Thermodynamics with Pressure and Volume in Charged AdS Black Hole

Based on B. Gwak, JHEP **1711**, 129 (2017).

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Motivation

The first law of charged AdS black hole thermodynamics is commonly given as

$$dM = dU = TdS + \Phi \mathrm{dQ}(+\Omega dJ).$$

There is a conjecture adding pressure term (cosmological constant) to the first law. Then,

$$dM = d(U - PV) = dH = TdS + VdP + \Phi \mathrm{dQ}.$$

Further, the thermodynamic volume of the black hole is just its conjugate property.

By adding a particle, we want the variation of the black hole such as thermodynamic process.

Then, this constructs the black hole to a thermal engine.

We consider D-dimensional electrically charged black holes in the AdS spacetime.

Variables: (M, Q, Λ).

Under the charged particle absorption, an irreversible process, the black hole changes:

- The reconstruction of the first law of thermodynamics.
- Validity of the second law of thermodynamics: violation.
- A process corresponding to particle absorption: isobaric process.
- Cosmic censorship conjecture is tested: valid

Charged AdS Black Hole in D dimensions

The gravity action coupled with Maxwell field in D dimensional spacetimes:

$$S = -\frac{1}{16\pi}\int d^D x\sqrt{-g}\left(R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda\right). \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A = -\frac{Q}{r^{D-3}}dt.$$

where the Maxwell field A_μ and electric charge Q.

The metric of Charged AdS black hole in Einstein frame:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{D-2}, \quad f(r) = 1 - \frac{2M}{r^{D-3}} + \frac{Q^2}{r^{2D-6}} + \frac{r^2}{\ell^2},$$

The function $f(r) = 0 \rightarrow$ two solutions: outer and inner horizons.

where the mass M and electric charge Q. The $D-2$ sphere is given as

$$d\Omega_{D-2} = \sum_{i=1}^{D-2}\left(\prod_{j=1}^{i}\sin^2\theta_{j-1}\right)d\theta_i^2, \quad \theta_0 \equiv \frac{\pi}{2}, \quad \theta_{D-2} \equiv \phi.$$

Thermodynamics

The mass, electric charge, and cosmological constant:

$$M_{\mathrm{b}} = \frac{(D-2)\Omega_{D-2}}{8\pi}M, \quad Q_{\mathrm{b}} = \frac{(D-2)\Omega_{D-2}}{8\pi}Q, \quad \Lambda = -\frac{(D-1)(D-2)}{2\ell^2},$$

where the Maxwell field A_μ and electric charge Q.

Temperature, entropy, electric potential, and pressure:

$$T_{\mathrm{h}} = \frac{1}{2\pi\ell^2}\left(r_{\mathrm{h}} - \frac{(D-3)Q^2\ell^2}{r_{\mathrm{h}}^{2D-5}} + \frac{(D-3)M\ell^2}{r_{\mathrm{h}}^{D-2}}\right), \quad S_{\mathrm{h}} = \frac{A_{\mathrm{h}}}{4} = \frac{\Omega_{D-2}r_{\mathrm{h}}^{D-2}}{4}, \quad \Phi_{\mathrm{h}} = \frac{Q}{r_{\mathrm{h}}^{D-3}}. \quad P = -\frac{\Lambda}{8\pi} = \frac{(D-1)(D-2)}{16\pi\ell^2},$$

The first law of thermodynamics for enthalpy:

$$dM_{\mathrm{b}} = T_{\mathrm{h}}dS_{\mathrm{h}} + \Phi_{\mathrm{h}}dQ + V_{\mathrm{b}}dP, \quad M_{\mathrm{b}} = U_{\mathrm{b}} + PV_{\mathrm{b}}.$$

The volume (conjugate variable of the pressure):

$$V_{\mathrm{b}} = \frac{\Omega_{D-2}}{D-1}r_{\mathrm{h}}^{D-1}.$$

Charged Particle Absorption

B. Gwak, *"Thermodynamics with Pressure and Volume under Charged Particle Absorption,"* JHEP 1711, 129 (2017).

We consider charged particle collision to the black hole.

The particle is added to the horizon surface.

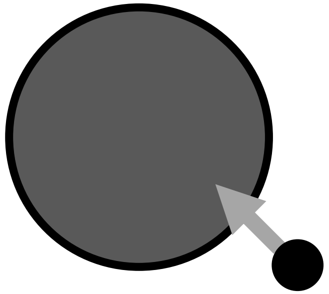
The black hole is assumed to be changed as much as the particle.

The absorption causes a change of volume of the black hole.

Then, the pressure of the system can be changed.

Thus, the cosmological constant can be changed.

Simple and analytical expression is possible without numerical method.



Charged Particle Equations of Motion

The particle will be absorbed to black hole when it pass through the black hole horizon.

We need a dispersion relation of the particle's energy at the outer horizon:

Hamiltonian and Hamilton-Jacobi action with a charged particle:

$$\mathcal{H} = \frac{1}{2}g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) \qquad S = \frac{1}{2}m^2\lambda - Et + L\phi + S_r(r) + S_\theta(\theta)\,,$$

The inverse metric is expressed as

$$g^{\mu\nu}\partial_\mu\partial_\nu = -f(r)^{-1}(\partial_t)^2 + f(r)(\partial_r)^2 + r^{-2}\sum_{i=1}^{D-2}\left(\prod_{j=1}^{i}\sin^{-2}\theta_{j-1}\right)(\partial_{\theta_i})^2.$$

The Hamilton-Jacobi equation:

$$-2\frac{\partial S}{\partial\lambda} = -m^2 = -\,f(r)^{-1}(-E - qA_t)^2 + f(r)(\partial_r S_r(r))^2 + r^{-2}\sum_{i=1}^{D-3}\left(\prod_{j=1}^{i}\sin^{-2}\theta_{j-1}\right)(\partial_{\theta_i}S_{\theta_i}(\theta_i))^2 + r^{-2}\left(\prod_{j=1}^{D-2}\sin^{-2}\theta_{j-1}\right)(L)^2,$$

The separate variables:

$$\mathcal{K} = -m^2r^2 + \frac{r^2}{f(r)}(-E + \frac{Qq}{r^{D-3}})^2 - r^2 f(r)(\partial_r S_r(r))^2, \quad R_i^2 = (\partial_i S_{\theta_i}(\theta_i))^2 + \sin^2\theta_i R_{i+1}^2, \quad \mathcal{K} = R_1^2, \quad L = R_{D-2}.$$

The Energy Relation

The equations of motion are

$$p^r \equiv \frac{\partial r}{\partial \lambda} = f(r)\sqrt{-\frac{\mathcal{K}+m^2r^2}{r^2 f(r)} + \frac{1}{f(r)^2}\left(E - \frac{Qq}{r^{D-3}}\right)^2}, \quad p^\theta \equiv \frac{\partial \theta}{\partial \lambda} = \frac{1}{r^2}\sqrt{\mathcal{K} - \sin^2\theta_1 R_2^2}.$$

In combination with these equations,

$$(p^r)^2 = -f(r)((r^2(p^\theta)^2 + \sin\theta_1^2 R_2^2) + m^2r^2) + \left(E - \frac{Qq}{r^{D-3}}\right)^2$$

At the outer horizon, the energy of the particle is given by combination of two equations:

$$E = \frac{Q}{r_{\mathrm{h}}^{D-3}} q + |p^r|,$$

We assume that the particle absorbs into the black hole when it passes through the horizon.

Then, the particle energy is the same to the variation of the internal energy of the black hole.

$$E = dU_{\mathrm{b}} = d(M_b - PV_{\mathrm{b}}), \quad q = dQ_{\mathrm{b}}.$$

The First Law of Thermodynamics

Then, the energy relation becomes a constraint to the change of the black hole:

$$dU_{\mathrm{b}} = \frac{Q}{r_{\mathrm{h}}^{D-3}} dQ_b + |p^r|. \quad U_{\mathrm{b}}(Q_{\mathrm{b}}, S_{\mathrm{b}}, V_{\mathrm{b}}),$$

The variation of the entropy is

$$dS_{\mathrm{h}} = \frac{1}{4}(D-2)\Omega_{D-2} r_{\mathrm{h}}^{D-3} dr_{\mathrm{h}},$$

Under the charged particle absorption at new horizon (combined with energy relation),

$$df_{\mathrm{h}} = \frac{\partial f_{\mathrm{h}}}{\partial M_{\mathrm{b}}} dM_{\mathrm{b}} + \frac{\partial f_{\mathrm{h}}}{\partial Q_{\mathrm{b}}} dQ_{\mathrm{b}} + \frac{\partial f_{\mathrm{h}}}{\partial \ell} d\ell + \frac{\partial f_{\mathrm{h}}}{\partial r_{\mathrm{h}}} dr_{\mathrm{h}} = 0, \quad f_{\mathrm{h}} = f(M_{\mathrm{b}}, Q_{\mathrm{b}}, \ell, r_{\mathrm{h}}), \longrightarrow dr_{\mathrm{h}} = \frac{16\pi r_{\mathrm{h}}^4 \ell^2 |p^r|}{\Omega_{D-2}(D-2)(D-3)(r_{\mathrm{h}}^{D+2} - 2M r_{\mathrm{h}}^3 \ell^2 + 2 r_{\mathrm{h}}^D \ell^2)}.$$

Then, the entropy and volume change

$$dS_{\mathrm{h}} = \frac{4\pi r_{\mathrm{h}}^{D+1} \ell^2 |p^r|}{(D-3)(r_{\mathrm{h}}^{D+2} - 2M r_{\mathrm{h}}^3 \ell^2 + 2 r_{\mathrm{h}}^D \ell^2)}, \quad dV_{\mathrm{h}} = \frac{16\pi r_{\mathrm{h}}^{D+1} \ell^2 |p^r|}{(D-2)(D-3)(r_{\mathrm{h}}^{D+2} - 2M r_{\mathrm{h}}^3 \ell^2 + 2 r_{\mathrm{h}}^D \ell^2)}.$$

The first law of thermodynamics is reconstructed as

$$dU_{\mathrm{h}} = \Phi_{\mathrm{h}} dQ_b + T_{\mathrm{h}} dS_{\mathrm{h}} - P dV_{\mathrm{h}}. \;\; dM_{\mathrm{b}} = T_{\mathrm{h}} dS_{\mathrm{h}} + \Phi_{\mathrm{h}} dQ + V_{\mathrm{h}} dP.$$

This result corresponds the result from (isobaric process)

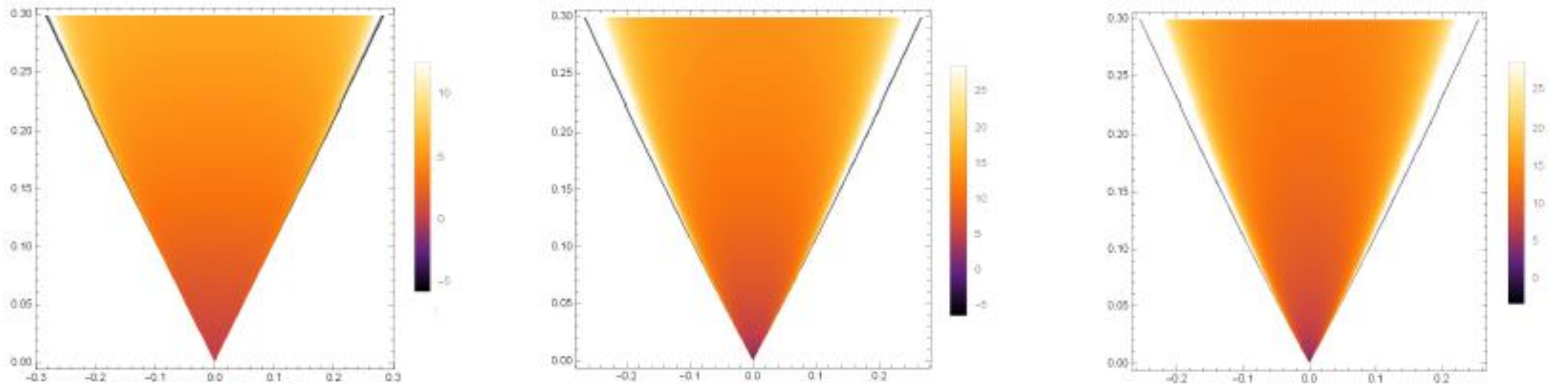
$$dP = 0$$

The Second Law of Thermodynamics

There is a specific example (extremal cases) for the violation of the second law:

$$dS_{\mathrm{h}} = \frac{4\pi r_{\mathrm{h}}^{D+1}\ell^2|p^r|}{(D-3)(r_{\mathrm{h}}^{D+2} - 2Mr_{\mathrm{h}}^3\ell^2 + 2r_{\mathrm{h}}^D\ell^2)}, \quad r_{\mathrm{h}}^{D+2} - 2Mr_{\mathrm{h}}^3\ell^2 + 2r_{\mathrm{h}}^D\ell^2 = -\frac{(D-1)r_{\mathrm{h}}^{D+2}}{(D-3)} < 0,$$

The diagrams in D-dimensional spacetime:



(a) $Q-M$ diagram for $D=4$. (b) $Q-M$ diagram for $D=5$. (c) $Q-M$ diagram for $D=6$.

Figure 1: The scaled dS_{h} in $Q-M$ diagrams with $\ell=1$.

More precisely,

$$S(\mathrm{BH{+}particle}) \to S(\mathrm{BH{+}\Delta BH}).$$

However, here, we consider

$$S(\mathrm{BH}) \to S(\mathrm{BH{+}\Delta BH})\colon S(\mathrm{BH}) > S(\mathrm{BH{+}\Delta BH}) \text{ for an extremal black hole.}$$

Generally, $S(\mathrm{BH})< S(\mathrm{BH{+}particle})$, so

$$S(\mathrm{BH{+}particle}) \ > S(\mathrm{BH}) > S(\mathrm{BH{+}\Delta BH}).$$

Then, we can conclude that the entropy decreases for an extremal case.

Investigation to Cosmic Censorship

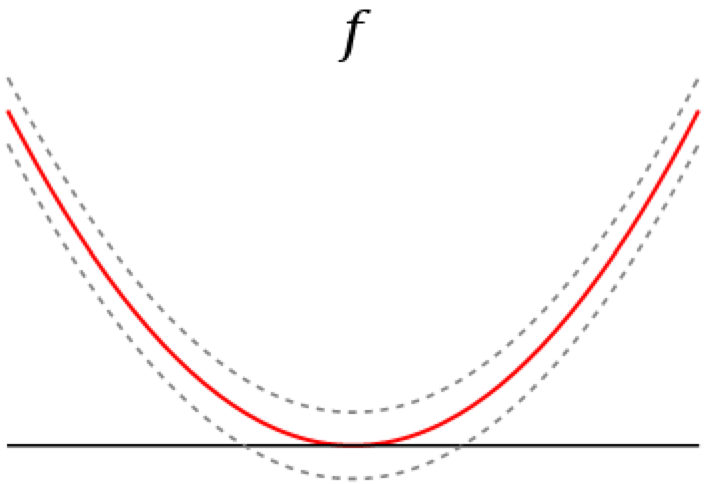
We will investigate whether the extremal black hole can be overcharged through particle absorption.

If the horizon disappears, the black hole becomes a naked singularity, and cosmic censorship is invalid.

The black hole horizon is determined by the function f.

The extremal black hole is changed as

$$(M_{\rm b}, Q_{\rm b}, \ell) \text{ into } (M_{\rm b} + dM_{\rm b}, Q_{\rm b} + dQ_{\rm b}, \ell + d\ell).$$



For Extremal and Near-extremal Cases

The extremal condition (and near-extremal condition, $|\delta| \ll 1$):

$$f(M_{\rm b}, Q_{\rm b}, \ell, r)|_{r=r_{\rm min}} \equiv f_{\rm min} = \delta \leq 0, \quad \partial_r f(M_{\rm b}, Q_{\rm b}, \ell, r)|_{r=r_{\rm min}} \equiv f'_{\rm min} = 0, \quad (\partial_r)^2 f(M_{\rm b}, Q_{\rm b}, \ell, r)|_{r=r_{\rm min}} > 0.$$

The change of the function f:

$$\begin{aligned} f(M_b + dM_b, Q_b + dQ_b, \ell + d\ell, r)|_{r=r_{\rm min}+dr_{\rm min}} &= f_{\rm min} + df_{\rm min} \\ &= \delta + \left(\frac{\partial f_{\rm min}}{\partial M_b} dM_b + \frac{\partial f_{\rm min}}{\partial Q_b} dQ_b + \frac{\partial f_{\rm min}}{\partial \ell} d\ell\right), \end{aligned}$$

The minimum point is located at:

$$\partial_r f(M_{\rm b} + dM_{\rm b}, Q_{\rm b} + dQ_b, \ell + d\ell, r)|_{r=r_{\rm min}+dr_{\rm min}} = f'_{\rm min} + df'_{\rm min} = 0,$$

$$df'_{\rm min} = \frac{\partial f'_{\rm min}}{\partial M_{\rm b}} dM_{\rm b} + \frac{\partial f'_{\rm min}}{\partial Q_{\rm b}} dQ_{\rm b} + \frac{\partial f'_{\rm min}}{\partial \ell} d\ell + \frac{\partial f'_{\rm min}}{\partial r_{\rm min}} dr_{\rm min} = 0,$$

For the near-extremal black holes:

$$\delta \to \delta_\epsilon, \quad r_{\rm h} \to r_{\rm min} + \epsilon.$$

The moved minimum value is

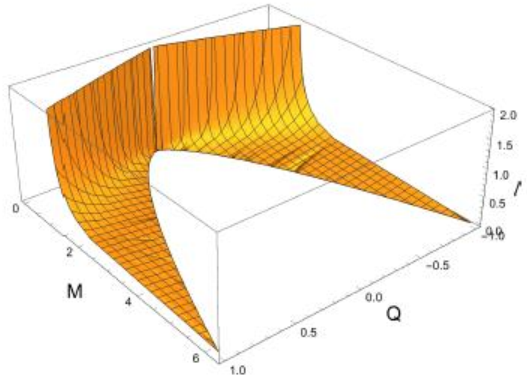
$$f_{\rm min} + df_{\rm min} = \left(\delta_\epsilon + \frac{32\pi r_{\rm min}^5(-1-(D-2)r_{\rm min}^{1-2D}(-Q^2 r_{\rm min}^3 + M r_{\rm min}^D)\ell^2)|p^r|}{\Omega_{D-2}(D-3)(D-2)(r_{\rm min}^{D+2} - 2M r_{\rm min}^3 \ell^2 + 2r_{\rm min}^D \ell^2)}\right) + \mathcal{O}(\epsilon),$$

Then, the near-extremal black hole:

$$f_{\rm min} + df_{\rm min} = \delta_\epsilon + \mathcal{O}(\epsilon^2).$$

The extremal black hole:

$$f_{\rm min} + df_{\rm min} = 0, \quad \delta_\epsilon = 0, \quad \epsilon = 0.$$



(a) (Q, M, ℓ) surface satisfying $\delta = 0$.

Extremal BH → Extremal BH, Near-Extremal BH→Near-Extremal BH.

Thus, the weak cosmic censorship conjecture is still valid.

Summary

We consider the charged particle absorption in D-dimensional charged AdS black holes.

There is a violation of the second law of thermodynamics.

However, the weak cosmic censorship conjecture is still valid.

In addition, the configurations of the extremal and near-extremal black hole are not much charged under the absorption.

THANK YOU!