Regularity properties of the $\bar{\partial}$-Neumann problem and the Kohn-Laplace equation

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In this talk, we will discuss about regularity properties for the $\bar{\partial}$-Neumann problem and the Kohn-Laplace equation. In particular, we focus on sufficient conditions for

- the L^2-existence
- the global regularity in C^∞
- the local regularity in C^∞
- the continuity/estimates in L^p and Hölder spaces.

1. The $\bar{\partial}$-Neumann problem
   - What is the $\bar{\partial}$-Neumann problem?
   - The L^2-existence of N
   - Global regulairity of N
   - Local regularity of N
   - L^p and Hölder estimate for P and N

2. The Kohn-Laplace equation
   - CR manifolds and Kohn-Laplacian
   - The L^2-existence of G
   - Global regularity of G
   - Local regularity of G

The $\bar{\partial}$-problem

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We first consider the **inhomogeneous Cauchy-Riemann equations**

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where $\varphi = \sum \varphi_j d\bar{z}_j$ is a $(0,1)$-form on Ω satisfying the compatibility condition $\bar{\partial}\varphi = 0$.
We observe that if u satisfies (CR-eq) then so does $v = u + h$ for any holomorphic function h in Ω since

$$\bar{\partial} v = \bar{\partial}(u + h) = \bar{\partial} u + \bar{\partial} h = \bar{\partial} u = \varphi.$$

So we do not look for *a solution* but for *the solution*. The optimal solution (the one of smallest in L^2-norm) is the solution orthogonal to the holomorphic functions; this is also called the *canonical solution*.

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If u satisfies (CR-eq), then so does $v = u - Pu$ and v is the unique function such that

$$\begin{cases} \bar\partial v = \varphi, \\ v \perp \mathcal{H}(\Omega). \end{cases}$$

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It then follows

$$Pu = u - K\bar{\partial}u.$$

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Before stating the $\bar\partial$-Neumann problem, we need definition of $\bar\partial^*$ the L^2-adjoint of $\bar\partial$. The $\bar\partial^*$ is defined as follows: Let

$$u = \sum u_j d\bar z_j \in \mathrm{Dom}(\bar\partial^*)$$

and $\bar\partial^* u = g$ if

$$(w, g) = (\bar\partial w, u) \quad \text{for all} \quad w \in C^\infty(\bar\Omega).$$

We see that $\bar\partial w = \sum \frac{\partial w}{\partial \bar z_j} d\bar z_j$ and

$$(\bar\partial w, u) = \sum (\frac{\partial w}{\partial \bar z_j}, u_j) \overset{Stokes}{=} -\sum (w, \frac{\partial u_j}{\partial z_j}) + \int_{b\Omega} \sum w \overline{u_j \frac{\partial r}{\partial z_j}} dS.$$

Hence $\sum w \overline{u_j \frac{\partial r}{\partial z_j}} = 0$ on $b\Omega$ for all w. i.e. $\sum_j u_j \frac{\partial r}{\partial z_j} = 0 \quad$ on $b\Omega$.
The condition $u \in \mathrm{Dom}(\bar\partial^*)$ implies boundary condition on u.

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We now state the $\bar{\partial}$-Neumann problem : Given $\varphi \in L^2_{(0,1)}(\Omega) \cap \mathcal{H}^{\perp}_{(0,1)}(\Omega)$, find $u \in L^2_{(0,1)}(\Omega)$ such that

$$\begin{cases} (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})u = \varphi \\ u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \\ \bar{\partial}u \in \mathrm{Dom}(\bar{\partial}^*), \bar{\partial}^* u \in \mathrm{Dom}(\bar{\partial}). \end{cases} \tag{2.1}$$

When the $\bar{\partial}$-Neumann problem is solvable for each $\varphi \in L^2_{(0,1)}(\Omega) \cap \mathcal{H}^{\perp}_{(0,1)}(\Omega)$, we define the $\bar{\partial}$-Neumann operator $N : L^2_{(0,1)}(\Omega) \to L^2_{(0,1)}(\Omega)$ by

$$N\varphi = \begin{cases} 0 & \text{if } \varphi \in \mathcal{H}_{(0,1)}(\Omega) \\ u & \text{if } \varphi \perp \mathcal{H}_{(0,1)}(\Omega). \end{cases}$$

Then N is a bounded self adjoint operator.

The canonical solution of the $\bar\partial$-problem

If $\bar\partial\alpha = 0$, then $v = \bar\partial^* N\alpha$ is the unique solution to the *$\bar\partial$-problem*

$$\begin{cases} \bar\partial v = \alpha, \\ v \text{ is orthogonal to } \operatorname{Ker}\bar\partial. \end{cases}$$

In fact,

$$\bar\partial v = \bar\partial\bar\partial^* N\alpha = \bar\partial\bar\partial^* N\alpha + \bar\partial^*\bar\partial N\alpha = \Box N\alpha = \alpha$$

and $(v, h) = (\bar\partial^* N\alpha, h) = (N\alpha, \bar\partial h) = 0$ for any h holomorphic. (i.e. $v \perp \operatorname{Ker}\bar\partial$).

Kohn's formula

$$P = I - \bar\partial^* N\bar\partial$$

where P is Bergman projection, the projection to holomorphic space.

General setting

- Let Ω be a domain in a complex hermitian manifold $\mathcal{X}$ of dimension n.
- For $0 \leq p \leq n$ and $1 \leq q \leq n$, we denote $L^2_{(p,q)}(\Omega)$ and $\mathcal{H}_{(p,q)}(\Omega)$ be the L^2 and harmonic spaces of (p, q)-forms, respectively.

The $\bar{\partial}$-Neumann problem can be stated as: Given $\varphi \in L^2_{(p,q)}(\Omega) \cap \mathcal{H}^{\perp}_{(p,q)}(\Omega)$, find $u \in L^2_{(p,q)}(\Omega)$ such that

$$\begin{cases} (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})u = \varphi \\ u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \\ \bar{\partial}u \in \mathrm{Dom}(\bar{\partial}^*), \bar{\partial}^* u \in \mathrm{Dom}(\bar{\partial}). \end{cases} \tag{2.2}$$

When the $\bar{\partial}$-Neumann problem is solvable for each $\varphi \in L^2_{(p,q)}(\Omega) \cap \mathcal{H}^{\perp}_{(p,q)}(\Omega)$, we define the $\bar{\partial}$-Neumann operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ by

$$N_{(p,q)}\varphi = \begin{cases} 0 & \text{if } \varphi \in \mathcal{H}_{(p,q)}(\Omega) \\ u & \text{if } \varphi \perp \mathcal{H}_{(p,q)}(\Omega). \end{cases}$$

Then $N_{(p,q)}$ is a bounded self adjoint operator and has the following properties

- $\Box N_{(p,q)} = N_{(p,q)}\Box = I - H_{(p,q)}$ where $H_{(p,q)}$ is the orthogonal projection from $L^2_{(p,q)}(\Omega)$ to the harmonic space $\mathcal{H}_{(p,q)}(\Omega)$;
- $\bar{\partial} N_{(p,q)} = N_{(p,q+1)}\bar{\partial}$; $\bar{\partial}^* N_{(p,q)} = N_{(p,q-1)}\bar{\partial}^*$
- $P_{(p,q-1)} = I - \bar{\partial}^* N_{(p,q)}\bar{\partial}$ the Bergman projection from $L^2_{(p,q-1)}(\Omega)$ to the kernel space of $\bar{\partial}$.

Questions:

The fundamental questions regarding the $\bar{\partial}$-Neumann problem can be asked, at level of understanding the regularity of N, as follows:

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- the local regularity in C^∞, i.e., if $\alpha \in C^\infty_c(U \cap \bar{\Omega})$ then $N\alpha \in C^\infty(U' \cap \bar{\Omega})$ where the sets $U' \subset U$ range in a system of neighborhoods of a point $z_0 \in \bar{\Omega}$.

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- the local regularity in C^∞, i.e., if $\alpha \in C_c^\infty(U \cap \bar\Omega)$ then $N\alpha \in C^\infty(U' \cap \bar\Omega)$ where the sets $U' \subset U$ range in a system of neighborhoods of a point $z_0 \in \bar\Omega$.
- the boundedness in the Sobolev spaces L_s^p and/or Holder space Λ_s.

The L^2-existence of N

Definition

Ω is pseudoconvex if the Levi matrix of defining function, $(r_{z_i\bar{z}_j})_{i,j}$, is semi-positive definite on the $T^{1,0}(b\Omega)$.

Proposition (Bochner-Hörmander-Kohn-Morrey formula)

Let $\rho(x) = -dist(x, b\Omega)$, and $\{L_1, \ldots, L_n\}$ is a local unitary frame of $T^{1,0}(\Omega)$. Then, for any (p,q)-form $u \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$ with $q \geq 1$ on Ω, we have

$$\|\bar{\partial}u\|^2_\phi + \|\bar{\partial}^*_\phi u\|^2_\phi = \|\overline{\nabla} u\|^2_\phi + \int_{b\Omega} \langle (i\partial\bar{\partial}\rho)u, u\rangle e^{-\phi}\, d\sigma$$

$$+\left((i\partial\bar{\partial}\phi)u, u\right)_\phi + (Eu, u)_\phi$$

where $d\sigma$ is the induced surface area measure,

$$\|\overline{\nabla} u\|^2_\phi = \int_\Omega \sum_{j=1}^n |\nabla_{\bar{L}_j} u|^2 e^{-\phi}\, dV,$$

and the error operator E satisfying

$$\|Eu\|_\phi \leq c\left(\|\overline{\nabla} u\|_\phi + \|u\|_\phi\right).$$

Here the constant c is independent of ϕ but dependent on the hermitian metric h. Furthermore, if M is a Kähler manifold, then we the error operator E coincides with the Ricci tensor $\Theta_{(p,q)}$.

Example

Let $\Omega \subset \mathbb{C}P^n$. The curvature term $\Theta_{(p,q)}$ with respect to the Fubini-Study metric $\langle\cdot,\cdot\rangle$ has the following equalities/inequality:

$$\begin{aligned}
\langle\Theta_{(0,q)}u,u\rangle &= q(2n+1)|u|^2 && \text{if } q\geq 1\\
\langle\Theta_{(n,q)}u,u\rangle &= 0 && \text{if } q\geq 1 \qquad (2.3)\\
\langle\Theta_{(0,q)}u,u\rangle &\geq 0 && \text{if } 1\leq p\leq n, q\geq 1
\end{aligned}$$

Corollary

Let $\mathcal{X}$ be a Kähler manifold which $\Theta_{(p,q)}$ is strictly positive and let Ω be a pseudoconvex domain in X. Then $\mathcal{H}_{(p,q)}(\Omega) = \{0\}$ and $N_{(p,q)}$ is continuous in L^2.

Theorem (Kohn 1973)

Let $\Omega \subset \mathcal{X}$ be a smooth bounded pseudoconvex domain which admits a strictly plurisubharmonic function in a neighborhood of $b\Omega$, $0 \leq p \leq n$ and $1 \leq q \leq n$. Then, $\mathcal{H}_{(p,q)}(\Omega)$ is finite dimensional and $N_{(p,q)}$ is continuous in L^2.
Additionally, the $\bar{\partial}$-equation $\bar{\partial}u = \varphi$ has a solution $u \in C^{\infty}_{p,q-1}(\bar{\Omega})$ if φ is a $\bar{\partial}$-closed form in $C^{\infty}_{p,q}(\bar{\Omega})$.

Global regularity of N

Definition (Compactness estimate)

For any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$\|u\|^2 \leq \epsilon \left(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2\right) + c_\epsilon \|u\|_{-1}^2$$

Theorem (Kohn and Nirenberg,1965)

Compactness estimate implies global regularity of N. Moreover $\varphi \in H^s(\bar{\Omega}) \Rightarrow N\varphi \in H^s(\bar{\Omega})$.

Definition (Property (P))

For any $\epsilon > 0$, there exists a uniformly bounded, psh function Φ_ϵ in a nbh of $b\Omega$ such that

$$\langle (i\partial\bar{\partial}\Phi_\epsilon)u, u\rangle \geq \frac{1}{\epsilon}|u|^2 \quad \text{on} \quad b\Omega.$$

Theorem (Catlin 1984)

Property (P) $\Rightarrow$ Compactness estimate

Remark

Catlin (1984) proved that Property (P) holds if domains is of finite type in $\mathbb{C}^n$. However, much more general than finite type, Sibony (1987) made a systematic study of the property (under the name of "*B*-regularity", allow a class domains of infinite type) for a sufficient condition of Property (P).

However, in the case $X = \mathbb{C}^n$, the compactness estimate/Property (P) is not a necessary condition for the global regularity of N. In particular, if $\Omega \subset \mathbb{C}^n$ has a plurisubharmonic defining function and/or a vector field that commutes approximately with $\bar{\partial}$, then the $\bar{\partial}$-Neumann operators are globally regular due to Boas and Straube 1990s.

Theorem (Boas and Straube 1990)

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ which admits a plurisubharmonic defining function. Then the Bergman projection $P_{(p,q)}$ is exactly regular.

Theorem (Boas and Straube 1991)

$N_{(p,q)}$ is exactly regular if and only if $P_{(p,q-1)}$, $P_{(p,q)}$, $P_{(p,q+1)}$ are exactly regular.

Straube developed the theory of global regularity of the $\bar{\partial}$-Neumann problem in $\mathbb{C}^n$ by unifying above conditions.

Theorem (Straube 2008)

Let $\Omega \subset\subset \mathbb{C}^n$ be a smooth pseudoconvex domain, ρ a defining function for Ω. Let $0 \leq p \leq n$ and $1 \leq q \leq n$. Assume that there is a constant C such that for all $\epsilon > 0$ there exist a defining function ρ_ϵ for Ω and a constant C_ϵ with $C^{-1} \leq |\nabla \rho_\epsilon| \leq C$ on $b\Omega$, and

$$\sum_{|K|=k-1}' \left\| \sum_{j,k=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial^2 \rho_\epsilon}{\partial z_k \partial \bar{z}_j} u_{kK} \right\|^2 \leq \epsilon(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2) + C_\epsilon \|u\|_{-1}^2$$

for all $u \in C^\infty_{(0,q)}(\bar{\Omega}) \cap Dom(\bar{\partial}^*)$. Then the $\bar{\partial}$-Neumann operator $N_{(0,q)}$ acting on $(0,q)$-forms is exactly regular.

Theorem

Let $\Omega \subset \mathcal{X}$ be a smooth bounded pseudoconvex domain which admits a strictly plurisubharmonic function acting (p_0, q_0)-forms in a neighborhood of $b\Omega$. Let ρ be a smooth defining function of Ω and denote $\gamma = \frac{i}{2}(\partial\rho - \bar{\partial}\rho)$. Assume that there is a constant C such that for all $\epsilon > 0$ there exist a purely imaginary vector field T_ϵ and a constant C_ϵ with $C^{-1} \leq |\gamma(T_\epsilon)| \leq C$ on $b\Omega$, and

$$\|\alpha_\epsilon \lrcorner u\|^2 \leq \epsilon(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2) + C_\epsilon \|u\|^2_{-1} \tag{2.4}$$

for all $u \in C^\infty_{(p_0,q_0)}(\bar{\Omega}) \cap Dom(\bar{\partial}^*)$, where α_ϵ is the $(1,0)$-part of the real form $\{Lie\}_{T_\epsilon}(\gamma)$.
Then, for $0 \leq p \leq n$ and $q_0 \leq q \leq n$, the space of L^2 harmonic (p,q)-forms $\mathcal{H}_{p,q}(\Omega) \subset C^\infty_{(p,q)}(\bar{\Omega})$ and the operators $N_{(p,q)}$ is exactly regular.

We can relax the "the existence of a strictly plurisubharmonic function" condition, but a stronger estimate is required.

Theorem

Let $\Omega \subset\subset \mathcal{X}$ be a smooth bounded pseudoconvex domain. Assume that there are constants C, c such that for all $\epsilon > 0$ there exist a purely imaginary vector field T_ϵ and a constant C_ϵ with

$$C^{-1} \le |\gamma(T_\epsilon)| \le C$$

on $b\Omega$, and

$$\|u\|^2 + \frac{1}{\epsilon}\left(\|\bar\alpha_\epsilon \wedge u\|^2 + \|\alpha_\epsilon \lrcorner u\|^2\right) \le c(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) + C_\epsilon \|u\|^2_{-1} \quad (2.5)$$

for all $u \in C^\infty_{(p,q)}(\bar\Omega) \cap Dom(\bar\partial^*)$. Then the space of L^2 harmonic (p,q)-forms $\mathcal{H}_{p,q}(\Omega) \subset C^\infty_{(p,q)}(\bar\Omega)$ and the operators $N_{(p,q)}$ is exactly regular.

Local regularity of N

- Let $z_o \in b\Omega$. Suppose U is a n.b.h. of z_o. Consider the local boundary coordinates (defined on U) denote by $(t, r) = (t_1, ..., t_{2n-1}, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ where r is defining function of Ω.

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- For $u \in C^\infty(\bar{\Omega} \cap U)$, the tangential Fourier transform of u, defined by

$$\mathcal{F}_t u(\xi, r) = \int_{\mathbb{R}^{2n-1}} e^{-i\langle t,x\rangle} u(t, r)dt.$$

The standard tangential pseudo-differential operator is expressed by

$$f(\Lambda)u(t, r) = \mathcal{F}_t^{-1}\Big(f((1 + |\xi|^2)^{1/2})\mathcal{F}_t u(\xi, r)\Big)$$

where $f : R^+ \to \mathbb{R}^+$.

Definition (Subelliptic estimate)

$$||\Lambda^\epsilon u||^2 \lesssim \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2, \qquad 0 < \epsilon \leq \frac{1}{2}$$

for any $u \in C_c^\infty(\bar{\Omega} \cap U) \cap \mathrm{Dom}(\bar{\partial}^*)$.

Theorem (Folland-Kohn 1972)

Subelliptic estimate $\Longrightarrow$ local regularity.
Moreover, $\varphi \in H^s(V) \Rightarrow N\varphi \in H^{s+2\epsilon}(V')$ for $V' \subset V$, where ϵ is order of subellipticity.

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Theorem (Kohn 1963-1964)

Strongly pseudoconvex $\Leftrightarrow$ $\frac{1}{2}$*-subelliptic estimates*

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Definition (Finite type : (D'Angelo finite type))

:

$$D(z_0) = \sup \frac{\text{ord}_{z_0}(r(\phi))}{\text{ord}_0 \phi}$$

where supremum is taken over all local holomorphic curves $\phi : \Delta \to \mathbb{C}^n$ with $\phi(0) = z_0$.

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- $r = \text{Re} z_3 + |z_2^b - z_1|^2$, then $D(z_0) = \infty$, since $\mathcal{C} : \{z_2^b - z_1 = 0, z_3 = 0\} \subset b\Omega$.

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- $r = \text{Re} z_3 + |z_1|^{2a} + |z_2^b - z_1|^2$, then $D(z_0) = 2ab$.
- $r = \text{Re} z_n + \exp(-\frac{1}{|z_j|^s})$ in $\mathbb{C}^n$, then $D(z_0) = \infty$

Theorem (Kohn 1979)

Let Ω be pseudoconvex+real analytic+finite type at $z_0 \in \Omega$. Then subelliptic estimate hold at z_0

Theorem (Caltin 1983, 1984 and 1987)

Let Ω be pseudoconvex at z_0. Then the following are equivalent:
(i) D'Angelo type is finite;
(ii) There exist a family of bounded, plurisubharmonic functions with large Levi form near boundary ($\cong |r|^{-2\epsilon}$);
(iii) ϵ-subelliptic hold

Proof : $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$

Domains of infinite type

Definition

Superlogarithmic estimate

$$\forall\eta > 0, \exists c_\eta > 0 : \|\log \Lambda u\|^2 \le \eta(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2) + c_\eta \|u\|^2_{-1}$$

for any $u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$.

Theorem (K.-Zampieri 2012 (inspired by Kohn 2002))

Superlogarithmic estimate $\Longrightarrow$ local regularity of N.

Proof. Let $R^s = \mathcal{F}^{-1}\left((1 + |\xi|^2)^{\frac{s}{2}\sigma(x)} \mathcal{F} u(\xi)\right)$. Using superlogarithmic estimate for $\zeta R^s \zeta$ where $\zeta = 1$ on supp(σ). The commutator term $[L, \zeta R^s \zeta] = \log \Lambda \zeta R^s \zeta + \text{Op}^{-\infty}$ can be absorted by superlogarithmic estimate.

Example

Let $M = \{(z = x + iy, w) \in \mathbb{C}^{n+1} : \text{Re } w = \sum_j \exp(-\frac{1}{|x_j|^{\alpha_j}})\}$, then N is local regularity if and only if $\alpha_j < 1$ for all j (iff superlogarithmic estimate holds).

Chirst 2002; Kohn 2000; Baracco-K.-Zampieri 2012

We observe that on the on both hypersurfaces

$$M_1 = \{(z = x + iy, w) \in \mathbb{C}^2 : \text{Re} w = e^{-\frac{1}{|x|^\alpha}}\}$$

$$M_2 = \{(z, w) \in \mathbb{C}^2 : \text{Re} w = e^{-\frac{1}{|z|^\alpha}}\},$$

superlogarithmic estimate does not hold if $\alpha > 1$. However, on M_2, N is locally regular for any $s > 0$.

L^p and Hölder estimate for P

Theorem (Fefferman 1974, Phong and Stein 1977, McNeal and Stein 1994, McNeal 1994, Cho 1994, 1996, 2002)

Let Ω be a bounded, pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that Ω satisfies one of the following conditions:

(a) *Ω is a strongly pseudoconvex domain;*

(b) *Ω is a pseudoconvex domain of finite type and $n = 2$;*

(c) *Ω is a convex domain of finite type;*

(d) *Ω is a decoupled domain of finite type;*

(e) *Ω is a pseudoconvex domain of finite type whose Levi-form has only one degenerate eigenvalue or comparable eigenvalues.*

Then the Bergman operator P maps from

- $L^p_s(\Omega)$ *to* $L^q_s(\Omega)$ *continuously with* $1 < p \leq \infty$, $s \geq 1$.
- $\Lambda_s(\Omega) \to \Lambda_s(\Omega)$ *continuously with* $s > 0$

L^p and Hölder estimate for N

Theorem (Griener and Stein 1977; Lieb and Range 1986, 1987; Beals, Greiner and Staton 1987, Fefferman and Kohn 1988, Change, Nagel and Stein 1992)

Let Ω be a bounded, pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that Ω satisfies one of the following conditions:

(a) *Ω is a strongly pseudoconvex domain; $(\epsilon = \frac{1}{2})$*

(b) *Ω is a pseudoconvex domain of finite type M and $n = 2$; $(\epsilon = 1/M)$*

Then the $\bar{\partial}$-Neumann operator N maps from

- $L^p_s(\Omega)$ *to* $L^p_{s+2\epsilon}(\Omega)$ *continuously with* $1 < p \leq \infty$, $s \geq 1$.
- $\Lambda_s(\Omega) \to \Lambda_{s+2\epsilon}(\Omega)$ *continuously with* $s > 0$

The method of proof of case (a) is to find explicit integral operators which is give good approximations of N.

CR manifolds

Let M be $2n+1$ dimensional differential manifold. We say M is a CR mainfold (of hypersurface type) if M is equipped a CR structure $T^{1,0}M \subset \mathbb{C}TM$ that has the following properties

1. $\dim_{\mathbb{C}}(T^{1,0}M) = n$;
2. $T^{1,0}M \cap T^{0,1}M = \{0\}$ where $T^{0,1}M = \overline{T^{1,0}M} = \{0\}$;
3. if $L, L' \in T^{1,0}M$, then the commutator $[L, L'] \in T^{1,0}M$ (this can be avoid when $n=1$).

M is pseudoconvex (resp. strictly pseudoconvex)if there is a nonvanishing real 1-form θ which annihilates $T^{1,0} \oplus T^{0,1}$ such that the Levi form

$$\sqrt{-1}d\theta(L, \bar{L}), \quad L \in T^{1,0}$$

is nonnegative (resp. positive).

Denote by $\mathcal{A}_b^{p,q}$ the bundle of (p,q)-forms on M. The operator $\bar{\partial}_b : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q+1}$ is defined as follows. If $u \in \mathcal{A}_b^{p,q}$, let u' be a (p,q)-form on M which $u'|_{\mathcal{A}_b^{p,q}} = u$. Then $\bar{\partial}_b u = du|_{\mathcal{A}_b^{p,q+1}}$.

Remark: if $M \subset X$, then $\bar{\partial}_b = \bar{\partial}|_M$.

The operator $\bar{\partial}_b^* : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q-1}$ is the L^2-adjoint of $\bar{\partial}_b$.

Let $L_1, \dots, L_n$ be a local basis of (1,0) vectorfields in a nbh U of $x_o \in M$ and $\omega_1, \dots, \omega_n$ be the dual basis of (1,0)-forms. If $u \in C^\infty(M)$, then $\bar{\partial}_b u = \sum_j \bar{L}_j(u)\bar{\omega}_j$.
If $u = \sum_j u_j \bar{\omega}_j$, then

$$
\begin{aligned}
\bar{\partial}_b u &= \sum_{j<i} \left(\bar{L}_j u_i - \bar{L}_i u_j + a_{ij}^k u_k \right) \bar{\omega}_j \wedge \bar{\omega}_i \\
\bar{\partial}_b^* u &= -\sum_j (L_j u_j + a_j u_j).
\end{aligned}
$$

The Kohn-Laplacian $\Box_b : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q}$ is defined by

$$\Box_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b.$$

This is NOT an elliptic operator. The complex energy is defined by $Q_b(u,u) = \|\bar{\partial}_b u\|^2 + \|\bar{\partial}_b^* u\|^2 = (\Box_b u, u)$.

Theorem

Let M be a CR manifold of hypersurface type and U a local patch. Let ϕ be a real C^2 function. There exists a constant C (independent of ϕ) such that for any $u \in C^\infty_{0,q}(M)$ with support in U,

$$\|\bar\partial_b u\|^2_{L^2_\phi} + \|\bar\partial^*_{b,\phi} u\|^2_{L^2_\phi} + C\|u\|^2_{L^2_\phi} \geq \frac{1}{2}\sum_{j=1}^{n}\|\bar L_j u\|^2_{L^2_\phi}$$

$$+ \sum_{I\in\mathbb{I}_{q-1}}\sum_{i,j=1}^{n}(\phi_{ij}u_{iI}, u_{jI})_\phi + Re\left\{\sum_{I\in\mathbb{I}_{q-1}}\sum_{i,j=1}^{n}(c_{ij}Tu_{iI}, u_{jI})_\phi\right\}$$

A great deal of work has been done for the L^2-existence of G on some classes of CR manifolds. In recent work with A. Raich, we generalise the result on abstract CR manifolds equipped with "good" CR-plurisubharmonic functions, which is a potential theoretical condition that we call a P-property.

Theorem (K. and Raich 2016)

Let M be a compact, pseudoconvex-oriented, $(2n+1)$-dimensional CR manifold equipped with a strictly CR-plurisubharmonic function on $(0, q_0)$-forms, i.e., there exist a global function λ and constant $\alpha > 0$ such that

$$\langle (\mathcal{L}_\lambda + d\theta)u, u\rangle \geq \alpha |u|^2 \quad \textit{holds on } M \textit{ for all smooth } (0,q)\textit{-forms } u,$$

then the operators G_q, $\bar{\partial}_b^* G_q$, $G_q \bar{\partial}_b^*$, $\bar{\partial}_b G_q$ $G_q \bar{\partial}_b$, $I - \bar{\partial}_b^* \bar{\partial}_b G_q$, $I - \bar{\partial}_b^* G_q \bar{\partial}_b$, $I - \bar{\partial}_b \bar{\partial}_b^* G_q$, $I - \bar{\partial}_b G_q \bar{\partial}_b^*$, $\bar{\partial}_b G_q^2 \bar{\partial}_b^*$ and $\bar{\partial}_b^* G_q^2 \bar{\partial}_b$ are L^2-bounded for all degrees $q_0 \leq q \leq n - q_0$.

Theorem (K. and Raich 2016)

Let M be a smooth, compact, pseudoconvex-oriented CR manifold of hypersurface type of dimension $2n+1$ which admits a strictly CR plurisubharmonic function on $(0, q_0)$-forms, $n \geq 2$. Assume that for every $\epsilon > 0$ there exist a C^∞ real valued function λ_ϵ, a purely imaginary vector field T_ϵ, and a constant $A_\epsilon > 0$ so that $|\lambda_\epsilon|$, $\gamma(T_\epsilon)$ are uniformly bounded, $\gamma(T_\epsilon)$ is bounded away from zero, and

$$\langle (\mathcal{L}_{\lambda_\epsilon} + A_\epsilon d\gamma) \lrcorner u, \bar{u} \rangle \geq \frac{1}{\epsilon} |\alpha_\epsilon|^2 |u|^2$$

for any $(0, q_0)$-forms u. The form α_ϵ is real and defined by $\alpha_\epsilon = -\{Lie\}_{T_\epsilon}(\gamma)$.
If $q_0 \leq q \leq n - q_0$, then the space of L^2 harmonic $(0, q)$ forms $\mathcal{H}_{0,q}(M) := \ker(\Box_b) \subset C^\infty_{0,q}(M)$. Additionally, the operators G_q, $\bar\partial_b G_q$, $G_q \bar\partial_b$, $\bar\partial_b^* G_q$, $G_q \bar\partial_b^*$, $I - \bar\partial_b^* \bar\partial_b G_q$, $I - \bar\partial_b^* G_q \bar\partial_b$, $I - \bar\partial_b^* \bar\partial_b G_q$, $I - \bar\partial_b G_q \bar\partial_b^*$ are both globally regular and exactly regular in the L^2-Sobolev space H^s, $s \geq 0$.

In the case $q=1$, the operators $G_0=\bar{\partial}_b^* G_1^2 \bar{\partial}_b$ and $G_n=\bar{\partial}_b G_{n-1}^2 \bar{\partial}_b^*$ and hence G_0, $\bar{\partial}_b G_0$, G_n, $\bar{\partial}_b^* G_n$ are both globally regular and exactly regular in H^s as well, $s \geq 0$.

Let $f : [1,+\infty) \to [1,+\infty)$, we define the pseudodifferential operator $f(\Lambda)$ by

$$f(\Lambda)u = \mathcal{F}^{-1}\left(f((1+|\xi|^2)^{1/2})\mathcal{F}(u)(\xi)\right) \quad u \in C_c^\infty.$$

Let $\mathcal{M} = (\mathcal{M}_{ij}) \geq 0$, denote by

$$(\mathcal{M}f(\Lambda)u, f(\Lambda)u) = \sum_{|K|=q-1}\sum_{ij}\int_M \mathcal{M}_{ij} f(\Lambda)u_{iK}\overline{f(\Lambda)u_{jK}}dS$$

where $u = \sum'_{|J|=q} u_J\bar\omega_J$.
If $\mathcal{M} = \rho^2 Id$, then $(\mathcal{M}f(\Lambda)u, f(\Lambda)u) = \|\rho f(\Lambda)u\|^2$
If $\mathcal{M} = \rho_i\bar\rho_j$, then $(\mathcal{M}f(\Lambda)u, f(\Lambda)u) = \sum_{|K|=q-1}\|\sum_j \rho_j f(\Lambda)u_{iK}\|^2$

We introduce a general estimate, called $(f\text{-}\mathcal{M})^q$, at $x_o \in M$ as

$$(\mathcal{M}f(\Lambda)u, f(\Lambda)) \lesssim Q_b(u,u) + \|u\|^2 + C_{\mathcal{M}}\|u\|^2_{-1}, \qquad (3.1)$$

for any $u \in C_c^\infty(U\cap M)^{0,q}$ where U is a nbh of x_o.

Example

- If $f(t) = t^\epsilon; \mathcal{M} = Id$, then $(f\text{-}\mathcal{M})^q$ consists of the subelliptic estimate, i.e.,

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ii If $f(t) = \log t$; $\mathcal{M} = \frac{1}{\eta} Id$ large constant, then $(f\text{-}\mathcal{M})^k$ coincides with the superlogarihmic estimate, i.e.,

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iii If $f(t) = 1$; $\mathcal{M} = \frac{1}{\eta} Id$ large constant, then $(f\text{-}\mathcal{M})^q$ is the compactness estimate, i.e.,

$$||u||^2 \lesssim \eta Q_b(u,u) + C_\eta \|u\|^2_{-1}, \qquad \forall \eta > 0$$

For $u \in C_c^\infty$, we decompose $u = u^+ + u^- + u^0$ where

$$Tu^+ = \left((T^+)^{1/2}\right)^* ((T^+)^{1/2})u^+; \quad Tu^- = -\left((T^-)^{1/2}\right)^* ((T^-)^{1/2})u^-$$

and elliptic estimate holds for u^0. Here T is a real vector field dual to contact form θ.
We consider microlocal version $(f\text{-}\mathcal{M})^q_{b,\pm}$ over the microlocal components $u^\pm$ of a form.

Theorem

$$(f\text{-}\mathcal{M})^q_{b,+} \Longleftrightarrow (f\text{-}\mathcal{M})^{n-q}_{b,-}$$

When $\mathcal{M} = \rho^2 Id$, then $(f\text{-}\mathcal{M})^q_b$ hold iff $(f\text{-}\mathcal{M})^q_{b,+}$ and $(f\text{-}\mathcal{M})^q_{b,-}$ hold.

$(f\text{-}\mathcal{M})^q$ estimate on hypersurface $M \subset X$

On a pseudoconvex hypersurface M of X, denote by Ω^+, Ω^- the pseudoconvex and pseudoconcave side of M, respectively.

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$$(\mathcal{M}f(\Lambda)u, f(\Lambda)u)_{L^2(\Omega^\pm)} \le c(\|\bar\partial u\|^2_{L^2(\Omega^\pm)} + \|\bar\partial^* u\|^2_{L^2(\Omega^\pm)}) + C_{\mathcal{M}}\|u\|^2_{-1,L^2(\Omega^\pm)}$$

for $u \in C^\infty_c(U \cap \bar\Omega^\pm)^{0,q} \cap \mathrm{Dom}(\bar\partial^*)$.

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for $u \in C^\infty_c(U \cap \bar{\Omega}^\pm)^{0,q} \cap \mathrm{Dom}(\bar{\partial}^*)$.

Theorem (K. 2015)

Let M be a hypersurface in X and $\Omega^{\pm}$ be the two sides of $X \setminus M$ (here, Ω^{+} is the pseudoconvex side). Then, we have the following system of equivalences

$$(f\text{-}\mathcal{M})^{q}_{\Omega^{+}} \iff (f\text{-}\mathcal{M})^{q}_{b,+} \iff (f\text{-}\mathcal{M})^{n-q}_{b,-} \iff (f\text{-}\mathcal{M})^{n-q}_{\Omega^{-}}.$$

We say that $\Box_b$ is (local) hypoelliptic at $x_o \in M$ if whenever $\Box_b u$ is C^∞ in some nbh of x_o then u is C^∞ in a nbh of x_o.

Theorem (Kohn-Nirenberg 1965)

Hypoellipticity is a consequence of subellipticity.

Example

$$\text{Im } z_{n+1} = \sum_{k=1}^{m} |h_k(z_1, \ldots, z_n)|^2, \tag{3.2}$$

where h_k's are holomorphic functions and whose common zeroes is only the origin of $\mathbb{C}^n$, the origin is a point of finite type.

Theorem (Kohn, 2002)

Superlogarithmic estimate $\Longrightarrow$ hypoellipticity.

Christ (2002) considered the model in $\mathbb{C}^2$ defined by

$$\text{Im } z_2 = \exp\left(-\frac{1}{|\text{Re } z_1|^{\alpha}}\right), \tag{3.3}$$

for some $\alpha > 0$, and showed that in a neighborhood of the origin, $\Box_b$ is locally hypoelliptic if and only if superlogarithmic estimate holds, i.e., $\alpha < 1$

The problem is to find conditions under which $\Box_b$ is locally hypoelliptic but superlogarithmicity fails.

However, that there exist many classes of hypersurfaces that superlogarithmicity might not hold but $\Box_b$ is still locally hypoelliptic. For example, $\Box_b$ is locally hypoelliptic on hypersurface models in $\mathbb{C}^{n+1}$ either defined by

$$\text{Im } z_{n+1} = \exp\left(-\frac{1}{\left(\sum_{k=1}^{m} |h_k(z_1,\ldots,z_n)|^2\right)^{\beta}}\right), \tag{3.4}$$

where h_k's are holomorphic functions whose common zeroes is only at the origin of $\mathbb{C}^n$, and $\beta > 0$ (by Kohn 2000 and Christ 2001); or

$$\text{Im } z_{n+1} = \sum_{j=1}^{n} |\text{Re } z_j|^{2m_j} \exp\left(-\frac{1}{|z_j|^{\beta_j}}\right), \tag{3.5}$$

where $m_j \in \mathbb{N}$ and $\beta_j > 0$, $j = 1, \ldots, n$ (by Baracco Pinton and Zampieri 2015).

The problem of local hypoellipticity of $\Box_b$ remained open for the model, defined by

$$\operatorname{Im} \; z_{n+1} = \sum_{j=1}^{n} \exp\left(-\frac{1}{|\operatorname{Re} \; z_j|^{\alpha_j}}\right) \exp\left(-\frac{1}{|z_j|^{\beta_j}}\right) \qquad (3.6)$$

where $0 < \alpha_j < 1$ and $\beta_j > 0$, $j = 1, \ldots, n$.

Theorem

Let M^{2n+1} be the hypersurface defined by

$$Im\ z_{n+1} = \sum_k H_k\left(\sum_j |h_{kj}|^2\right) F_k\left(\sum_j |Re\, h_{kj}|^2\right) \tag{3.7}$$

where $H_k, F_k : \mathbb{R}^+ \to \mathbb{R}^+$ are increasing and convex functions such that $H_k(0)F_k(0) = 0$; and the h_{kj}'s are holomorphic functions in $\mathbb{C}^n$ with an isolated zero at the origin. If

$$\lim_{\delta\to 0} \delta \ln\left(F_k(\delta^2)\right) = 0 \quad \text{for all } k,$$

then $\Box_b$'s acting on forms of any degree are locally hypoelliptic.

Definition

Let σ be a cutoff function in U. We say that the σ-superlogarithmic property on $(0,q_0)$-forms holds if there exist a pair of rate functions $f, f_0 : \mathbb{R}^+ \to \mathbb{R}^+$ so that $f(t), f_0(t) \nearrow \infty$ and $t^{-\frac{1}{2}}f(t), t^{-\frac{1}{2}}f_0(t) \searrow 0$ as $t \nearrow \infty$, and a family of smooth real-valued function $\{\lambda_t\}_{t\geq 1}$ defined on U such that

$$\begin{cases} \langle(\mathcal{L}_{\lambda_t} + td\theta)\lrcorner u, u\rangle \geq f_0^2(t)\left(\ln t|\langle\mathcal{L}_\sigma \lrcorner u, u\rangle| + \ln^2 t|\partial_b\sigma\lrcorner u|^2\right) + f^2(t)|u|^2 \\ |T^m(\lambda_t)| \leq c_m t^{\frac{m}{2}}, \text{for all } m \in \mathbb{N}, \end{cases}$$

holds at any $x \in U$ for all smooth $(0,q_0)$-forms u. The functions f, f_0 are called the independent and dependent rate, respectively.

This definition is inspired by the P-property defined by Catlin.

Theorem (K. 2016)

Let M be a pseudoconvex CR manifold of dimension $(2n+1)$ such that $\bar{\partial}_b$ has closed range in L^2 spaces for all degrees of forms (if $n=1$ one assumes in addition that the solution operator to $\Box_b$ is globally regular). Suppose that the σ-superlogarithmic property on $(0,q_0)$-forms holds. Then, for any $q_0 \leq q \leq n-q_0$, if $u \in L^2_{0,q}(M)$ such that $\Box_b u \in L^2_{0,q}(M) \cap C^\infty_{0,q}(V^\sigma_1)$ then $u \in C^\infty_{0,q}(V^\sigma_0)$, where V^σ_0 the interior of $\{x \in M : \sigma(x)=1\}$ and V^σ_1 be an open set containing $supp(\sigma)$.

The main tool

$$\|u\|^2_{g\text{-}\mathcal{A}^{\pm,s\sigma}_\zeta} := \sum_{k=1}^{\infty} g^2(e^k)\|e^{ks\sigma}u^{\pm}_{\zeta,k}\|^2_{L^2}$$

The norm defined by

$$\|\cdot\|^2_{g\text{-}\mathcal{A}^{s\sigma}_\zeta} := \|\cdot\|^2_{g\text{-}\mathcal{A}^{+,s\sigma}_\zeta} + \|\cdot\|^2_{g\text{-}\mathcal{A}^{-,s\sigma}_\zeta} + \|g(\Lambda)\Psi^0\zeta\cdot\|^2_{H^s}$$

is an intermediate norm between $\|g(\Lambda)\zeta_0\cdot\|^2_{H^s}$ and $\|g(\Lambda)\zeta\cdot\|^2_{H^s}$ for any $\zeta_0 \prec \sigma \prec \zeta$. Instead of working on the norms of H^s we will work on $\|\cdot\|^2_{g\text{-}\mathcal{A}^{\pm,s\sigma}_\zeta}$.

Theorem (K.2016)

Let M be a pseudoconvex, CR manifold of dimension $(2n+1)$ such that $\bar{\partial}_b$ has closed range in L^2 spaces for all degrees of forms. Suppose that the σ- and $\hat{\sigma}$-superlogarithmic property on $(0, q_0)$ forms holds for $supp(\sigma) \cap supp(\hat{\sigma}) = \emptyset$. Then

$$\mathcal{G}_q \in C^\infty_{0,q;q,0}\left(\left(V_0^\sigma \times V_0^{\hat{\sigma}}\right) \cup \left(V_0^{\hat{\sigma}} \times V_0^\sigma\right)\right) \qquad \text{for all } q_0 \leq q \leq n - q_0.$$

If $q_0 = 1$ and $n \geq 2$, or $q_0 = n = 1$ with the extra assumption that G_0 is globally regular, then this conclusion also holds for $q = 0$ and $q = n$.

THANK YOU