Einstein Double Field Equations

Stephen Angus
Ewha Woman's University
based on arXiv:1804.00964 in collaboration with
Kyoungho Cho and Jeong-Hyuck Park (Sogang Univ.)

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Outline

Introduction

General Relativity is a successful theory of gravity.

- Equivalence Principle: gravity = acceleration; at every spacetime point, $\exists$ local inertial frame in which laws of Physics are invariant.
- Mathematically: spacetime is a Riemannian manifold, endowed with a dynamical metric, $g_{\mu\nu}$, and associated covariant derivative

  $$\nabla_\mu = \partial_\mu + \gamma_\mu + \omega_\mu \ ,$$

  where
  - Christoffel symbols $\gamma_\mu \Rightarrow$ diffeomorphism invariance,
  - spin connection $\omega_\mu \Rightarrow$ local Lorentz invariance.
- Geometry $\Leftrightarrow$ Matter; expressed via Einstein's equations

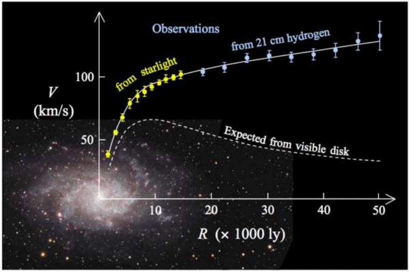
  $$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ .$$

Motivation: dark matter

- GR accurately describes astrophysical/cosmological phenomena: perihelion precession, gravitational lensing, Big Bang cosmology...
- However, some results cannot be explained by GR + visible matter alone, eg. galaxy rotation curves.
- Kepler/Newton/GR: orbital velocity

$$V^2 = \frac{GM}{R} ,$$

does not match observations.



Broadly, two classes of solutions:

1. GR is correct, but there is additional invisible dark matter present.
2. Theory of gravity should be modified for appropriate R/MG.

Motivation: string theory

In GR, the metric $g_{\mu\nu}$ is the only gravitational field.
In string theory, the closed-string massless sector always includes:

- the metric, $g_{\mu\nu}$;
- an antisymmetric 2-form potential, $B_{\mu\nu}$;
- the dilaton, ϕ.

Furthermore, these fields transform into each other under T-duality.

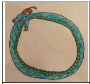
Natural stringy extension of General Relativity:

Consider $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ as the fundamental gravitational multiplet.

This is the idea of Stringy Gravity, which can be realized using the mathematical formalism of Double Field Theory (DFT).

Why Stringy Gravity?

- In Stringy Gravity, the additional degrees of freedom $B_{\mu\nu}$ and ϕ augment gravity beyond GR, allowing new types of solutions.
- E.g. $D = 4$, spherical, static case: Stringy Gravity has 4 free parameters (c.f. 1 parameter in GR, the Schwarzschild mass).
- Gravity is modified at "short" distances (Ko, Park, Suh; 2017); best expressed in terms of the dimensionless variable R/MG.
- Anomalous behavior of large astrophysical objects corresponds to this parameter range, as very large $M \Rightarrow$ small $R/MG \lesssim 10^7$.

	Electron ($R \simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1\mathrm{AU}/M_\odot G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M \propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim 10^{5}$	0^+

'Uroboros' spectrum of the dimensionless Radial variable normalized by Mass in natural units.
The orbital speed of rotation curves is also dimensionless, and depends on the single variable, $R/(MG)$.

A brief introduction to Double Field Theory

Let us review some key features of Double Field Theory, and its application to Stringy Gravity (see Kanghoon's talk for more details).

- In Double Field Theory we describe D-dimensional physics using twice as many coordinates: $x^A = (\tilde{x}_\mu, x^\nu)$, $A = 1, \ldots, D+D$.
- $\exists$ an $\mathbf{O}(D, D)$ T-duality gauge symmetry; doubled vector indices are raised and lowered using the $\mathbf{O}(D, D)$-invariant metric:
$$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
- $\exists$ doubled diffeomorphisms acting on vectors ξ^A, etc.
- $\exists$ twofold local Lorentz symmetry: $\mathbf{Spin}(1, D{-}1) \times \mathbf{Spin}(D{-}1, 1)$, with local metrics $\eta_{pq} = \mathrm{diag}(-++\cdots+)$, $\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+--\cdots-)$.

This formalism was originally motivated in $D = 10$ to develop a unified description of manifest T-duality in string theory (Hull, Zwiebach; 2009). However, other choices are also possible: here I will focus on $D = 4$.

Section condition

- The doubled coordinates satisfy a gauge equivalence relation

$$x^A \sim x^A + \Delta^A(x) \ ,$$

where Δ^A is a derivative-index-valued $\mathbf{O}(D,D)$ vector; for example $\Delta^A(x) = \mathcal{J}^{AB}\partial_B\Phi(x)$ for some function $\Phi(x)$, where $\partial_A = (\tilde{\partial}^\mu, \partial_\nu)$.

- All fields and functions in DFT should thus be gauge invariant,

$$\Phi(x+\Delta) = \Phi(x) \quad \Longleftrightarrow \quad \Delta^A\partial_A = 0 \ .$$

- This is equivalent to the section condition: $\partial_A\partial^A = 2\,\partial_\mu\tilde{\partial}^\mu = 0$.
- Natural choice: $\tilde{\partial}^\nu = 0$. Thus the D coordinates $\{\tilde{x}_\mu\}$ are gauged, and their gauge orbits correspond to points in the resulting D-dimensional spacetime which is spanned by $\{x^\nu\}$.

Basic ingredients of Double Field Theory

- The basic fields of Double Field Theory are: $\{d, \mathcal{H}_{AB}\}$, the DFT dilaton and the symmetric $\mathbf{O}(D, D)$ metric, respectively.
- After imposing the section condition, these reduce to the closed string massless sector $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$, e.g. $e^{-2d} \simeq e^{-2\phi}\sqrt{-g}$.
- Define projectors, $P_{AB} = \frac{1}{2}(\mathcal{J}_{AB} + \mathcal{H}_{AB}), \bar{P}_{AB} = \frac{1}{2}(\mathcal{J}_{AB} - \mathcal{H}_{AB})$, which satisfy relations like $P^2 = P$, $\bar{P}^2 = \bar{P}$, $P\bar{P} = 0$, $P + \bar{P} = \mathcal{J}$.
- Furthermore, it is sometimes useful to express these further in terms of a pair of DFT vielbeins $\{V_{Ap}, \bar{V}_{A\bar{p}}\}$, satisfying

$$P_A{}^B = V_{Ap}V^{Bp}\,, \qquad \bar{P}_A{}^B = \bar{V}_{A\bar{p}}\bar{V}^{B\bar{p}}\,.$$

- Semi-covariant derivative ∇_A and master derivative $\mathcal{D}_A$:

$$\nabla_A = \partial_A + \Gamma_A\,, \qquad \mathcal{D}_A = \nabla_A + \Phi_A + \bar{\Phi}_A\,,$$

where the Γ_{ABC} are DFT Christoffel symbols, while Φ_{Apq} and $\bar{\Phi}_{A\bar{p}\bar{q}}$ are spin connections for the two local Lorentz groups.

Curvature

- $\nexists$ normal coordinates where $\Gamma_{CAB} = 0 \Rightarrow$ no equivalence principle! ($B_{\mu\nu}$ sources string; EP does not apply to extended objects.)
- Semi-covariant Riemann curvature (Jeon, Lee, Park; 2011),

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) .$$

- Construct fully covariant objects using projectors P and $\bar{P}$ (equivalently V and $\bar{V}$), e.g. DFT Ricci tensor and scalar,

$$S_{p\bar{q}} := V^{A}{}_{p}\bar{V}^{B}{}_{\bar{q}}S^{C}{}_{ACB} , \qquad S_{(0)} := \left(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{CD}\right)S_{ABCD} .$$

- Upon Riemannian backgrounds ($\tilde{\partial}^{\mu} = 0$), reduces to e.g.

$$S_{(0)} = R + 4\Box\phi - 4\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} .$$

This gives the spacetime Lagrangian for Stringy Gravity.

Stringy energy-momentum tensor

We will now consider DFT as Stringy Gravity coupled to matter.
The action is

$$\int_{\Sigma} e^{-2d}\left[\,\frac{1}{16\pi G}S_{(0)} + L_{\mathrm{matter}}(\Upsilon_a)\,\right] ,$$

where the integral is performed over a D-dimensional section Σ.

Note: $\mathbf{O}(D,D)$ gauge symmetry $\Rightarrow$ proper distance, geodesic motion, etc. have a natural covariant definition in string (Jordan) frame.

The resulting equations of motion are

$$S_{p\bar{q}} = 8\pi G K_{p\bar{q}} , \qquad S_{(0)} = 8\pi G T_{(0)} , \qquad \frac{\delta L_{\mathrm{matter}}}{\delta \Upsilon_a} \equiv 0 ,$$

where the stringy energy-momentum tensor has components

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\mathrm{matter}}}{\delta \bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\mathrm{matter}}}{\delta V_A{}^{p}} \right) , \quad T_{(0)} := e^{2d} \times \frac{\delta\left(e^{-2d}L_{\mathrm{matter}}\right)}{\delta d} .$$

Einstein Double Field Equations

Analogously to GR, we can define the stringy Einstein curvature tensor which is covariantly conserved,

$$G_{AB} = 4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)} \,, \qquad \mathcal{D}_A G^{AB} = 0 \qquad \text{(off-shell)} \,.$$

This implies that the energy-momentum tensor can be written similarly,

$$T_{AB} := 4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)} \,, \qquad \mathcal{D}_A T^{AB} \equiv 0 \qquad \text{(on-shell)} \,.$$

Hence the Einstein Double Field Equations can be summarized as

$$G_{AB} = 8\pi G T_{AB} \,.$$

Note: unlike in GR, the DFT Ricci tensor is traceless $\Rightarrow$ the $S_{(0)} \propto T_{(0)}$ part is an essential and independent component of the equations.

Riemannian backgrounds

Riemannian backgrounds, section condition $\tilde{\partial}^{\mu} = 0$:

- Einstein Double Field Equations reduce to the usual closed-string equations, plus source terms from $K_{\mu\nu} = 2e_{\mu}{}^{p}\bar{e}_{\nu}{}^{q}K_{p\bar{q}}$ and $T_{(0)}$:

$$\begin{array}{rcl} R_{\mu\nu} + 2\bigtriangledown_{\mu}(\partial_{\nu}\phi) - \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} & = & 8\pi G K_{(\mu\nu)}\ ; \\ \bigtriangledown^{\rho}\left(e^{-2\phi}H_{\rho\mu\nu}\right) & = & 16\pi G e^{-2\phi}K_{[\mu\nu]}\ ; \\ R + 4\Box\phi - 4\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} & = & 8\pi G T_{(0)}\ . \end{array}$$

- Asymmetric $K_{\mu\nu}$ possible (e.g. fermions, strings) $\rightarrow$ source for H.
- In addition, the conservation laws reduce to

$$\begin{array}{r} \nabla^{\mu}K_{(\mu\nu)} - 2\partial^{\mu}\phi\, K_{(\mu\nu)} + \frac{1}{2}H_{\nu}{}^{\lambda\mu}K_{[\lambda\mu]} - \frac{1}{2}\partial_{\nu}T_{(0)} \equiv 0\ , \\ \nabla^{\mu}\left(e^{-2\phi}K_{[\mu\nu]}\right) \equiv 0\ , \end{array}$$

which are automatically satisfied on-shell.

Spherical symmetry

Finally, we wish to look for solutions to these equations.
Basic example: spherically symmetric, asymptotically flat, static, Riemannian, regular solutions in $D = 4$.

- Spherical symmetry $\Rightarrow$ isometries parametrized by three DFT-Killing vectors, $\xi_a^A = (\tilde{\xi}_{a\mu}, \xi_a^\nu)$, $a = 1, 2, 3$, where

$$\begin{aligned} \tilde{\xi}_1 &= \frac{\cos\varphi}{\sin\vartheta}\left[h\mathrm{d}t + B(r)\mathrm{d}r\right] , & \xi_1 &= \sin\varphi\partial_\vartheta + \cot\vartheta\cos\varphi\partial_\varphi \ , \\ \tilde{\xi}_2 &= \frac{\sin\varphi}{\sin\vartheta}\left[h\mathrm{d}t + B(r)\mathrm{d}r\right] , & \xi_2 &= -\cos\varphi\partial_\vartheta + \cot\vartheta\sin\varphi\partial_\varphi \ , \\ \tilde{\xi}_3 &= 0 \ , & \xi_3 &= -\partial_\varphi \ . \end{aligned}$$

- These form an $\mathbf{so}(3)$ algebra under the so-called C-bracket, i.e.

$$[\xi_a, \xi_b]_{\mathbf{C}} = \sum_c \epsilon_{abc}\xi_c \ .$$

Spherical, static field ansatz

- On Riemannian backgrounds, solving the DFT-Killing equations for for the above **so**(3) algebra plus $\partial_t \equiv 0$ gives a spherical, static ansatz for $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ (we can set $g_{tr} \equiv 0$ by diffeomorphisms):

$$\mathrm{d}s^2 = e^{2\phi(r)}\left[-A(r)\mathrm{d}t^2 + A^{-1}(r)\mathrm{d}r^2 + A^{-1}(r)C(r)\,\mathrm{d}\Omega^2\right];$$

$$B_{(2)} = B(r)\cos\vartheta\,\mathrm{d}r \wedge \mathrm{d}\varphi + h\cos\vartheta\,\mathrm{d}t \wedge \mathrm{d}\varphi\,; \qquad \phi = \phi(r)\,,$$

  where $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2$ and $B_{(2)} = \frac{1}{2}B_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.
- Solution determined by $A(r)$, $B(r)$, $C(r)$, $\phi(r)$, and parameter h.
- Note: physical distance determined by “areal radius", R:

$$e^{-2d} = e^{-2\phi}\sqrt{-g} = R^2\sin\vartheta\,; \qquad R := e^{\phi}\sqrt{C/A}\,.$$

We will now plug this ansatz into the Einstein Double Field Equations.

Solving the Einstein Double Field Equations

Consider solutions with matter only up to a cutoff radius, r_c. Solution for $r \geq r_\mathrm{c}$ is thus the known DFT vacuum solution (Ko, Park, Suh; 2017):

$$C(r) = (r-\alpha)(r+\beta) \ ; \ A(r) = \left(\frac{r-\alpha}{r+\beta}\right)^{\frac{a}{\sqrt{a^2+b^2}}} \ ; \ B_{(2)} = h\cos\vartheta\,\mathrm{d}t\wedge\mathrm{d}\varphi \ ;$$

$$e^{2\phi} = \gamma_+\left(\frac{r-\alpha}{r+\beta}\right)^{\frac{b}{\sqrt{a^2+b^2}}} + \gamma_-\left(\frac{r+\beta}{r-\alpha}\right)^{\frac{b}{\sqrt{a^2+b^2}}} \ , \ \gamma_\pm := \frac{1}{2}\left(1 \pm \sqrt{1-\frac{h^2}{b^2}}\right) \ ,$$

where $b^2 := (\alpha+\beta)^2 - a^2$. (Note: for real solutions, require $b^2 \geq h^2$.)

- Boundary cond's: flat spacetime at $r \to \infty$; regular sol'n at $r = 0$.
- New feature: matching inner ($r < r_\mathrm{c}$) and outer ($r \geq r_\mathrm{c}$) region gives $\{\alpha, a, b, h\}$ in terms of the interior energy-momentum tensor.
- Note: for consistency with flat spacetime at $r \to \infty$, require $B(r) = 0$ for $r \geq r_\mathrm{c}$: i.e. only "electric" H-flux, $h = H_{t\vartheta\varphi}$, for $r \geq r_\mathrm{c}$.
- At $r = \alpha$, $A = C = R = 0$ and $\phi \to \pm\infty \Rightarrow$ require $r_\mathrm{c} > \alpha$.

Energy conditions

For reasonable physics, we should assume some energy conditions:

- the weak energy condition with electric H-flux,

$$\int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \, e^{-2d} \left(-K_t{}^t + \frac{1}{16\pi G} \left| H_{t\vartheta\varphi} H^{t\vartheta\varphi} \right| \right) \geq 0 \, ;$$

- the strong energy condition, with magnetic H-flux,

$$\int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \, e^{-2d} \left(K_i{}^i - K_t{}^t - T_{(0)} + \frac{1}{8\pi G} \left| H_{r\vartheta\varphi} H^{r\vartheta\varphi} \right| \right) \geq 0 \, ;$$

- the pressure condition, with magnetic H-flux and no integration,

$$K_r{}^r + K_\vartheta{}^\vartheta - T_{(0)} + \frac{1}{16\pi G} \left| H_{r\vartheta\varphi} H^{r\vartheta\varphi} \right| \geq 0 \, .$$

If strong energy and pressure conditions are satisfied, then $\alpha, \beta, a \geq 0$.

Effective mass

- Define an effective mass $M(r)$ via the centripetal acceleration measured in Newtonian gravity with the areal radius R. From the radial geodesic equation,

$$\frac{GM(r)}{R^2} \equiv R\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 .$$

- When $r \to \infty$,

$$\begin{aligned} M_\infty &\equiv \lim_{r\to\infty} M(r) = \frac{a + b\sqrt{1 - h^2/b^2}}{2G} \\ &= \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, e^{-2d}\left(\frac{1}{8\pi G}\left|H_{t\vartheta\varphi}H^{t\vartheta\varphi}\right| - 2K_t{}^t\right) . \end{aligned}$$

- Thus $M_\infty \geq 0 \Leftrightarrow$ weak energy condition satisfied.
- Note that b can take either sign $\Rightarrow b < 0$ may be problematic.

Small-radius behavior

- The effective mass $M(r)$ appears in the expansion of g_{tt} and corresponds to the gravitational "force".
- In particular, if $M(r) = 0$ for some finite $r = r_M \geq r_{\mathrm{c}}$, then for $r < r_M$ the gravitational force may become repulsive!
- On the other hand, $M(r) \propto \Omega(r)^{-1}$, where

$$\Omega(r) := r - \alpha + \tfrac{1}{2}\sqrt{a^2 + b^2} - \tfrac{1}{2}a \pm \tfrac{1}{2}|b|\sqrt{1 - e^{-4\phi}\, h^2/b^2} \sim \frac{\mathrm{d}R}{\mathrm{d}r} \,.$$

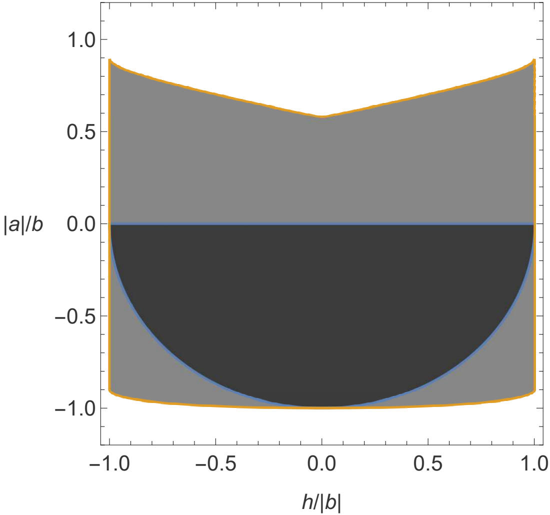
  For sufficiently small $r > \alpha$, the sign of the last term is always negative $\Rightarrow \exists\ r_\Omega > \alpha$ such that $\Omega(r_\Omega) = 0$ and $M(r)$ diverges.

The small-r (and thus small-R) behavior depends on the relative size of r_M and r_Ω, which in turn depends on the parameters $\{a, b, h\}$:

1. $r_M > r_\Omega \Rightarrow$ gravity repulsive for $r_\Omega \leq r < r_M$, repulsive wall at $r = r_\Omega$;
2. $r_M \leq r_\Omega$ or $\nexists \Rightarrow M(r) \geq 0$ for $r > r_\Omega$, attractive singularity at $r = r_\Omega$.

Constraints on parameters

- Parameter space for $h/\,|b|$ and $|a|\,/b$.
- In the gray region $r_M > r_\Omega$, whereas this is not satisfied in the outer white region.
- The black region corresponds to violation of the weak energy condition ($M_\infty < 0$) $\Rightarrow$ excluded.
- For physical solutions, expect $r_c > r_\Omega$.



Summary

- Stringy Gravity considers the closed string massless sector $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ to be the fundamental gravitational multiplet $\Rightarrow (D^2+1)$ degrees of freedom, thus richer spectrum.
- We studied Double Field Theory as Stringy Gravity in the presence of matter. Imposing on-shell energy-momentum conservation gives the Einstein Double Field Equations,

$$G_{AB} = 8\pi G T_{AB} \ .$$

- For spherically symmetric regular solutions in $D=4$, at small radii and for certain matter distributions, the gravitational force can become repulsive. Applications to modified gravity?

Possible future directions: non-Riemannian spacetimes; applications to cosmology; stringy thermodynamics; tests against observations.