The Simplest Massive S-matrix

Yu-tin Huang (NTU)

In collaboration with
Jong-Wook Kim (SNU) Sangmin Lee (SNU)
Ming-Zhi Chung (NTU)

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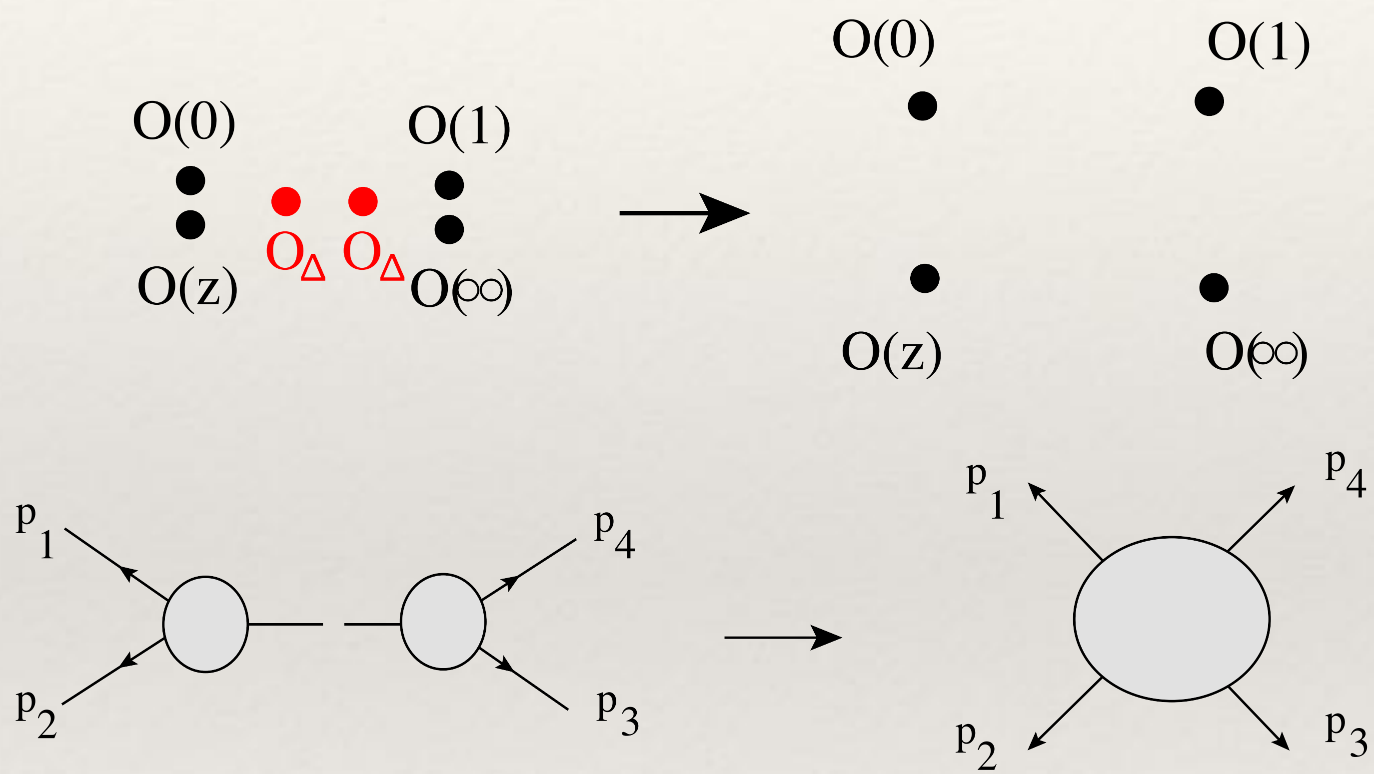
The massively rich real world

- On-shell observables such as S-matrix, correlation functions are derivable from the simplest interactions,



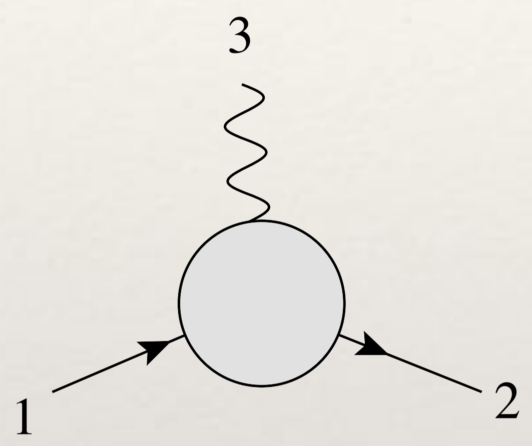
Start with the fundamental interaction Physical principles (Unitarity, Locality, Symmetry) fixes the answer.

For massless particles or CFTs, this is completely fixed by symmetries

The massively rich real world

Start with the fundamental interaction physical principles (Unitarity, Locality, Symmetry) fixes the answer.

For massive particles, the fundamental interactions are not fixed



→ 2s+1 distinct interactions

- What physics can we learn from employing consistency conditions on these interactions ?
- Is there a notion of simplest massive amplitude ? (Similar question for massless amplitudes led to dual conformal symmetry of N=4 SYM, and maximal SUGRA)

The massively rich real world

Consider an amplitude for massive states. Since it is a scalar function that carries the quantum number of the physical state (Little group)

$$\langle in\ t \to +\infty | out\ t \to -\infty \rangle \to M_n^{\{I_1 I_2 \cdots I_{2s_1}\},\{J_1 J_2 \cdots J_{2s_2}\}\cdots}$$

I =1,2 are doublets of SU(2) Little group.

We introduce $$p^{\alpha\dot{\alpha}} = \lambda^{\alpha I}\tilde{\lambda}_I^{\dot{\alpha}}$$

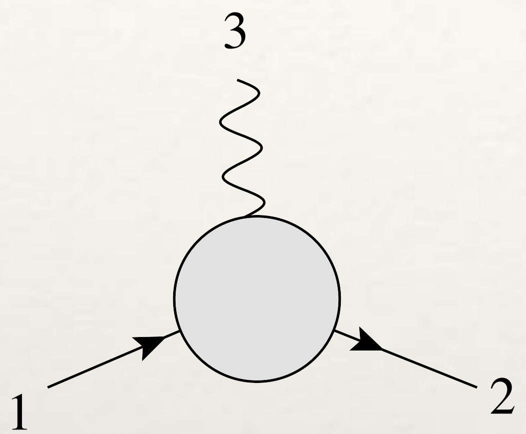
In doing so we simply have:

$$M_n^{\{I_1 I_2 \cdots I_{2s_1}\},\cdots} = \lambda_1^{I_1\alpha_1}\lambda_1^{I_2\alpha_2}\cdots\lambda_1^{I_{2s_1}\alpha_{2s_1}} M_{n,\ \{\alpha_1\alpha_2\cdots\alpha_{2s_1}\}\cdots}$$

- The decomposition is unambiguous
- The problem reduces to finding the Irreps of SL(2,C)

The massively rich real world

Consider the three point amplitude with one massless and two equal mass



$$M^h{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}}$$

We need two vectors to span the SL(2,C) space:

$$(u_\alpha, v_\alpha) = (\lambda_{3,\alpha}, \epsilon_{\alpha\beta}\lambda_3^\beta)$$

We also need to have variables that carry the opposite helicity weight of the massless leg

m1=m2 $\longrightarrow$ $2p_2 \cdot p_3 = \langle 3|p_2|3] = 0$ $\longrightarrow$ $x\lambda_3^\alpha = \frac{p_2^{\alpha\dot\alpha}\tilde\lambda_{3\dot\alpha}}{m}$

This allows us to define the x factor which carries positive helicity

Three point amplitude is constructed from (x, λ, ε)

The massively rich real world

Consider the three point amplitude with one massless and two equal mass

$$
\begin{aligned}
s=\frac{1}{2}: &\qquad x\left(\epsilon_{\alpha\beta}+g_1\, x\frac{\lambda_\alpha\lambda_\beta}{m}\right)\\
s=1: &\qquad x\left(\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}+g_1\epsilon_{\alpha_2\beta_2}\, x\frac{\lambda_{\alpha_1}\lambda_{\beta_1}}{m}+g_2\, x^2\frac{\lambda_{\alpha_1}\lambda_{\beta_1}\lambda_{\alpha_2}\lambda_{\beta_2}}{m^2}\right)\\
&\qquad\vdots\\
s=2: &\qquad x\left(\epsilon^4+g_1\epsilon^3 x\frac{\lambda^2}{m}+g_2\,\epsilon^2x^2\frac{\lambda^4}{m^2}+g_3\,\epsilon\frac{\lambda^6}{m^3}+g_4\frac{\lambda^8}{m^3}\right)\\
&\qquad\vdots
\end{aligned}
$$

Putting back the massive spinors

$$
M_3^h=g_0x^h\frac{\langle\mathbf{21}\rangle^{2s}}{m^{2s-1}}+g_1x^{h+1}\frac{\langle\mathbf{21}\rangle^{2s-1}\langle\mathbf{2}3\rangle\langle3\mathbf{1}\rangle}{m^{2s}}+\cdots+g_{2s}x^{h+2s}\frac{\langle\mathbf{2}3\rangle^{2s}\langle3\mathbf{1}\rangle^{2s}}{m^{4s-1}}
$$

For exp

$$
\mathcal{M}_3^{\rm QED}=-ie\,\varepsilon_\mu^+(k_3)\bar{v}(p_2)\gamma^\mu u(p_1)=-i\sqrt{2}e\frac{[\mathbf{2}3]\langle\zeta\mathbf{1}\rangle+\langle\mathbf{2}\zeta\rangle[3\mathbf{1}]}{\langle3\zeta\rangle}
$$

$$
=i\sqrt{2}ex\frac{\langle3\zeta\rangle\langle\mathbf{21}\rangle}{\langle3\zeta\rangle}=i\sqrt{2}ex\langle\mathbf{21}\rangle
$$

The x-factor

Let's first consider the case where all gi=0 for i>0

- What characterizes it in the UV ?

In the UV we approach the massless limit m->0

$$M_3^h = g_0 x^h \frac{\langle \mathbf{21} \rangle^{2s}}{m^{2s-1}} + g_1 x^{h+1} \frac{\langle \mathbf{21} \rangle^{2s-1} \langle \mathbf{23} \rangle \langle \mathbf{31} \rangle}{m^{2s}} + \cdots + g_{2s} x^{h+2s} \frac{\langle \mathbf{23} \rangle^{2s} \langle \mathbf{31} \rangle^{2s}}{m^{4s-1}}$$

All gi have more singular for m->0, bad UV behaviours. Indeed consider

$$S = \frac{1}{M_{pl}} \int \sqrt{-g} \frac{(-1)^s}{2} \left(D^\mu \phi^{\nu_1 \cdots \nu_s} D_\mu \phi_{\nu_1 \cdots \nu_s} - m^2 \phi^{\nu_1 \cdots \nu_s} \phi_{\nu_1 \cdots \nu_s} \right)$$

The three-pt amp

$$x^2 \frac{m^2}{M_{pl}} \left(\frac{\langle \mathbf{21} \rangle}{m^2} \right)^{2s} - \frac{s(s-1)}{2} \frac{1}{M_{pl} m^2} \left(\frac{\langle \mathbf{21} \rangle}{m} \right)^{2s-2} \langle \mathbf{23} \rangle^2 \langle \mathbf{31} \rangle^2 + \cdots$$

contains bad H.E. behaviors, with the leading term removable by

$$h \frac{s(s-1)}{2} \phi^{\mu\rho\mu_3 \cdots \mu_s} R_{\mu\nu\rho\sigma} \phi^{\nu\sigma}{}_{\mu_3 \cdots \mu_s}$$

with h=-1 (Ioannis Giannakis, James T. Liu, Massimo Porrati)

The x-factor

Let's first consider the case where all g=0

- What characterizes it in the UV ?

In the UV we approach the massless limit:

$$\lambda^I_\alpha = \lambda_\alpha \xi^{-I} + \eta_\alpha \xi^{+I}\,, \quad \tilde{\lambda}^I_{\dot{\alpha}} = \tilde{\eta}_{\dot{\alpha}} \xi^{-I} + \tilde{\lambda}_{\dot{\alpha}} \xi^{+I}$$

where <η λ>= [η λ] =m and

$$(mx)^h \left(\frac{\langle \mathbf{12} \rangle}{m}\right)^s \Bigg|_{m\to 0} = \left(\frac{[3|p_1|\xi\rangle}{\langle 3\xi\rangle}\right)^h \left(\frac{\langle \eta_1 2\rangle}{m}\right)^s \Bigg|_{m\to 0} = \left(\frac{[23][31]}{[12]}\right)^h \left(\frac{[13]}{[23]}\right)^s$$

This is the amplitude for minimal coupling between a positive h helicity state with +s and -s helicity.

The x-factor

Let's first consider the case where all gi=0

- What characterizes it in the IR ?

In the IR we can consider the magnetic moment. Lets start non-relativistically

$$V_Z := -\vec{\mu}\cdot\vec{B} = -\frac{ge}{m}\vec{S}\cdot\vec{B}\,.$$

Taking the relativistic form

$$V_Z = -\frac{ge}{m}S_\mu B^\mu = -\frac{ge}{4m^3}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\tau\eta\chi}p_1^\nu p_{1\tau}J^{\rho\sigma}F_{\eta\chi} \\ = \frac{ge}{2m}J^{\mu\nu}F_{\mu\nu} + \frac{ge}{m^3}p_1^\tau F_{\tau\eta}J^{\eta\chi}p_{1\chi}\,,$$

$$\longrightarrow \quad \left(V^+_{Z,2s}\right) = s\frac{ge}{m}[\mathbf{13}][\mathbf{32}]\left(\frac{[\mathbf{12}]}{m}\right)^{s-1}$$

Let's convert the minimal coupling

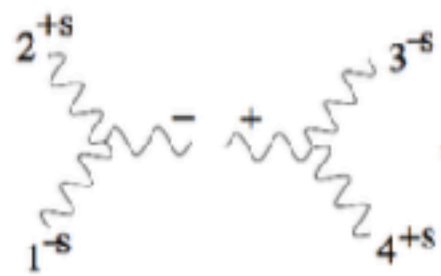
$$\bar{M}_3^{+1} = emx\frac{[\mathbf{21}]^{2s}}{m^{2s}} + 2s\frac{[\mathbf{21}]^{2s-1}[\mathbf{23}][\mathbf{31}]}{m^{2s}} + \cdots\,.$$

we see that g=2! In other words good H.E. behavior is tied to the classical magnetic moment being 2. Indeed this is the case for W bosons.

The x-factor - under factorization

Let us now impose factorization constraints

We expect consistent factorization to impose non-trivial constraint on the three-point



$$, \quad R_s = \left(\frac{\langle 1I\rangle^3}{\langle 12\rangle\langle I2\rangle}\right)^s \left(\frac{[I4]^3}{[I3][43]}\right)^s = \left(\frac{\langle 13\rangle^2[24]^2}{u}\right)^s$$

Start with an ansatz

$$\langle 13\rangle^2[24]^2\left(\frac{A^{a_1a_2a_3a_4}}{st}+\frac{B^{a_1a_2a_3a_4}}{tu}+\frac{C^{a_1a_2a_3a_4}}{us}\right)$$

The constraint of matching to all channels

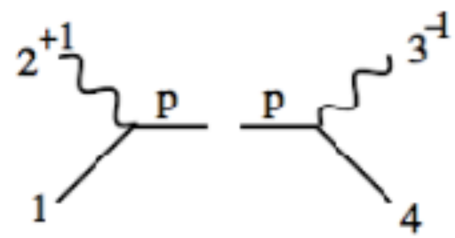
$$\begin{aligned} C^{a_1a_2a_3a_4}-A^{a_1a_2a_3a_4} &= f^{a_1a_2e}f^{ea_3a_4} \\ A^{a_1a_2a_3a_4}-B^{a_1a_2a_3a_4} &= f^{a_2a_3e}f^{ea_4a_1} \\ B^{a_1a_2a_3a_4}-C^{a_1a_2a_3a_4} &= f^{a_3a_1e}f^{ea_2a_4} \end{aligned}$$

Leads to the constraint

$$f^{a_1a_2e}f^{ea_3a_4}+f^{a_2a_3e}f^{ea_4a_1}+f^{a_3a_1e}f^{ea_2a_4}=0$$

The x-factor - under factorization

Consider compton scattering



$$\sim \frac{1}{m^{2(S-1)}}\frac{x_{12}}{x_{34}}(\langle \mathbf{1}P^I\rangle[P_I\mathbf{4}])^{2S}\,.$$

The presence of $\frac{x_{12}}{x_{34}}$ introduces constraint in the other channel since

$\frac{x_{12}}{x_{34}}m^2 = -\langle 3|p_1|2]^2/t$. Importing the solution to the factorization:

$$\frac{\langle \mathbf{1}|P_I|\mathbf{4}]}{m} = m\frac{\langle \mathbf{4}3\rangle[\mathbf{1}2] + \langle \mathbf{1}3\rangle[\mathbf{4}2]}{\langle 3|p_1|2]}$$

we find

$$\frac{\langle 3|p_1|2]^{2-2S}}{(s-m^2)(u-m^2)}(\langle \mathbf{4}3\rangle[\mathbf{1}2] + \langle \mathbf{1}3\rangle[\mathbf{4}2])^{2S}$$

\

Indeed for S=1/2, 1 yields

$$M(\mathbf{1}^{\frac{1}{2}},\gamma_2^{+1},\gamma_3^{-1},\mathbf{4}^{\frac{1}{2}}) = \frac{\langle 3|p_1-p_4|2]}{2(s-m^2)(u-m^2)}(\langle \mathbf{4}3\rangle[\mathbf{1}2] + \langle \mathbf{1}3\rangle[\mathbf{4}2])$$

$$M(\mathbf{1}^{1},\gamma_2^{+1},\gamma_3^{-1},\mathbf{4}^{1}) = \frac{(\langle \mathbf{4}3\rangle[\mathbf{1}2] + \langle \mathbf{1}3\rangle[\mathbf{4}2])^2}{(s-m^2)(u-m^2)}\,.$$

The x-factor - under factorization

Consider compton scattering

Indeed for S>1 the naive form becomes nonlocal

$$\frac{\langle 3|p_1|2]^{2-2S}}{(s-m^2)(u-m^2)}\left(\langle 43\rangle[12]+\langle 13\rangle[42]\right)^{2S}$$

We use s-channel identities

$$\frac{\langle 43\rangle[12]+\langle 13\rangle[42]}{\langle 3|p_1|2]}=\left(\frac{[14]}{m}+\frac{\langle 42\rangle[21]-\langle 12\rangle[24]}{2m^2}\right)+\frac{t[21]\langle 34\rangle}{2m^2\langle 3|p_1|2]}\equiv A+B$$

On the s-channel kinematics we can write:

$$\left.\frac{t[21]\langle 34\rangle}{2m^2\langle 3|p_1|2]}\right|_{s=m^2}=-\frac{\langle 43\rangle[32]\langle 21\rangle}{2m^3}$$

Photon Compton Amplitude for $S>1$

$$\begin{aligned}M(1^S,2^{+1},3^{-1},4^S)=&\frac{\langle 3|p_1|2]^2}{(s-m^2)(u-m^2)}A^{2S}\\&-\left\{\frac{\langle 3|p_1|2]\langle 34\rangle[21]}{2m^2(s-m^2)}\left[\sum_{r=1}^{2S}\binom{2S}{r}A^{2S-r}\left(\frac{-\langle 43\rangle[32]\langle 21\rangle}{4m^3}-\frac{[43]\langle 32\rangle[21]}{4m^3}\right)^{r-1}\right]\right.\\&\left.+\frac{\langle 3|p_1|2]\langle 31\rangle[24]}{2m^2(u-m^2)}\left[\sum_{r=1}^{2S}\binom{2S}{r}(-1)^rA^{2S-r}\left(\frac{-\langle 13\rangle[32]\langle 24\rangle}{4m^3}-\frac{[13]\langle 32\rangle[24]}{4m^3}\right)^{r-1}\right]\right\}\end{aligned}\qquad(4.14)$$

The x-factor - under factorization

Consider gravitational compton scattering

Graviton Compton Amplitude for $S > 2$

$$
\begin{aligned}
M_4(S>2) = &-\frac{\langle 3|p_1|2]^4}{(s-m^2)(u-m^2)tM_{pl}^2}\mathcal{A}^{2S} \\
&+\frac{2S\langle 3|p_1|2]^3}{t(s-m^2)}\frac{\langle 34\rangle[21]}{2m^2M_{pl}^2}\mathcal{A}^{2S-1}-\frac{2S\langle 3|p_4|2]^3}{t(u-m^2)}\frac{\langle 31\rangle[24]}{2m^2M_{pl}^2}\mathcal{A}^{2S-1} \\
&+\left\{\frac{\langle 3|p_1|2]^2\langle 34\rangle^2[21]^2}{4m^4(s-m^2)M_{pl}^2}\left[\sum_{r=2}^{2S}\binom{2S}{r}\mathcal{A}^{2S-r}\left(\frac{-\langle 43\rangle[32]\langle 21\rangle}{4m^3}-\frac{[43]\langle 32\rangle[21]}{4m^3}\right)^{r-2}\right]\right. \\
&\quad\left.+\frac{\langle 3|p_1|2]^2\langle 31\rangle^2[24]^2}{4m^4(u-m^2)M_{pl}^2}\left[\sum_{r=2}^{2S}\binom{2S}{r}(-1)^r\mathcal{A}^{2S-r}\left(\frac{-\langle 13\rangle[32]\langle 24\rangle}{4m^3}-\frac{[13]\langle 32\rangle[24]}{4m^3}\right)^{r-2}\right]\right\} \\
&-\frac{Poly+Poly_{[23]}+Poly_{\langle 23\rangle}}{tM_{pl}^2}
\end{aligned}
\tag{4.24}
$$

Non-minimal couplings

- We know that for photon couplings $\lambda\lambda$ deformations corresponds to g-2 factors, what does such deformations correspond to ? (take s=1/2)

$$\mathcal{M}_4^{(NM-Min)} = \mathcal{M}^{(l=1,r=0)} + \mathcal{M}^{(l=0,r=1)}$$
$$= \frac{\langle 3|p_1|2]^3}{(s-m^2)(u-m^2)t}[12]\langle 23\rangle[24] + \frac{\langle 3|p_1|2]^3}{(s-m^2)(u-m^2)t}\langle 13\rangle[23]\langle 34\rangle$$

and

$$\mathcal{M}_4^{(NM-NM)} = -\frac{\langle 3|p_1|2]^3}{2(s-m^2)(u-m^2)}([12]\langle 34\rangle + [42]\langle 31\rangle)$$
$$+\frac{\langle 3|p_1|2]^3}{2(s-m^2)t}([12]\langle 34\rangle - [42]\langle 31\rangle) - \frac{\langle 3|p_4|2]^3}{2(u-m^2)t}([42]\langle 31\rangle - [12]\langle 34\rangle)$$

But there are excessive terms that do not match with the *t*-channel residue!

$$\frac{1}{t} \times \quad \frac{\langle 3|p_1|2]^3}{(s-m^2)}([12]\langle 34\rangle - [42]\langle 31\rangle)$$

One cannot make factorization consistent!

Non-minimal couplings

- We know that for photon couplings $\lambda\lambda$ deformations corresponds to g-2 factors, what does such deformations correspond to ? They are simply not allowed!

$$M_3 = x^2(\epsilon^{2S} + \sum_i a_i \epsilon^{2S-i} x^i \lambda^{2i})$$

we must have a1=0! The gravi-magnetic moment is zero

- If we have a system where the gravitational couplings are given by the double copy of gauge couplings (string theory), the magnetic moment for the gauge coupling must be 2!

$$x^2(\epsilon^{2S} + a_1 \epsilon^{2S-1} x\lambda^2 + \cdots)^2 = x^2(\epsilon^{4S} + 2a_1 \epsilon^{4S-1} x\lambda^2 + \cdots)$$

Fundamental charged particles must have classical g=2 is a prediction of string theory.

The simplest amplitude

- Are there particles in nature with s>2 and has minimal coupling to graviton and photons (just x) ?
- String theory ?

The simplest amplitude

- Are there particles in nature with s>2 and has minimal coupling to graviton and photons (just x) ?
- String theory ? No!

$$\frac{x\langle 12\rangle^n}{m^{2n-1}}\sum_{k=0}^{n} C_k^n\,(k-1)\,\langle 12\rangle^{n-k}{}_{2}\left(\frac{4x}{m}\langle 23\rangle\langle 31\rangle\right)^k$$

The simplest amplitude

- Are there particles in nature with s>2 and has minimal coupling to graviton and photons (just x) ?
- Since the state is likely composite, we consider the one body EFT

$$S = \int d\sigma \; -m\sqrt{u^2} - \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} + L_{SI}\left[u^\mu, S_{\mu\nu}, g_{\mu\nu}(y^\mu)\right]$$

$$L_{SI} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\frac{C_{\mathrm{ES}^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n-1}} S^{\mu_{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{C_{\mathrm{BS}^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n}} S^{\mu_{2n+1}}$$

where

$$E_{\mu\nu} := R_{\mu\alpha\nu\beta} u^\alpha u^\beta$$
$$B_{\mu\nu} := \frac{1}{2}\epsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu} u^\gamma u^\delta$$

for Kerr Black hole C#=1

The simplest amplitude

- Are there particles in nature with s>2 and has minimal coupling to graviton and photons (just x) ?
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$$\begin{aligned}L_{SI}=&\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\frac{C_{\mathrm{ES}^{2n}}}{m^{2n-1}}D_{\mu_{2n}}\cdots D_{\mu_3}\frac{E_{\mu_1\mu_2}}{\sqrt{u^2}}S^{\mu_1}S^{\mu_2}\cdots S^{\mu_{2n-1}}S^{\mu_{2n}}\\&+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{C_{\mathrm{BS}^{2n+1}}}{m^{2n}}D_{\mu_{2n+1}}\cdots D_{\mu_3}\frac{B_{\mu_1\mu_2}}{\sqrt{u^2}}S^{\mu_1}S^{\mu_2}\cdots S^{\mu_{2n}}S^{\mu_{2n+1}}\end{aligned}$$

where

$$\begin{aligned}E_{\mu\nu}&:=R_{\mu\alpha\nu\beta}u^\alpha u^\beta\\B_{\mu\nu}&:=\frac{1}{2}\epsilon_{\alpha\beta\gamma\mu}R^{\alpha\beta}{}_{\delta\nu}u^\gamma u^\delta\end{aligned}$$

for Kerr Black hole C#=1

The simplest amplitude

We derive the three-point amplitude from the 1 BD EFT

$$
\begin{aligned}
L_{SI} = &\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{\mathrm{ES}^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1 \mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \ldots S^{\mu_{2n-1}} S^{\mu_{2n}} \\
&+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{\mathrm{BS}^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1 \mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \ldots S^{\mu_{2n}} S^{\mu_{2n+1}}
\end{aligned}
$$

translating into amplitudes:

$$
\begin{aligned}
u^\mu &\to \frac{1}{m} p_1^\mu \\
D_\mu &\to -i k_{3\mu} \\
h_{\mu\nu} &\to 2 \epsilon_\mu^+ \epsilon_\nu^+ \\
E_{\mu\nu} &\to \frac{\kappa x^2}{2} k_{3\mu} k_{3\nu} \\
B_{\mu\nu} S^\mu &\to \frac{\kappa x}{2} \left[k_{3\alpha} (\sqrt{2} \epsilon_\beta^+ - x u_\beta) J^{\alpha\beta} \right] k_{3\nu}
\end{aligned}
$$

The simplest amplitude

We derive the three-point amplitude from the 1 BD EFT

$$L_{SI} \to -\sum_{n=1}^{\infty} \frac{C_{\mathrm{ES}^{2n}}}{(2n)!} \frac{\kappa m x^2}{2} \left(\frac{k_3 \cdot S}{m}\right)^{2n} - \sum_{n=1}^{\infty} \frac{C_{\mathrm{BS}^{2n+1}}}{(2n+1)!} \frac{\kappa x}{2} \left[-i k_{3\alpha} (\sqrt{2}\epsilon_\beta^+ - x u_\beta) J^{\alpha\beta}\right] \left(\frac{k_3 \cdot S}{m}\right)^{2n}$$
$$= -\sum_{n=2}^{\infty} \frac{\kappa m x^2}{2} \frac{C_{\mathrm{S}^n}}{n!} \left(-\frac{k_3 \cdot S}{m}\right)^n$$

But we need to add the canonical term in the Lagrangian as well:

$$-m\sqrt{u^2} = -m\sqrt{\eta_{\mu\nu}u^\mu u^\nu + \kappa h_{\mu\nu} u^\mu u^\nu} \to -\frac{\kappa m}{2} h_{\mu\nu} u^\mu u^\nu + \cdots \to -\frac{\kappa m x^2}{2}$$

$$L = -\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} = -\frac{1}{2} S_{AB} \omega_\mu{}^{AB} u^\mu$$

which translate to

$$L \to -\frac{\kappa}{2} x \left[-i k_{3\mu} (\sqrt{2}\epsilon_\nu^+ - x u_\nu) J^{\mu\nu}\right]$$
$$= -\frac{\kappa m x^2}{2} \left(-\frac{k_3 \cdot S}{m}\right)$$

The simplest amplitude

We derive the three-point amplitude from the 1 BD EFT

Converting to spin-*s* amplitudes

- The spin-independent piece $-m\sqrt{u^2}$ yields

$$A_3^+ \supset x^2 \langle \mathbf{21} \rangle^s [\mathbf{21}]^s$$
$$A_3^- \supset x^{-2} [\mathbf{21}]^s \langle \mathbf{21} \rangle^s$$

- The spin-dependent piece: acting on spin-s

$$\frac{k_3 \cdot S}{m} = \frac{xs}{m} |3\rangle\langle 3|$$
$$\frac{k_3 \cdot S}{m} = -\frac{s}{mx} |3]|3|$$

generalizing to arbitrary powers

$$\left[\left(\frac{k_3 \cdot S}{m}\right)^2\right]_{\alpha_1 \cdots \alpha_{2s}}^{\beta_1 \cdots \beta_{2s}} = 2s(2s-1)\frac{x^2}{(2m)^2} |3\rangle_{\alpha_1} |3\rangle_{\alpha_2} \langle 3|^{\beta_1} \langle 3|^{\beta_2}$$
$$\left[\left(\frac{k_3 \cdot S}{m}\right)^3\right]_{\alpha_1 \cdots \alpha_{2s}}^{\beta_1 \cdots \beta_{2s}} = 2s(2s-1)(2s-2)\frac{x^3}{(2m)^3} |3\rangle_{\alpha_1} |3\rangle_{\alpha_2} |3\rangle_{\alpha_3} \langle 3|^{\beta_1} \langle 3|^{\beta_2} \langle 3|^{\beta_3}$$

The simplest amplitude

We derive the three-point amplitude from the 1 BD EFT

Putting everything together, we find:

$$A_3^+ \supset x^2 C_{\mathrm{S}^{a+b}} \tilde{A}^s_{a,b} \langle \mathbf{21} \rangle^{s-a} \left(-\frac{x \langle \mathbf{23} \rangle \langle \mathbf{31} \rangle}{2m} \right)^a [\mathbf{21}]^{s-b} \left(\frac{[\mathbf{23}][\mathbf{31}]}{2mx} \right)^b$$

$$\tilde{A}^s_{a,b} = \frac{1}{(a+b)!} \begin{pmatrix} a+b \\ a \end{pmatrix} \frac{s!}{(s-a)!} \frac{s!}{(s-b)!} = \begin{pmatrix} s \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix}$$

Now lets compare to our “minimal coupling”

The simplest amplitude

Let us compare with minimal coupling

First rewrite:

$$\langle \mathbf{21}\rangle^2 = \langle \mathbf{21}\rangle[\mathbf{21}] + \langle \mathbf{21}\rangle \frac{[\mathbf{23}][\mathbf{31}]}{2mx} - \frac{x\langle \mathbf{23}\rangle\langle \mathbf{31}\rangle}{2m}[\mathbf{21}] - \frac{2\langle \mathbf{23}\rangle\langle \mathbf{31}\rangle[\mathbf{23}][\mathbf{31}]}{(2m)^2}$$

Then the spin-*s* interaction becomes:

$$\langle \mathbf{21}\rangle^{2s} = \left(\langle \mathbf{21}\rangle^2\right)^s = \sum_{a,b=0}^{s} A^s_{a,b}\langle \mathbf{21}\rangle^{s-a}\left(-\frac{x\langle \mathbf{23}\rangle\langle \mathbf{31}\rangle}{2m}\right)^a [\mathbf{21}]^{s-b}\left(\frac{[\mathbf{23}][\mathbf{31}]}{2mx}\right)^b$$

Where

$$A^s_{a,b} = \sum_{c=0}^{\min(a,b)} \frac{2^c s!}{(s-a-b+c)!(a-c)!(b-c)!c!}$$

The simplest amplitude

Let us compare with minimal coupling

Let's compare

EFT $\quad x^2 C_{S^{a+b}} \tilde{A}^s_{a,b} \langle \mathbf{21} \rangle^{s-a} \left(-\frac{x \langle \mathbf{23} \rangle \langle \mathbf{31} \rangle}{2m} \right)^a [\mathbf{21}]^{s-b} \left(\frac{[\mathbf{23}][\mathbf{31}]}{2mx} \right)^b$

"Minimal coupling" $\quad \langle \mathbf{21} \rangle^{2s} = \left(\langle \mathbf{21} \rangle^2 \right)^s = \sum_{a,b=0}^{s} A^s_{a,b} \langle \mathbf{21} \rangle^{s-a} \left(-\frac{x \langle \mathbf{23} \rangle \langle \mathbf{31} \rangle}{2m} \right)^a [\mathbf{21}]^{s-b} \left(\frac{[\mathbf{23}][\mathbf{31}]}{2mx} \right)^b$

This is exactly the same if $A^s_{a,b} = \tilde{A}^s_{a,b}$!

The simplest amplitude

Let us compare with minimal coupling

However, there is a very simple relationship:

$$A^s_{a,b} = \sum_{i=0}^{\min(a,b)} \binom{s}{i} \tilde{A}^{s-i}_{a-i,b-i}$$
$$= \tilde{A}^s_{a,b} + s\tilde{A}^{s-1}_{a-1,b-1} + \frac{s(s-1)}{2}\tilde{A}^{s-2}_{a-2,b-2} + \cdots$$

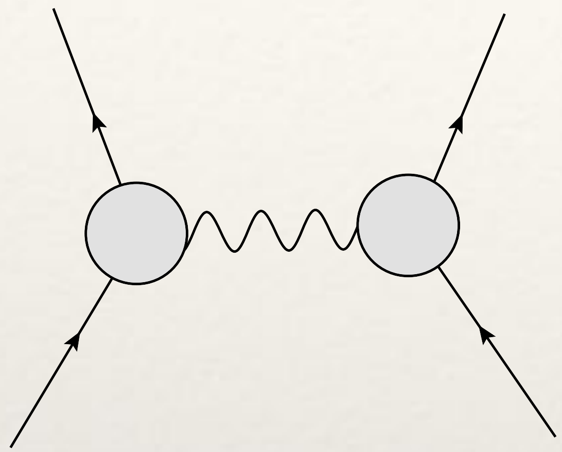
Since $A^s_{a,b} = \binom{s}{a}\binom{s}{b}$ scales as $\frac{s^{a+b}}{a!b!}$, we see that in the large s limit

$$A^s_{a,b} = \tilde{A}^s_{a,b}(1 + \mathcal{O}(1/s))$$

The coupling $x^2\langle 12\rangle^s$ marches with Kerr black hole coefficients as $s \to \infty$! See also Alfredo Guevara, Justine Vine and Alexander Ochirov

The inspired gravitational potential

see also (Alfredo Guevara 1706.02314)



$$\frac{\mathcal{M}}{4E_aE_b} \xrightarrow{NR} V_{Cl}$$

$$\begin{aligned}\frac{\mathcal{M}}{4E_aE_b}\bigg|_{S_a^1 S_b^0} &= \frac{8\pi i G\,(m_a+m_b)}{q^2 m_a}(\vec{S}_a\cdot\vec{p_a}\times\vec{q})+\cdots\\ &= -\frac{2G}{r^2}\frac{m_a+m_b}{m_a}(\vec{S}_a\cdot\vec{p_a}\times\hat{n})+\cdots\end{aligned}$$

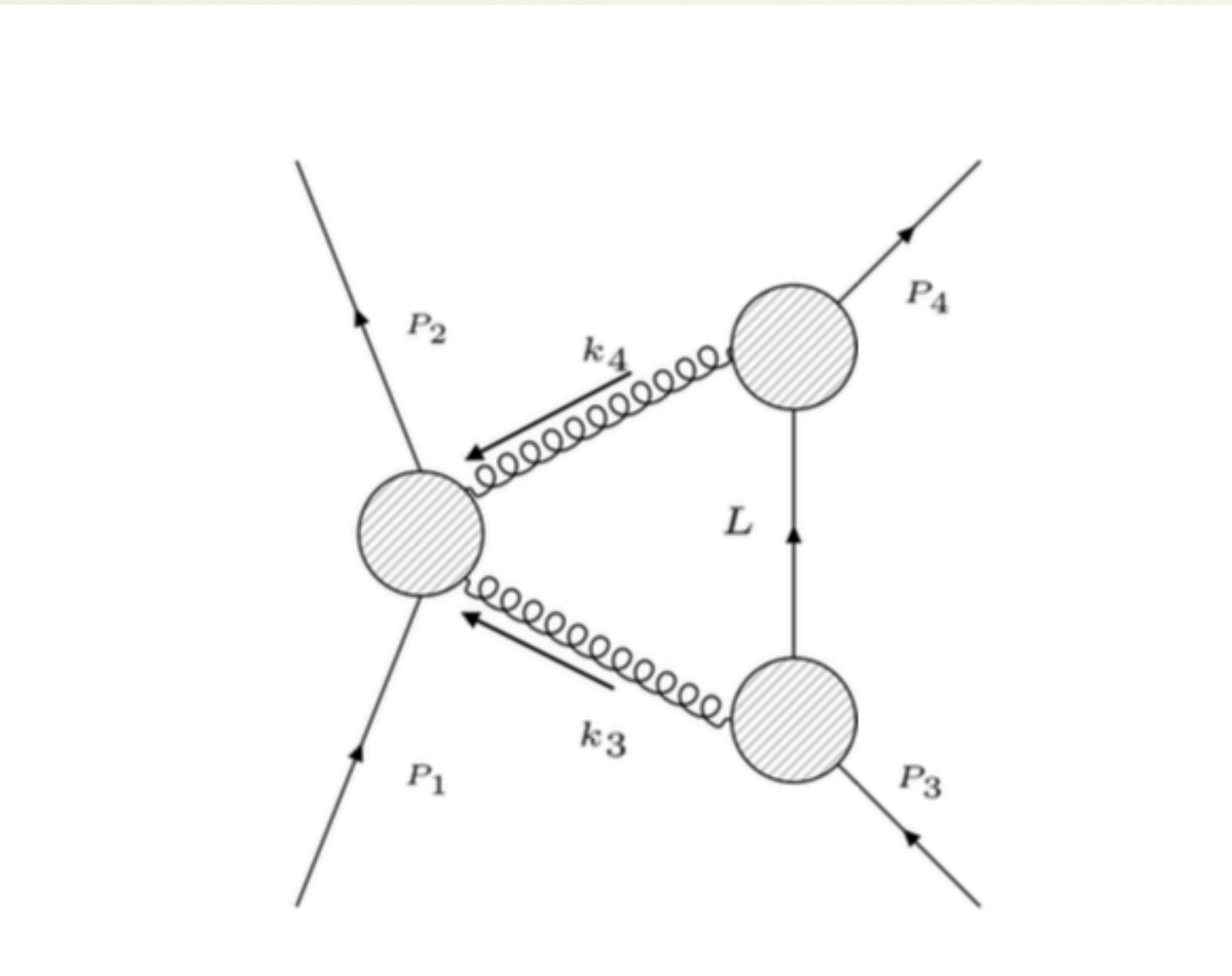
$$\begin{aligned}\frac{\mathcal{M}}{4E_aE_b}\bigg|_{S_a^1 S_b^1} &= -\frac{4\pi G}{q^2}(\vec{S}_a\cdot\vec{q})(\vec{S}_b\cdot\vec{q})+\cdots\\ &= -\frac{G}{r^3}\left(\vec{S}_a\cdot\vec{S}_b-3(\vec{S}_a\cdot\vec{n})(\vec{S}_b\cdot\vec{n})\right)+\cdots\end{aligned}$$

$$\begin{aligned}\frac{\mathcal{M}}{4E_aE_b}\bigg|_{S_a^2 S_b^0} &= -\frac{2\pi G m_b}{q^2 m_a}(\vec{S}_a\cdot\vec{q})^2+\cdots\\ &= -\frac{G m_b}{2m_a r^3}\left(\vec{S}_a\cdot\vec{S}_a-3(\vec{S}_a\cdot\vec{n})^2\right)+\cdots\end{aligned}$$

$$\begin{aligned}\frac{\mathcal{M}}{4E_aE_b}\bigg|_{S_a^3 S_b^0} &= \frac{4i\pi G\,(m_a+m_b)}{3q^2 m_a^3}(\vec{S}_a\cdot\vec{q})^2(\vec{S}_a\cdot\vec{p_a}\times\vec{q})+\cdots\\ &= -\frac{G(m_a+m_b)}{r^4 m_a^3}(\vec{S}_a\cdot\vec{p_a}\times\vec{n})\left(\vec{S}_a\cdot\vec{S}_a-5(\vec{S}_a\cdot\vec{n})^2\right)+\cdot\cdot\end{aligned}$$

The inspired gravitational potential

With the Compton amplitude, we can now compute the NLO spin effects



$$\frac{\mathcal{M}}{4E_aE_b} \xrightarrow{NR} V_{Cl}$$

Out look

- There is a natural expansion basis for the scattering of massive states that makes physical properties of the interaction transparent. UV behaviors, magnetic moments
- Show that all charged perturbative string states must have g=2
- Kerr-black holes have the simplest spinning amplitude (supersymmetric?)
- Leads to on-shell approach to compute spin-dependent pieces of graviational potential
- For fixed spins, subleading trajectories are degenerate, and should be the dominant contributions to BH microstate counting (Horowitz-Polchinski)$\rightarrow$ amplitudes of subleading trajectories are simpler than leading ones?
- We see that Kerr-Black hole has x^2 coupling. How is this protected under quantum corrections?