phi+\psi)} e^{\psi-\phi-\varphi} \tag{0.6}$$

for every continuous function $\kappa \geq 0$ on Ω.

Proof. We use a trick due to Berndtsson. Since $u \perp A^2_{\phi+\varphi}(\Omega)$, so $ue^{\psi} \perp A^2_{\phi+\psi+\varphi}$. Applying Theorem 1.1 with $\Theta = i\partial\bar{\partial}(\phi+\psi)$, we have

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$$
\begin{aligned}
\int_\Omega |u|^2 e^{\psi-\phi-\varphi} &\le \int_\Omega |\bar\partial(ue^{\psi})|^2_{i\partial\bar\partial(\phi+\psi)} e^{-\psi-\phi-\varphi} \\
\le \quad & \int_\Omega (1+\kappa^{-1})|v|^2_{i\partial\bar\partial(\phi+\psi)} e^{\psi-\phi-\varphi} \\
& + \int_\Omega |\bar\partial\psi|^2_{i\partial\bar\partial(\phi+\psi)} |u|^2 e^{\psi-\phi-\varphi} \\
& + \int_{\operatorname{supp} v} \kappa |\bar\partial\psi|^2_{i\partial\bar\partial(\phi+\psi)} |u|^2 e^{\psi-\phi-\varphi} \quad \text{Q.E.D.}
\end{aligned}
$$

We apply Lemma 2.3 with $\kappa \equiv r$, $\phi = 0$ and φ **replaced** by $\varphi + 2\log|z_n|$. The key is choice of ψ.

Ohsawa-Takegoshi extension theorem

As I mentioned before, a natural choice is

$$\psi = -\log\left[-\log(|z_n|^2 + \varepsilon^2)\right].$$

But (0.5) would imply that

$$1 - |\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} = \frac{\varepsilon^2 |\log(\varepsilon^2 + |z_n|^2)|}{|z_n|^2 + \varepsilon^2 |\log(\varepsilon^2 + |z_n|^2)|}$$

so that the LHS of (0.6) is bounded above by

$$\int_\Omega |u|^2 e^{-\varphi} \cdot \frac{\varepsilon^2}{|z_n|^2 [|z_n|^2 + \varepsilon^2 |\log(\varepsilon^2 + |z_n|^2)|]}$$

⇒ LHS of (0.6) can **not** be bounded below by

$$\text{const.} \int_\Omega |u|^2 e^{-\varphi}!$$

Ohsawa-Takegoshi extension theorem

Thus one needs to modify the choice of ψ, e.g.

$$\psi = -\log \eta$$

where

$$\eta = -\rho + \log(-\rho), \ \rho = \log(|z_n|^2 + \epsilon^2).$$

Then

$$i\partial\bar{\partial}\psi = \frac{i\partial\bar{\partial}\rho}{\eta} + \frac{i\partial\bar{\partial}[-\log(-\rho)]}{\eta} + \frac{i\partial\eta \wedge \bar{\partial}\eta}{\eta^2}$$

$$\Rightarrow i\partial\bar{\partial}\psi \geq \left[1 + \frac{\eta}{(-\rho+1)^2}\right]\frac{i\partial\eta \wedge \bar{\partial}\eta}{\eta^2}$$

$$\Rightarrow |\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} \leq \frac{1}{1 + \frac{\eta}{(-\rho+1)^2}}.$$

Ohsawa-Takegoshi extension theorem

$$i\partial\bar\partial\psi \geq \frac{i\partial\bar\partial\rho}{\eta} = \frac{\epsilon^2 i dz_n \wedge d\bar z_n}{\eta(|z_n|^2+\epsilon^2)^2},$$

$$\Rightarrow |\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq \frac{4}{\eta} \quad \text{on supp}\, v_\varepsilon, \quad \text{for} \quad \varepsilon \ll 1.$$

$$\text{and} \quad \Rightarrow \int_\Omega |v_\varepsilon|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi-2\log|z_n|}$$

$$\leq \int_{\varepsilon^2/2\leq|z_n|^2\leq\varepsilon^2} |\chi'|^2 \frac{|z_n|^2}{\epsilon^4} \frac{\eta(|z_n|^2+\epsilon^2)^2}{\epsilon^2} \frac{|f|^2}{\eta|z_n|^2} e^{-\varphi}$$

$$\leq C_0 \int_V |f|^2 e^{-\varphi}, \quad \text{for} \quad \varepsilon \ll 1.$$

Ohsawa-Takegoshi extension theorem

By (0.6), we have

$$\int_\Omega \left[\frac{\frac{\eta}{(-\rho+1)^2}}{1+\frac{\eta}{(-\rho+1)^2}} - \frac{4r}{\eta} \right] \frac{|u|^2}{|z_n|^2} e^{\psi-\varphi} \le (1+r^{-1}) C_0 \int_V |f|^2 e^{-\varphi}.$$

Since $\eta \sim -\rho$ $(\varepsilon \to 0)$, we may choose $r \ll 1$ s.t.

$$\text{LHS} \ge c_0 \int_\Omega \frac{|u|^2 e^{-\varphi}}{|z_n|^2 [\log(\varepsilon^2 + |z_n|^2)]^2}.$$

In order to get the extension, it suffices to take a weak limit of

$$F_\epsilon = \chi(|z|^2/\epsilon^2) f - u. \quad \text{Q.E.D.}$$

Ohsawa-Takegoshi extension theorem

We conclude this section by giving a lovely consequence of OT.

Corollary 2.4

$\Omega \subset\subset \mathbb{C}^n$: *pseudoconvex*, $\partial\Omega \in \mathrm{Lip} \Rightarrow$

$$K_\Omega(z) \geq \mathrm{const.}\delta(z)^{-2},$$

where K_Ω : *Bergman kernel*, $\delta(z) = d(z, \partial\Omega)$.

Proof. For (fixed) z close to $\partial\Omega$, $\exists$ a cone Λ with vertex at $z^* \in \partial\Omega$ and apical angel $\alpha_0 > 0$ s.t. z lies on the axis ($\Rightarrow |z - z^*| \asymp \delta(z)$). One can show

$$K_{\Omega\cap H}(z) \geq \mathrm{const.}|z - z^*|^{-2}.$$

Ohsawa-Takegoshi extension theorem

OT $\Rightarrow \forall f \in A^2(\Omega \cap H)$, $\exists F \in A^2(\Omega)$ s.t. $F(z) = f(z)$ and $\|F\|_{L^2} \leq \text{const.}\|f\|_{L^2}$. Thus

$$K_\Omega(z) \geq \frac{|F(z)|^2}{\|F\|_{L^2}^2} \geq \text{const.}\frac{|f(z)|^2}{\|f\|_{L^2}^2},$$

so that $K_\Omega(z) \geq \text{const.}K_{\Omega\cap H}(z)$. Q.E.D.

Problem 2.5

Does one have $S_\Omega(z) \geq \text{const.}\delta(z)^{-1}$?

Here S_Ω is the Szegö kernel, i.e. reproducing kernel of the Hardy space $H^2(\Omega)$. It is even not known where $S_\Omega(z)$ is an exhaustion function.

Ohsawa-Takegoshi type theorems

After the paper of OT, a number of variations, refinements and generalisations appeared. We call them OT type theorems.

Here we shall focus on domains in $\mathbb{C}^n$, and have to miss the rather important generalisations of OT to complex manifolds.

In particular, I could not introduce Siu's proof (simplified by Paun) of invariance of plurigenera, which from my viewpoint is the most significant application of OT.

We refer the audience to those beautiful articles/monographs of Demailly/Berndtsson.

Ohsawa-Takegoshi type theorems

Theorem 3.1 (Ohsawa 1995)

$\Omega \subset \mathbb{C}^n$: *pseudoconvex*, H : *complex hyperplane*, $\varphi, \psi \in PSH(\Omega)$ *with*

$$\sup_{z \in \Omega} [\psi(z) + 2 \log d(z, H)] \leq 0.$$

Then $\forall f \in \mathcal{O}(\Omega \cap H)$ *with* $\int_{\Omega \cap H} |f|^2 e^{-\varphi-\psi} < \infty$, $\exists$ $F \in \mathcal{O}(\Omega)$ *s.t.* $F|_{\Omega \cap H} = f$ *and*

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq C_0 \int_{\Omega \cap H} |f|^2 e^{-\varphi-\psi},$$

C_0 : *universal constant.*

Ohsawa-Takegoshi type theorems

Theorem 3.1 can also be used to improve the estimate in Corollary 2.4 for domains of finite type.

Corollary 3.2

$\Omega \subset \mathbb{C}^n$: *bounded smooth pseudoconvex domain of finite type* $\Rightarrow \exists\, \alpha > 0$ *s.t.*

$$K_\Omega(z) \geq \text{const.}\delta(z)^{-2-\alpha}.$$

A major subject in SCV is to understand the boundary behavior of the Bergman kernel for smooth pseudoconvex domains.

Ohsawa-Takegoshi type theorems

Theorem 3.3 (Ohsawa 2002)

$\Omega \subset \mathbb{C}^n$: *bounded,* **strongly** *pseudoconvex,* H : *complex hyperplane intersecting* $\partial\Omega$ *transversally. Then* $\exists\, C > 0$ *s.t.* $\forall\, \varphi \in PSH(\Omega)$ *which extends continuously to* $\partial\Omega$ *and* $f \in \mathcal{O}(\Omega \cap H)$ *with*

$$\int_{\Omega\cap H} |f|^2 e^{-\varphi} < \infty,$$

$\exists\, F \in H^2(\Omega)$ *s.t.* $F|_{\Omega\cap H} = f$ *and*

$$\int_{\partial\Omega} |F|^2 e^{-\varphi} \le C \int_{\Omega\cap H} |f|^2 e^{-\varphi}.$$

Ohsawa-Takegoshi type theorems

Although the result above is quite impressive, it brought nothing new on the Szegö kernel, since it only implies that if $z \in \Omega \cap H$ then

$$S_\Omega(z) \geq \text{const}.K_{\Omega \cap H}(z) \geq \text{const}.\delta(z)^{-n},$$

which is well-known.

Problem 3.4

$\Omega \subset \mathbb{C}^n$: bounded, smooth pseudoconvex, H : complex hyperplane intersecting $\partial\Omega$ transversally. Does there exist $C > 0$ s.t. $\forall f \in H^2(\Omega \cap H)$, $\exists F \in H^2(\Omega)$ s.t.

$$\int_{\partial\Omega} |F|^2 \leq C \int_{\partial(\Omega \cap H)} |f|^2?$$

Ohsawa-Takegoshi type theorems

In April 2, 2012, many experts of SCV received an e-mail from Zbigniew Blocki, who proved the following version of OT with **optimal** constant, in a preprint posed at his homepage.

Theorem 3.5

$\Omega \subset \mathbb{C}^{n-1} \times D$: pseudoconvex, $0 \in D \subset\subset \mathbb{C} \Rightarrow$ $\forall\, \varphi \in PSH(\Omega)$ and $f \in \mathcal{O}(\Omega')$ with $\int_{\Omega'} |f|^2 e^{-\varphi} < \infty$, where $\Omega' = \Omega \cap \{z_n = 0\}$, $\exists\, F \in \mathcal{O}(\Omega)$ s.t. $F|_{\Omega'} = f$,

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi}.$$

Here $c_D(z) = \exp[\lim_{w \to z}(g_D(w,z) - \log|w - z|)]$, g_D : Green function of D.

Ohsawa-Takegoshi type theorems

A direct consequence of Theorem 3.5 is the following result conjectured by Suita (1972).

Corollary 3.6

$D \subset \mathbb{C}$: *bounded domain* $\Rightarrow$

$$c_D^2 \leq \pi K_D. \tag{0.7}$$

Proof. Theorem 3.5 $\Rightarrow$ given $z \in D$, $\exists f \in \mathcal{O}(D)$ s.t. $f(z) = 1$ and

$$\int_D |f|^2 \leq \frac{\pi}{c_D(z)^2}. \quad \text{Q.E.D.}$$

The relation between Suita's conjecture and OT was first noticed by Ohsawa (1995).

Ohsawa-Takegoshi type theorems

Now we discuss about Blocki's proof. His new L^2–estimate for the $\bar{\partial}$–equation is as follows.

Lemma 3.7

$\varphi \in PSH(\Omega)$, ϕ, $\psi \in C^2(\overline{\Omega})$ with $\phi \in PSH(\Omega)$ s.t.

$$|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\phi} < 1, \quad |\bar{\partial}\psi|^2_{i\partial\bar{\partial}\phi} \leq \gamma < 1 \text{ on supp } v.$$

Then $\exists$ a solution of $\bar{\partial}u = v$ s.t.

$$\int_\Omega |u|^2 e^{2\psi-\phi-\varphi}\left[1 - |\bar{\partial}\psi|^2_{i\partial\bar{\partial}\phi}\right] \leq \frac{1+\sqrt{\gamma}}{1-\sqrt{\gamma}} \int_\Omega |v|^2_{i\partial\bar{\partial}\phi} e^{2\psi-\phi-\varphi}.$$

Proof. With ϕ replaced by $\phi - \psi$ in Lemma 2.3, we have

Ohsawa-Takegoshi type theorems

$$\int_\Omega |u|^2 e^{2\psi-\phi-\varphi}\left[1 - |\bar\partial\psi|^2_{i\partial\bar\partial\phi} - \kappa\chi_{\mathrm{supp}\, v}|\bar\partial\psi|^2_{i\partial\bar\partial\phi}\right]$$
$$\leq \int_\Omega (1+\kappa^{-1})|v|^2_{i\partial\bar\partial\phi} e^{2\psi-\phi-\varphi}.$$

It suffices to choose

$$\kappa = \frac{1 - |\bar\partial\psi|^2_{i\partial\bar\partial\phi}}{\sqrt{\gamma}(1+\sqrt{\gamma})}. \quad \text{Q.E.D.}$$

Set $g := g_D(\cdot, 0)$. We choose

$$\phi = \eta(-2g), \quad \psi = \zeta(-2g)$$
$$v_\varepsilon = \bar\partial[f\chi(-2g)]$$

where $\eta \in C^{1,1}(\mathbb{R}^+)$, $\zeta \in C^{0,1}(\mathbb{R}^+)$ and $\chi \in C^2(\mathbb{R})$ s.t.

Ohsawa-Takegoshi type theorems

$\chi(\infty) = 1, \chi|_{(-\infty,-2\log\varepsilon]} = 0$ (to be determined later). We need that η is convex decreasing s.t.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\phi} \leq \frac{(\zeta')^2}{\eta''} \circ (-2g). \tag{0.8}$$

To apply Lemma 3.7 with φ **replaced** by $\varphi + 2g$, we need the conditions:

$$|\bar\partial\psi|^2_{i\partial\bar\partial\phi} \leq \gamma \quad \text{on } \operatorname{supp} v_\varepsilon \tag{0.9}$$

$$\left[1 - |\bar\partial\psi|^2_{i\partial\bar\partial\phi}\right] e^{2\psi-\phi-2g} \geq 1 \quad \text{on } \Omega \tag{0.10}$$

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for they would imply $\exists$ a solution of $\bar{\partial}u = v_\varepsilon$ s.t.

$$\int_\Omega |u|^2 e^{-\varphi} \leq \int_\Omega |u|^2 \left[1 - |\bar{\partial}\psi|^2_{i\partial\bar{\partial}\phi}\right] e^{2\psi-\phi-2g-\varphi}$$
$$\leq \frac{1+\sqrt{\gamma}}{1-\sqrt{\gamma}} \int_\Omega |v_\varepsilon|^2_{i\partial\bar{\partial}\phi} e^{2\psi-\phi-2g-\varphi}.$$

We need to choose η, ζ, χ satisfying (0.9)~(0.10), $\gamma = \gamma_\varepsilon \to 0$ and

$$\limsup_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^2_{i\partial\bar{\partial}\phi} e^{2\psi-\phi-2g-\varphi} \leq \frac{\pi}{c_D(0)^2} \int_V |f|^2 e^{-\varphi}. \tag{0.11}$$

Ohsawa-Takegoshi type theorems

From (0.8), we see that if

$$\left[1 - \frac{(\zeta')^2}{\eta''}\right] e^{2\zeta - \eta + t} \geq 1,$$

then (0.10) is satisfied. Thus it suffices to solve the following ODE problem:

$$\left[1 - \frac{(\zeta')^2}{\eta''}\right] e^{2\zeta - \eta + t} = 1. \tag{0.12}$$

With $\zeta = \log(-\eta')$, one reduces (0.12) to

$$(e^{-\eta})'' = e^{-t}.$$

Ohsawa-Takegoshi type theorems

One may choose

$$\begin{aligned} \eta(t) &= -\log(t + e^{-t} - 1) \\ \zeta(t) &= -\log(t + e^{-t} - 1) + \log(1 - e^{-t}). \end{aligned}$$

We need to adjust the definition of η, ζ on the part $\{t : t \geq M := -2\log\varepsilon\}$ by simpler ones

$$\begin{aligned} \eta(t) &= -\gamma\log(t - M + a) + b \\ \zeta(t) &= -\gamma\log(t - M + a) + c \end{aligned}$$

in order to match (0.9) and (0.11). Then

$$a = -\frac{\gamma}{\eta'(M)},\ b = \eta(M) + \gamma\log a,\ c = \zeta(M) + \gamma\log a.$$

Ohsawa-Takegoshi type theorems

For $t \geq M$, one has

$$\frac{(\zeta')^2}{\eta''} = \gamma,$$

which implies (0.9).

We may choose $\gamma := M^{-1/2} \to 0$ s.t. $a \to \infty$ as $\varepsilon \to 0$. Thus $2\zeta - \eta + t \sim t$ as $t \to +\infty$, so that for $t \geq M \gg 1$,

$$\left[1 - \frac{(\zeta')^2}{\eta''}\right] e^{2\zeta - \eta + t} \geq 1$$

and (0.10) is still satisfied.

Ohsawa-Takegoshi type theorems

We are left to verify (0.11). Recall that

$$\int_\Omega |v_\varepsilon|^2_{i\partial\bar{\partial}\phi} e^{2\psi-\phi-2g-\varphi} \le \int_{g\le\log\varepsilon} |f|^2 e^{-\varphi}\left[\frac{(\chi')^2}{\eta''}e^{2\zeta-\eta+t}\right]\circ(-2g).$$

Since $g(z_n) = \log|z_n| + \log c_D(0) + o(1)$ as $z_n \to 0$, the RHS is bounded by

$$(B_\varepsilon + o(1))\left[\int_V |f|^2 e^{-\varphi} + o(1)\right]$$

where

$$B_\varepsilon := \frac{1}{c_D(0)^2}\int_{\log|z_n|\le\log\varepsilon}\left[\frac{(\chi')^2}{\eta''}e^{2\zeta-\eta+t}\right]\circ(-2\log|z_n|)$$

Ohsawa-Takegoshi type theorems

Let $t = -2\log|z_n|$. Then

$$B_\varepsilon = \frac{\pi}{c_D(0)^2}\int_M^\infty \frac{(\chi')^2 e^{2\zeta-\eta}}{\eta''}.$$

Now choose χ s.t. $\chi|_{(-\infty,M]} = 0$ and

$$\chi(t) = \int_M^t \eta'' e^{\eta-2\zeta}\left[\int_M^\infty \eta'' e^{\eta-2\zeta}\right]^{-1}$$

which implies $\chi(\infty) = 1$. Thus

$$B_\varepsilon = \frac{\pi}{c_D(0)^2}\left[\int_M^\infty \eta'' e^{\eta-2\zeta}\right]^{-1}.$$

Ohsawa-Takegoshi type theorems

Note that

$$\int_M^\infty \eta'' e^{\eta - 2\zeta} = \gamma e^{b-2c} \int_0^\infty (t+a)^{\gamma-2}$$

$$= \frac{1}{1-\gamma} e^{\eta(M) - 2\zeta(M) + \log(-\eta'(M))}.$$

Since

$$\eta(M) - 2\zeta(M) + \log(-\eta'(M)) = \eta(M) - \zeta(M) \to 0$$

as $M \to \infty$, i.e. $\varepsilon \to 0$, it follows that

$$B_\varepsilon \to \frac{\pi}{c_D(0)^2}. \quad \text{Q.E.D.}$$

Ohsawa-Takegoshi type theorems

After Blocki's work, Guan-Zhou obtained some results analogous to Theorem 3.5. Here we only mention one of them.

Theorem 3.8 (Guan-Zhou 2013)

$\Omega \subset \mathbb{C}^n$: *pseudoconvex*, $\varphi, \psi \in PSH(\Omega)$ *with*

$$\sup_{z \in \Omega} [\psi(z) + 2 \log |z_n|] \leq 0.$$

Then $\forall f \in \mathcal{O}(\Omega')$ *with* $\int_{\Omega'} |f|^2 e^{-\varphi} < \infty$ *where* $\Omega' = \Omega \cap \{z_n = 0\}$, $\exists\, F \in \mathcal{O}(\Omega)$ *s.t.* $F|_{\Omega'} = f$ *and*

$$\int_{\Omega} |F|^2 e^{-\varphi+\psi} \leq \pi \int_{\Omega'} |f|^2 e^{-\varphi}.$$

Ohsawa-Takegoshi type theorems

Based on Theorem 3.8, Guan-Zhou found an interesting new proof of the following

Theorem 3.9 (Berndtsson 2006)

$\Omega \subset \mathbb{C}^n_z \times \mathbb{C}^m_t$: pseudoconvex domain, $\varphi \in PSH(\Omega)$. Set $\Omega_t = \Omega \cap \{w = t\}$, $\varphi_t = \varphi|_{\Omega_t}$. Let K_t denote the Bergman kernel of $A^2_{\varphi_t}(\Omega_t)$. Then

$$\log K_t(z) \in PSH(\Omega).$$

Proof. It suffices to consider the case $m = 1$. One needs to get a mean value inequality near any fixed $(z_0, t_0) \in \Omega$. Assume $t_0 = 0$ and set

$$f(z) = K_0(z, z_0) / \sqrt{K_0(z_0)}.$$

Ohsawa-Takegoshi type theorems

Theorem 3.8 (with $\psi := -2\log r$)$\Rightarrow \exists\, F \in \mathcal{O}(\Omega^r)$ where $\Omega^r := \Omega \cap \{(z,t) : |t| < r\}$, s.t. $F|_{\Omega_0} = f$ and

$$\int_{\Omega^r} |F|^2 e^{-\varphi} \leq |\mathbb{D}_r| \tag{0.13}$$

where $\mathbb{D}_r = \{t \in \mathbb{C} : |t| < r\}$. Since

$$K_t(z) \geq |F(z,t)|^2 / \|F(\cdot,t)\|^2_{L^2_{\varphi_t}(\Omega_t)},$$

it follows from (0.13) and Fubini's theorem that

$$\begin{aligned} 1 &\geq \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} \|F(\cdot,t)\|^2_{L^2_{\varphi_t}(\Omega_t)} \geq \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} |F(z_0,t)|^2/K_t(z_0) \\ &\geq \exp\left[\frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} \log|F(z_0,t)|^2/K_t(z_0)\right] \end{aligned}$$

in view of Jensen's inequality.

Together with the mean value inequality, we get

$$\log K_0(z_0) \leq \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} \log|F(z_0,t)|^2 \leq \frac{1}{|\mathbb{D}_r|}\int_{\mathbb{D}_r} \log K_t(z_0).$$

Thus $\log K_t(z)$ is subharmonic in t. Q.E.D.

Remark. The crucial inequality (0.13) can also be derived from Theorem 3.5, for

$$c_{\mathbb{D}_r}(0) = \exp\left[\lim_{z\to 0}(\log|z|/r - \log|z|)\right] = r^{-1}$$

$$\Rightarrow \frac{\pi}{c_{\mathbb{D}_r}(0)^2} = |\mathbb{D}_r|.$$

Ohsawa-Takegoshi type theorems

Berndtsson' original proof is based on an ingenious application of Theorem 1.1, is already very beautiful.

More surprisingly, a variant of Theorem 3.9 was used by Berndtsson-Lempert (2015) to give a entirely new approach of OT with optimal constant.

Another interesting constructive proof was posed by Ohsawa (2015).

Ohsawa-Takegoshi type theorems

Guan-Zhou (2013) (see also Lempert (2014), Heip (2014)) discovered that OT can be used to prove the following

Theorem 3.10 (Strong Openness Theorem)

$\int_U |f|^2 e^{-\varphi} < \infty$ *for fixed* $f \in \mathcal{O}(U)$ *and* $\varphi \in PSH(U)$ $\Rightarrow \forall\, V \subset\subset U, \exists\, p > 1$ *s.t.* $\int_V |f|^2 e^{-p\varphi} < \infty$.

The special case $f = 1$ was first proved by Berndtsson (2013) by using Theorem 3.9, which solved the openness conjecture of Demailly-Kollár (2001).

Another approach using only Theorem 1.1 was posed by Chen (2016).

Ohsawa-Takegoshi type theorems

Definition 3.11

A domain $\Omega \subset \mathbb{C}^n$ is called complete Kähler (CK) if it admits a complete Kähler metric.

Lemma 3.12

Pseudoconvex $\Rightarrow$ *CK.*

Proof.

Take a smooth strictly psh exhaustion function $\rho \geq 1$, s.t. $ds^2 = \partial\bar{\partial}\rho^2$ gives a CK metric. □

On the other hand, the Poincaré metric

$$ds^2 = -\partial\bar{\partial}\log(-\log|z|^2)$$

is CK on $\mathbb{B}^n \backslash \{0\}$, which is not pseudoconvex if $n \geq 2$.

Ohsawa-Takegoshi type theorems

CK ⇒ pseudoconvex under one of the following conditions

1. $\partial\Omega \in C^\omega$ (Grauert '56).
2. $\partial\Omega \in C^1$ (Ohsawa '80).
3. $\overline{\Omega}^\circ = \Omega$ (Diederich-Pflug 81).

A large class of CK domains are given by $\Omega \backslash E$, where $\Omega \subset \mathbb{C}^n$: pseudoconvex, $E \subset \Omega$: closed and complete pluripolar, i.e. $\forall\, a \in E$, $\exists\, \psi \in PSH(U)$ where $U \ni a$: neighborhood, s.t.

$$E \cap U = \psi^{-1}(-\infty).$$

Ohsawa-Takegoshi type theorems

Problem 3.13 (Ohsawa 1995)

$\Omega \subset \mathbb{C}^n$: bounded CK, H : complex hyperplane $\Rightarrow$ $\forall f \in A^2(\Omega \cap H)$, $\exists F \in A^2(\Omega)$ s.t. $F|_{\Omega \cap H} = f$?

Problem 3.14

Can the OT type theorems be generalized to bounded CK domains?

The basic difference between CK domains and pseudoconvex domains is that the former can not be exhausted by relatively compact CK domains in general, so that one can not first solve extension problem on relatively compact ones then passing to a weak limit.

Ohsawa-Takegoshi type theorems

Theorem 3.15 (Chen-Wang-Wu 2015)

$\Omega \subset \mathbb{B}^n$: *bounded CK*, $\varphi \in PSH(\Omega) \Rightarrow \forall\, a \in \Omega$, $\forall\, c \in \mathbb{C}$ *with* $|c|^2 \leq e^{\varphi(a)}$, $\exists\, f \in \mathcal{O}(\Omega)$ *s.t.* $f(a) = c$,

$$\int_\Omega |f|^2 e^{-\varphi} \leq C_n.$$

Remark. Actually, OT can be generalized to bounded CK domains Ω and complex hyperplanes H s.t. Ω is pseudoconvex in a neighborhood of $\partial\Omega \cap H$.

Ohsawa-Takegoshi type theorems

Let us explain briefly the idea of the proof. The key point is to generalize Theorem 1.1 to bounded CK domains. Then Theorem 3.15 can be derived similarly as the classical OT.

The case of smooth weight φ is only a special case of the general theory of Andreotti-Vesentini (1965). For general $\varphi \in PSH(\Omega)$, one has smooth psh functions $\varphi_j \downarrow \varphi$, but only **locally** on Ω. Thus Hörmander's approximation procedure breaks down.

The difficulty was overcome by Demailly in 1982, who used a rather involved approximation argument, however.

Ohsawa-Takegoshi type theorems

A more transparent approach was posed by Chen-Wang-Wu (2015) as follows.

We exhaust Ω by a sequence $\{\Omega_j\}$ of bounded subdomains with smooth boundaries, and take $\varphi_j \in SPSH(\overline{\Omega}_j)$ s.t. $\varphi_j \downarrow \varphi$, $i\partial\bar{\partial}\varphi_j \geq \Theta$.

By using Friedrichs mollifier, we have smooth $(0, 1)$ form v_j on Ω s.t.

$$\int_\Omega |\bar{\partial} v_j|^2_\omega e^{-\varphi}, \int_\Omega |v_j - v|^2_\Theta e^{-\varphi} \to 0$$

where $\omega \geq \Theta$ is a CK metric. Instead of solving

$$\bar{\partial} u_j = v_j$$

on Ω_j, we solve the Laplace-Beltrami equation

Ohsawa-Takegoshi type theorems

$$\Box_{\varphi_j} w = \tilde{v}_j := dz \wedge v_j$$

on Ω_j s.t.

$$||\bar{\partial} w_j||^2_{\varphi_j} + ||\bar{\partial}^*_{\varphi_j} w_j||^2_{\varphi_j} \le \int_\Omega |v_j|^2_\Theta e^{-\varphi}.$$

where $\Box_{\varphi_j}$ is the Laplace operator w.r.t. weight φ_j and metric ω. The (unique) solution exists by the Bochner-Kodaira-Nakano inequality.

Put $\tilde{u}_j = \bar{\partial}^*_{\varphi_j} w_j$ on Ω_j. We may choose a weak limit $\tilde{u} = u dz$ in $L^2_{(n,0)}(\Omega, \mathrm{loc})$ s.t.

$$\int_\Omega |u|^2 e^{-\varphi} \le \liminf_{j\to\infty} \int_\Omega |v_j|^2_\Theta e^{-\varphi} = \int_\Omega |v|^2_\Theta e^{-\varphi}.$$

Ohsawa-Takegoshi type theorems

Since $\tilde{v}_j = \bar{\partial}\tilde{u}_j + \bar{\partial}^*_{\varphi_j}\bar{\partial}w_j$, so

$$\bar{\partial}\tilde{u} = \tilde{v} := dz \wedge v \iff \bar{\partial}^*_{\varphi_j}\bar{\partial}w_j \to 0$$

weakly in $L^2_{(n,0)}(\Omega, \mathrm{loc})$.

The conditions $\bar{\partial}v = 0$ and ω is complete are used only to verify

$$\bar{\partial}^*_{\varphi_j}\bar{\partial}w_j \to 0.$$

In such a way, one can show that Theorem 1.1 holds for all CK domains. Q.E.D.

Ohsawa-Takegoshi type theorems

As an application of Theorem 3.15, one has

Theorem 3.16 (Chen-Wang-Wu 2015)

$E \subset \mathbb{D}^n \times \mathbb{D}_r$ $(r < 1)$, *closed complete pluripolar,* $\varphi \in PSH(\mathbb{D}^{n+1} \backslash E)$. *Suppose* $\exists\, A \subset \mathbb{D}^n$ *with* $|A|_n > 0$,

$$\forall\, z' \in A,\ \varphi(z', \cdot) \in SH(\mathbb{D}).$$

Then φ *can be extended to a psh function on* $\mathbb{D}^{n+1}$.

The important case when E is an analytic set was proved by Siu (1974). Such theorems are called Thullen type extension theorem for psh functions in the literature.

Ohsawa-Takegoshi type theorems

Proof. It suffices to show that $\varphi \in L^\infty_{\mathrm{loc}}$ near every $a \in E$. Let $B \subset\subset \mathbb{D}^n$: ball, $z^0 \in (B \times \mathbb{D}_r)\backslash E$.
Theorem 3.15 $\Rightarrow \exists\, f \in \mathcal{O}(\mathbb{D}^{n+1}\backslash E), f(z^0) = e^{\frac{\varphi(z^0)}{2}}$,

$$\int_{\mathbb{D}^{n+1}\backslash E} |f|^2 e^{-\varphi} \le C_n.$$

Fubini's theorem $\Rightarrow \exists\, Z_1 \subset \mathbb{D}^n$, $|Z_1|_n = 0$, s.t.

$$\int_{\mathbb{D}} |f(z', \cdot)|^2 e^{-\varphi(z', \cdot)} < \infty,\ z' \in \mathbb{D}^n\backslash Z_1.$$

E is pluripolar $\Rightarrow \exists\, Z_2 \subset \mathbb{D}^n$, $|Z_2|_n = 0$, s.t.

$$E_{z'} := (\{z'\} \times \mathbb{D}) \cap E \text{ is polar },\ z' \in \mathbb{D}^n\backslash Z_2.$$

Ohsawa-Takegoshi type theorems

Thus

$$f(z',\cdot)\in L^2_{\mathrm{loc}},\quad \forall\, z'\in A-Z_1-Z_2,$$

which implies $f(z',\cdot)\in\mathcal{O}(\mathbb{D})$, in view of

Theorem 3.17 (Carleson 1967)

$\Omega\subset\mathbb{C}$: *open set,* $E\subset\Omega$ *closed. Then*

$$A^2(\Omega\backslash E)=A^2(\Omega)\iff E \text{ is polar.}$$

By the Hartogs theorem, we get

$$f\in\mathcal{O}(\mathbb{D}^{n+1}).$$

Ohsawa-Takegoshi type theorems

Fix $r < r'' < r' < 1$, $B \subset\subset B' \subset\subset \mathbb{D}^n$.

$$
\begin{aligned}
|f(z^0)|^2 &\leq C \int_{B' \times (\mathbb{D}_{r'} \setminus \mathbb{D}_{r''})} |f|^2 \\
&\leq C \sup_{B' \times (\mathbb{D}_{r'} \setminus \mathbb{D}_{r''})} e^{\varphi} \int_{B' \times (\mathbb{D}_{r'} \setminus \mathbb{D}_{r''})} |f|^2 e^{-\varphi} \\
&\leq C \sup_{B' \times (\mathbb{D}_{r'} \setminus \mathbb{D}_{r''})} e^{\varphi}
\end{aligned}
$$

i.e.

$$
\varphi(z^0) \leq \log C + \sup_{B' \times (\mathbb{D}_{r'} \setminus \mathbb{D}_{r''})} \varphi.
$$

Q.E.D.

Ohsawa-Takegoshi type theorems

It is natural to ask

Problem 3.18

Can the complete pluripolarity be replaced by pluripolarity?

It seems interesting to develop an L^2–theory for the $\bar{\partial}$–equation on domains $\Omega \backslash E$ where E is a closed "thin" set, e.g. a pluripolar set, and Ω is a bounded pseudoconvex domain.

On the other hand, suppose the weighted L^2 estimate in Theorem 1.1 holds on $\Omega \backslash E$, then what kind of analytic structure should E have?