Comments on Crossing Kernel in d-dimensional CFTs

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- Let us focus on scalar four point correlation function in d-dimensional CFT, it is often decomposed in particular OPE channel say (12)(34) or s-channel:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\{\mathcal{O}_{\Delta,J}\}} c^{(\mathrm{s})}_{\Delta,J} W^{(\mathrm{s})}_{\Delta,J}(x_i) = \mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i) \sum_{\{\mathcal{O}_{\Delta,J}\}} c^{(\mathrm{s})}_{\Delta,J} G^{(\mathrm{s})}_{\Delta,J}(\mathrm{u},\mathrm{v})$$

where $c^{(\mathrm{s})}_{\Delta,J}$ are product OPE coefficients and the kinematic pre-factor is:

$$\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i) = \frac{1}{(x_{12}^2)^{\frac{\Delta^+_{12}}{2}}(x_{34}^2)^{\frac{\Delta^+_{34}}{2}}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^{\mathrm{a}^{(\mathrm{s})}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\mathrm{b}^{(\mathrm{s})}}, \quad \mathrm{a}^{(\mathrm{s})} = \frac{\Delta^-_{21}}{2}, \ \mathrm{b}^{(\mathrm{s})} = \frac{\Delta^-_{34}}{2};$$

where $\Delta^{\pm}_{ij} = \Delta_i \pm \Delta_j$. $W^{(\mathrm{s})}_{\Delta,J}(x_i)$ and $G^{(\mathrm{s})}_{\Delta,J}(z,\bar{z})$ are the s-channel "conformal partial wave" and "conformal block" for $\mathcal{O}_{\Delta,J}$ family.

- We have also introduced the conformally invariant cross ratios:

$$\mathrm{u} = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad \mathrm{v} = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad x_{ij}^2 = (x_i - x_j)^2.$$

$G^{(\mathrm{s})}_{\Delta,J}(z,\bar{z})$ almost behaves as natural kinematic basis and it satisfies the quadratic Casimir equation" [Dolan-Osborn 2003, 2011]

$$\Delta_2^{(\varepsilon)}(a^{(s)},b^{(s)},0)G^{(s)}_{\Delta,J}(z,\bar{z}) = C_2(\Delta,J)G^{(s)}_{\Delta,J}(z,\bar{z}), \quad C_2(\Delta,J) = \frac{\Delta(\Delta-d)}{2} + \frac{J(J+\varepsilon)}{2}$$

Here $d-2=2\varepsilon$ and the second order partial differential operator is:

$$\Delta_2^{(\varepsilon)}(a,b,c) = D_z(a,b,c) + D_{\bar{z}}(a,b,c) + 2\varepsilon\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\frac{\partial}{\partial z} - (1-\bar{z})\frac{\partial}{\partial \bar{z}}\right)$$

$$D_z(a,b,c) = z^2(1-z)\frac{\partial^2}{\partial z^2} - ((a+b+1)z^2 - cz)\frac{\partial}{\partial z} - abz,$$

This equation has 8-fold degeneracies, i.e. eigenvalue is invariant under:

$$\Delta \leftrightarrow d-\Delta, \quad J \leftrightarrow 2-d-J, \quad \Delta \leftrightarrow 1-J$$

While $G^{(\mathrm{t})}_{\Delta,J}(z,\bar{z})$ and $G^{(\mathrm{u})}_{\Delta,J}(z,\bar{z})$ can be obtained from the crossing trans:

$$\mathrm{s} \leftrightarrow \mathrm{t} \,:\, (x_2,\Delta_2) \leftrightarrow (x_4,\Delta_4), \quad \mathrm{s} \leftrightarrow \mathrm{u} \,:\, (x_2,\Delta_2) \leftrightarrow (x_3,\Delta_3).$$

$$\mathrm{s} \leftrightarrow \mathrm{t} \,:\, (\mathrm{u},\mathrm{v}) \to (\mathrm{v},\mathrm{u});\, (z,\bar{z}) \to (1-z,1-\bar{z}), \quad \mathrm{s} \leftrightarrow \mathrm{u} \,:\, (\mathrm{u},\mathrm{v}) \to (1/\mathrm{u},\mathrm{v}/\mathrm{u});\, (z,\bar{z}) \to (1/z,1/\bar{z})$$

- For even d, $G^{(s)}_{\Delta,J}(z,\bar{z})$ is given by finite sum of hypergeometric functions:

$$d=2\ :\ G^{(s)}_{\Delta,J}(z,\bar{z}) = \frac{1}{(-2)^J(1+\delta_{J,0})}(k_{\Delta+J}(z)k_{\Delta-J}(\bar{z})+k_{\Delta+J}(\bar{z})k_{\Delta-J}(z)),$$
$$d=4\ :\ G^{(s)}_{\Delta,J}(z,\bar{z}) = \frac{z\bar{z}}{(-2)^J(z-\bar{z})}(k_{\Delta+J}(z)k_{\Delta-J-2}(\bar{z})-k_{\Delta+J}(\bar{z})k_{\Delta-J-2}(z)),$$
$$k_\beta(x) = x^{\frac{\beta}{2}}{}_2F_1\left(\frac{\beta}{2}+\mathrm{a}^{(s)},\frac{\beta}{2}+\mathrm{b}^{(s)},\beta,x\right)$$

- For general d, quadratic Casimir equation has integral solution:

$$\hat{\Psi}^{(s)}_{\nu,J}(x_i) \propto \int_{\mathbb{R}^d} d^dx_0\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{h+i\nu,J}(x_0)\rangle\langle\tilde{\mathcal{O}}_{h-i\nu,J}(x_0)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle,\quad h=\frac{d}{2}.$$

  Constructed from 3 point functions involving symmetric-traceless primary $\mathcal{O}_{h+i\nu,J}(x_0)$ and its shadow $\tilde{\mathcal{O}}_{h-i\nu,J}(x_0)$.

- The x_0-integration yields a symmetric linear combination of direct and shadow CPWs, we can use $\frac{1}{\nu^2+(\Delta-h)^2}$ to project out one or the other in ν-integration.

- After contracting the tensor structures from three point functions yielding Gegenbauer polynomial, the explicit x_0 can be done via Symanzik star-formula:

$$\int_{\mathbb{R}^d} d^d x_0 \prod_{i=1}^{n} \frac{1}{(x_{i0}^2)^{\delta_i}} = \frac{\pi^h}{\prod_{i=1}^{n}\Gamma(\delta_i)} \int_{-i\infty}^{i\infty} [d\delta]_{\frac{n(n-3)}{2}} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(x_{ij}^2)^{\delta_{ij}}}$$

- This leads us to Mellin representation of $W^{(s)}_{\Delta,J}(x_i)$ [Mack, Penedones + others]:

$$W^{(\mathrm{s})}_{\Delta,J}(x_i) \propto \int_{-\infty}^{\infty} d\nu\, \tilde{\mu}^{(\mathrm{s})}_{\Delta,J}(\nu)\Psi^{(\mathrm{s})}_{\nu,J}(x_i), \quad \tilde{\mu}^{(\mathrm{s})}_{\Delta,J}(\nu) = \frac{1}{2\pi i((\Delta-h)^2+\nu^2)\Gamma(\pm i\nu)(h\pm i\nu-1)_J}$$

$$\Psi^{(\mathrm{s})}_{\nu,J}(x_i) = \mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i) \int_{-i\infty}^{i\infty} \frac{ds}{(4\pi i)} \int_{-i\infty}^{i\infty} \frac{dt}{(4\pi i)} \mathrm{u}^{\frac{s}{2}} \mathrm{v}^{\frac{t}{2}} \rho^{(\mathrm{s})}_{\Delta_i}(s,t) \mathcal{M}^{(\mathrm{s})}_{\nu,J}(s,t)$$

where s and t are the so-called Mellin variables.

Here the integration measure over Mellin space is given by:

$$\rho^{(s)}_{\Delta_i}(s,t) = \prod_{i<j} \Gamma\left(\delta^{(s)}_{ij}\right)$$

$$\delta^{(s)}_{12} = \frac{\Delta^+_{12} - s}{2},\ \delta^{(s)}_{34} = \frac{\Delta^+_{34} - s}{2},\ \delta^{(s)}_{13} = \frac{s+t}{2} + \mathrm{b}^{(s)},\ \delta^{(s)}_{24} = \frac{s+t}{2} + \mathrm{a}^{(s)},\ \delta^{(s)}_{14} = -\frac{t}{2} - \mathrm{a}^{(s)} - \mathrm{b}^{(s)},\ \delta^{(s)}_{23} = -\frac{t}{2}$$

The remaining integrand is called "Mellin partial amplitude":

$$\mathcal{M}^{(s)}_{\nu,J}(s,t) = \frac{\Gamma\left(\tau_{\pm\nu} - \frac{s}{2}\right)}{\Gamma\left(\delta^{(s)}_{12}\right)\Gamma\left(\delta^{(s)}_{34}\right)} \tilde{P}^{(s)}_{\nu,J}(s,t), \quad \tau_{\pm\nu} = \frac{h \pm i\nu - J}{2}$$

$$\tilde{P}^{(s)}_{\nu,J}(s,t) =$$

$$\sum_{r=0}^{[\frac{J}{2}]} \frac{(-1)^r J!(J+h-1)_{-r}}{2^J r!} \sum_{\sum k_{ij} = J-2r} \frac{(-1)^{k_{13}+k_{24}}}{\prod_{(ij)} k_{ij}!} \left(\tau_{\pm\nu} - \frac{s}{2}\right)_r \prod_{(ij)} \left(\delta^{(s)}_{ij}\right)_{k_{ij}} \prod_{i=1}^{4} \left(1 - \gamma^{(s)}_i\right)_{J-r-\sum_j k_{ji}}$$

where $\tilde{P}^{(s)}_{\nu,J}(s;t)$ is Mack polynomial which contains no poles in s, t; $\{\gamma^{(s)}_i\}$ are also functions of $\tau_{\pm\nu}$.

- Explicitly integrating over both s and t, we obtain: [Chen-Kyono]

$$\Psi^{(s)}_{\nu,J}(x_i) = \mathcal{T}^{(s)}_{\Delta_i}(x_i) \sum_{\sigma_s=\pm} \widetilde{\sum_{r,k} \prod_{i=1}^{4}} \left(1-\gamma_i^{(s)}\right)_{J-r-\sum_j k_{ji}} \sum_{n_s=0}^{\infty} \frac{(-1)^{n_s}}{n_s!} \Gamma(-i\sigma_s\nu - n_s) \frac{\mathrm{u}^{\tau_{\sigma_s\nu}+n_s+r}}{\mathrm{v}^{\frac{\mathrm{a}^{(s)}+\mathrm{b}^{(s)}}{2}}}$$
$$\times \left[\mathrm{v}^{k_{14}-\frac{\mathrm{a}^{(s)}+\mathrm{b}^{(s)}}{2}} {}_2\tilde{\mathbf{F}}_1 \left[\begin{matrix} \kappa^{1(s)}_{\sigma_s\nu}+n_s+r,\ \kappa^{4(s)}_{\sigma_s\nu}+n_s+r \\ 1+\varpi^{(s)}_{14} \end{matrix} ; \mathrm{v} \right] + \mathrm{v}^{k_{23}+\frac{\mathrm{a}^{(s)}+\mathrm{b}^{(s)}}{2}} {}_2\tilde{\mathbf{F}}_1 \left[\begin{matrix} \kappa^{2(s)}_{\sigma_s\nu}+n_s+r,\ \kappa^{3(s)}_{\sigma_s\nu}+n_s+r \\ 1+\varpi^{(s)}_{23} \end{matrix} ; \mathrm{v} \right] \right]$$

$${}_2\tilde{\mathbf{F}}_1 \left[\begin{matrix} a,\ b \\ c \end{matrix} ; x \right] = \Gamma(a)\Gamma(b)\Gamma(1-c)\, {}_2F_1 \left[\begin{matrix} a,\ b \\ c \end{matrix} ; x \right] = \frac{\pi}{\sin \pi c} \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)} (-x)^{-s}$$

This is valid for all d, for even d where finite sum results are known, we obtain non-trivial identities among hypergeometric functions!

- Even more excitingly, the infinite summation over n_s can also be expressed in terms of Appell's hypergeometric functions $\mathbf{F}_4$:

$$\Psi^{(s)}_{\nu,J}(x_i) = \mathcal{T}^{(s)}_{\Delta_i}(x_i) \sum_{\sigma_s=\pm} \widetilde{\sum_{r,k} \prod_{i=1}^{4}} \left(1-\gamma_i^{(s)}\right)_{J-r-\sum_j k_{ji}} \frac{\mathrm{u}^{\tau_{\sigma_s\nu}+r}}{\mathrm{v}^{\frac{\mathrm{a}^{(s)}+\mathrm{b}^{(s)}}{2}}}$$
$$\times \left[\mathrm{v}^{k_{14}-\frac{\mathrm{a}^{(s)}+\mathrm{b}^{(s)}}{2}} \tilde{\mathbf{F}}_4 \left[\begin{matrix} \kappa^{1(s)}_{\sigma_s\nu}+r,\ \kappa^{4(s)}_{\sigma_s\nu}+r \\ 1+\varpi^{(s)}_{14},\ 1+i\sigma_s\nu \end{matrix} ; \mathrm{u}, \mathrm{v} \right] + \mathrm{v}^{k_{23}+\frac{\mathrm{a}^{(s)}+\mathrm{b}^{(s)}}{2}} \tilde{\mathbf{F}}_4 \left[\begin{matrix} \kappa^{2(s)}_{\sigma_s\nu}+r,\ \kappa^{3(s)}_{\sigma_s\nu}+r \\ 1+\varpi^{(s)}_{23},\ 1+i\sigma_s\nu \end{matrix} ; \mathrm{u}, \mathrm{v} \right] \right]$$

- The Appell function $\mathbf{F}_4$ is one of the four possible two variable generalization of hypergeometric function:

$$\tilde{\mathbf{F}}_4\left[\begin{matrix} a_1, \, a_2 \\ c_1, \, c_2 \end{matrix}; x, y\right] = \Gamma(a_1)\Gamma(a_2)\Gamma(1-c_1)\Gamma(1-c_2)\mathbf{F}_4\left[\begin{matrix} a_1, \, a_2 \\ c_1, \, c_2 \end{matrix}; x, y\right],$$

$$\mathbf{F}_4\left[\begin{matrix} a_1, \, a_2 \\ c_1, \, c_2 \end{matrix}; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}}{m!n!(c_1)_m(c_2)_n}x^m y^n, \quad |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1.$$

- In particular, it satisfies the following partial differential equations:

$$x(1-x)\frac{\partial^2\mathbf{F}_4}{\partial x^2} - y^2\frac{\partial^2\mathbf{F}_4}{\partial y^2} - 2xy\frac{\partial^2\mathbf{F}_4}{\partial x\partial y} + (c_1-(a_1+a_2+1)x)\frac{\partial\mathbf{F}_4}{\partial x} - (a_1+a_2+1)y\frac{\partial\mathbf{F}_4}{\partial y} - a_1a_2\mathbf{F}_4 = 0,$$

$$y(1-y)\frac{\partial^2\mathbf{F}_4}{\partial y^2} - x^2\frac{\partial^2\mathbf{F}_4}{\partial x^2} - 2xy\frac{\partial^2\mathbf{F}_4}{\partial x\partial y} + (c_2-(a_1+a_2+1)x)\frac{\partial\mathbf{F}_4}{\partial y} - (a_1+a_2+1)x\frac{\partial\mathbf{F}_4}{\partial x} - a_1a_2\mathbf{F}_4 = 0,$$

  and they reduce to various defining equation for generalized hypergeometric function e.g. ${}_3F_2$ when $(x, y) = (t^2, (1-t)^2)$ and $(a_{1,2}, c_{1,2})$ take special values. It would be very interesting to understand their connections with conformal Casimir equation.

Another cute connection with mathematics, consider 2×2 matrix (zonal) generalization of hypergeometric function [Koornwinder, Spinkjuizen-Kuyper 1978]:

$$
{}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; \begin{bmatrix} z, & 0 \\ 0, & \bar{z} \end{bmatrix}\right] =
$$
$$
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\mathbf{F}_4\left[\begin{matrix} a,\ b \\ 1+a+b-c,\ c-\frac{1}{2} \end{matrix}; \mathrm{u}, \mathrm{v}\right] + \mathrm{v}^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\mathbf{F}_4\left[\begin{matrix} c-a,\ c-b \\ 1-a-b+c,\ c-\frac{1}{2} \end{matrix}; \mathrm{u}, \mathrm{v}\right]
$$

where LHS can be expressed in terms of Legendre polynomials:

$$
{}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; \begin{bmatrix} z, & 0 \\ 0, & \bar{z} \end{bmatrix}\right] = \sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{(a)_m(a-\frac{1}{2})_l(b)_m(b-\frac{1}{2})_l(\frac{3}{2})_{m-l}}{(c)_m(c-\frac{1}{2})_l(\frac{3}{2})_m l!(\frac{1}{2})_{m-l}}\mathrm{P}_{m-l}\left(\frac{1}{2}\frac{z+\bar{z}}{\sqrt{z\bar{z}}}\right)
$$

This implies in our case:

$$
a = \kappa^{1(\mathrm{s})}_{\sigma_s\nu} + r, \quad b = \kappa^{4(\mathrm{s})}_{\sigma_s\nu} + r, \quad c = \frac{3}{2} + i\sigma_s\nu \quad (\text{i.e. } d = 3)
$$

we can express $G^{(\mathrm{s})}_{\Delta,J}(\mathrm{u},\mathrm{v})$ in terms of 2×2 zonal hypergeometric function.

- Summarizing here, we have obtained an expression for s-channel scalar conformal block:

$$G^{(\mathrm{s})}_{\Delta,J}(z,\bar{z}) = \frac{1}{\mathbf{c}^{(\mathrm{s})}_{\Delta,J}} \widetilde{\sum_{r,k}} \prod_{i=1}^{4} \left(1-\gamma_i^{(\mathrm{s})}\right)_{J-r-\sum_j k_{ji}} \frac{\mathrm{u}^{\frac{\Delta-J}{2}+r}}{\mathrm{v}^{\frac{\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})}}{2}}}$$
$$\times \left[\mathrm{v}^{k_{14}-\frac{\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})}}{2}} \tilde{\mathbf{F}}_4 \left[\begin{matrix} \kappa^{1(\mathrm{s})}(\tau)+r,\ \kappa^{4(\mathrm{s})}(\tau)+r \\ 1+\varpi^{(\mathrm{s})}_{14},\ 1+h-\Delta \end{matrix}; \mathrm{u},\mathrm{v}\right] + \mathrm{v}^{k_{23}+\frac{\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})}}{2}} \tilde{\mathbf{F}}_4 \left[\begin{matrix} \kappa^{2(\mathrm{s})}(\tau)+r,\ \kappa^{3(\mathrm{s})}(\tau)+r \\ 1+\varpi^{(\mathrm{s})}_{23},\ 1+h-\Delta \end{matrix}; \mathrm{u},\mathrm{v}\right]\right]$$
$$\mathbf{c}^{(\mathrm{s})}_{\Delta,J} = \frac{1}{(-2)^J c_J} \frac{(d-1-\Delta)_J \Gamma(h-\Delta)}{\Gamma(\Delta+J)} \Gamma\left(\frac{\Delta+J}{2} \pm \mathrm{a}^{(\mathrm{s})}\right) \Gamma\left(\frac{\Delta+J}{2} \pm \mathrm{b}^{(\mathrm{s})}\right),$$

in terms of *finite sum* over Appell's hypergeometric function $\mathbf{F}_4$.

- It has correct $|\mathrm{u}| \ll 1$ expansion, and satisfies the desired property such as:

$$G^{(\mathrm{s})}_{\Delta,J}(z,\bar{z}) \mid_{\Delta_i \to \bar{\Delta}_i = d-\Delta_i} = ((1-z)(1-\bar{z}))^{\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})}} G^{(\mathrm{s})}_{\Delta,J}(z,\bar{z}),$$

and matches with known expressions for $d = 2, 4$ etc.
Other t- and u- channels are obtained through crossing trans.

- To discuss crossing kernel, it is convenient to consider spectral representation of four point scalar correlation function:

$$\langle \prod_{i=1}^{4} \mathcal{O}_{\Delta_i, J_i}(x_i) \rangle = \sum_{J=0}^{\infty} \int_{-\infty}^{\infty} d\nu \, \hat{b}_J^{(s)}(\nu) \hat{\Psi}_{\nu,J}^{(s)}(x_i)$$

  where $\hat{b}_J(\nu)$ is "spectral function" whose poles and residues encode the spectrum and OPE coefficients.

- We can regard $\hat{\Psi}_{\nu,J}^{(s)}(x_i)$ as orthonormal basis satisfying orthogonality condition:

$$\left(\hat{\Psi}_{\nu,J}^{(s)}(x_i), \hat{\Psi}_{\nu',J'}^{(s)}(x_i)\right) = \int \frac{\prod_{i=1}^{4} d^d x_i}{\mathrm{Vol}(SO(1,d+1))} \hat{\Psi}_{\nu,J}^{(s)}(x_i) \overline{\hat{\Psi}_{\nu',J'}^{(s)}(x_i)} = \frac{1}{2} \left(\delta(\nu+\nu') + \delta(\nu-\nu')\right) \delta_{J,J'},$$

  where $\bar{\Psi}$ implies $\Delta_i \to \bar{\Delta}_i = d - \Delta_i$, $\mathrm{Vol}(\mathrm{SO}(1, \mathrm{d}+1))$ is the volume of conformal transformation for gauge fixing. We can have similar expansion and orthogonal basis for t- and u-channels.

- We can think of crossing kernel as the expansion coefficients/mixing matrix when changing the basis, e.g.:

$$\hat{\Psi}^{(\mathrm{t})}_{\nu,J}(x_i) = \sum_{J'=0}^{\infty} \int_{-\infty}^{\infty} d\nu' \mathbf{K}^{(\mathrm{ts})}_{JJ'}(\nu,\nu') \hat{\Psi}^{(\mathrm{s})}_{\nu',J'}(x_i)$$

The crossing kernel is defined as the overlap of the different basis:

$$\mathbf{K}^{(\mathrm{ts})}_{J,J'}(\nu,\nu') = \left(\hat{\Psi}^{(\mathrm{t})}_{\nu,J}(x_i), \hat{\Psi}^{(\mathrm{s})}_{\nu',J'}(x_i)\right) = \frac{1}{\mathcal{N}_d} \int \frac{\prod_{i=1}^{4} d^d x_i}{\mathrm{Vol}(SO(1,d+1))} \Psi^{(\mathrm{t})}_{\nu,J}(x_i) \overline{\Psi^{(\mathrm{s})}_{\nu',J'}(x_i)}.$$

Can directly relate $\hat{\Psi}^{(s),\Delta_i}_{\nu,J}(x_i)$ and $\hat{\Psi}^{(t),\Delta_i}_{\nu',J'}(x_i)$ using crossing kernel.

- Moreover we can also recast the crossing equation as relation between spectral functions in different OPE channels:

$$\hat{b}^{(\mathrm{s})}_{J}(\nu) = \sum_{J'=0}^{\infty} \int_{-\infty}^{\infty} d\nu' \, \hat{b}^{(t)}_{J'}(\nu') \mathbf{K}^{(\mathrm{ts})}_{JJ'}(\nu',\nu)$$

The simplest case is the 1-dim. crossing kernel, in this case quadratic Casimir equation for $SL(2,\mathbb{R})$ simplifies to ODE [Van Rees-Hogervorst, 2017] :

$$D_z(a^{(s)}, b^{(s)}, 0)g_\lambda^{(s)}(z) = \lambda(\lambda-1)g_\lambda^{(s)}(z), \quad z = \frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}$$
$$g_\lambda^{(s)}(z) = z^\lambda {}_2F_1(a^{(s)}+\lambda, b^{(s)}+\lambda, 2\lambda, z).$$

To cure singular behavior $z=1$ for ${}_2F_1(a,b,c,z)$, instead we consider symmetric combination:

$$\Psi_\alpha^{(s)}(z) = Q_s(+\alpha)g^{(s)}_{\frac{1}{2}+\alpha}(z) + Q_s(-\alpha)g^{(s)}_{\frac{1}{2}-\alpha}(z), \quad Q_s(\pm\alpha) = \frac{\Gamma(\mp 2\alpha)\Gamma(1+a^{(s)}+b^{(s)})}{\Gamma(\frac{1}{2}\mp\alpha+a^{(s)})\Gamma(\frac{1}{2}\mp\alpha+b^{(s)})}$$

where $\alpha \in i\mathbb{R}$ acts as spectral parameter. The orthogonality condition is:

$$(\Psi_\alpha^{(s)}(z), \Psi_\beta^{(s)}(z)) = \int_0^1 dz w_s(z)\bar{\Psi}_\alpha^{(s)}(z)\Psi_\beta^{(s)}(z) = N_s(\alpha)\delta(\alpha-\beta), \quad w_s(z) = \frac{(1-z)^{a^{(s)}+b^{(s)}}}{z^2}$$

- We can now expand an arbitrary function $f(z)$ using this basis:

$$f(z) = \int_{-i\infty}^{+i\infty} \frac{d\alpha}{2\pi i N_s(\alpha)} \hat{f}^{(s)}(\alpha)\Psi_\alpha^{(s)}(z) \Longleftrightarrow \hat{f}^{(s)}(\alpha) = \int_0^1 dz w_s(z) f(z) \Psi_\alpha^{(s)}(z)$$

where $N_s(\alpha) = \frac{|Q_s(\alpha)|^2}{2}$. $\hat{f}_s(\alpha)$ is the Jacobi transformation of $f(z)$.

- Similar basis can be defined for other channels, this allows us to define the crossing kernel from crossing equation:

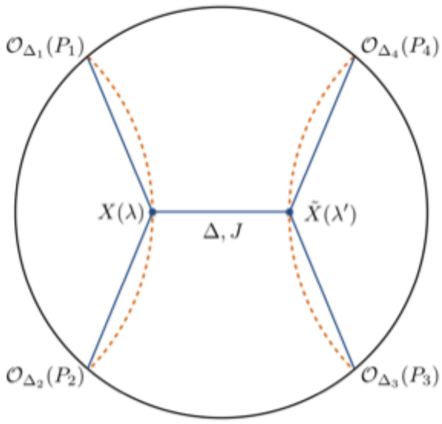
$$\left(\frac{z}{1-z}\right)^{\Delta_2} \Psi_\beta^{(t)}(1-z) = \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i N_s(\alpha)} K(\beta,\alpha;\Delta_i)\Psi_\alpha^{(s)}(z)$$

The final answer is expressed in terms of Wilson function $\mathbf{W}(\alpha,\beta;\Delta_i)$ which is a linear combination of ${}_4F_3$ contains no poles in spectral parameters α and β:

$$K(\beta,\alpha;\Delta_i) = \Gamma\left(1+a^{(s)}+b^{(s)}\right)\Gamma\left(1+a^{(t)}+b^{(t)}\right)\Gamma\left(\frac{\Delta_1+\Delta_2}{2}-\frac{1}{2}\pm\alpha\right)\Gamma\left(-\frac{(\Delta_1+\Delta_4)}{2}+\frac{3}{2}\mp\beta\right)\mathbf{W}(\alpha,\beta;\Delta_i)$$

One natural application for crossing kernel is in AdS_{d+1}/CFT_d:

- The holographic dual configuration for the basis $\hat{\Psi}_{\Delta,J}(x_i)$: [Hijano et al 2015]



  Namely so-called "Geodesic Witten Diagram" (GWD).

- Quick Justification: What else? Other than entire AdS_{d+1}, the only possible remaining trajectories preserving the AdS-isometries/conformal symmetries are the "geodesics" connecting the pairs of CFT primaries inserted at $\{P_i\}$.

- Using kinematical GWDs, we can decompose single s-channel spin-J exchange Witten Diagram (WD) containing dynamical information about large *N* CFTs into infinite sum of GWDs for single and double trace operators [Chen et al 2017]:

$$\mathcal{I}^{4\mathrm{pt}}_{\mathrm{WD}}(P_i) = a_{\Delta,J}\mathcal{W}_{\Delta,J}(P_i)+\sum_{l=0}^{J}\left(\sum_{m_l=0}^{\infty} b_{m_l}\mathcal{W}_{\Delta_1+\Delta_2+l+2m_l,l}(P_i) + \sum_{n_l=0}^{\infty}\tilde{b}_{n_l}\mathcal{W}_{\Delta_3+\Delta_4+l+2n_l,l}(P_i)\right)$$

  This allows us to identify its precise contributions to the (12)(34) OPE coefficients of the four point CFT correlation function.

- For a given s, t, u-channel, the corresponding GWDs is the natural kinematic basis for expanding the Witten diagram in *that* channel.

- However, to identify the individual s, t, u-channel WD contributions to dynamical CFT data $\{\Delta\}$ or $\{\lambda_{12\mathcal{O}}\}$, it is necessary to recast them into the GWD in *the same* OPE channel, e.g. s-channel for (12)(34).

- Contrast with normalization, the inner product for crossing kernel in *Euclidean* signature is somewhat ambiguous, as $0 \leq \mathrm{u}, \mathrm{v} \leq \infty$, we necessarily cross the branch cuts of ${}_2F_1$ in either *s*- or *t*-channels.
- Instead we Wick rotate into *Lorentzian* signature [Caron-Huot, Simmons-Duffin-Stanford-Witten 2017]:

$$\left(\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle\,, \Psi^{(\mathrm{s})}_{i(h-\Delta),J}(x_i)\right) = \alpha^{(\mathrm{s})}_{\Delta,J}\Big[(-1)^J \int_0^1\int_0^1 \frac{dzd\bar{z}}{(z\bar{z})^d}|z-\bar{z}|^{d-2} G^{(\mathrm{s}),\bar{\Delta}_i}_{\tilde{\Delta},\tilde{J}}(z,\bar{z})\frac{\langle[\mathcal{O}_3,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_4]\rangle}{\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i)}$$
$$+\int_{-\infty}^0\int_{-\infty}^0 \frac{dzd\bar{z}}{(z\bar{z})^d}|z-\bar{z}|^{d-2}\hat{G}^{(\mathrm{s}),\bar{\Delta}_i}_{\tilde{\Delta},\tilde{J}}(z,\bar{z})\frac{\langle[\mathcal{O}_4,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_3]\rangle}{\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i)}\Big]$$

where the s-channel conformal block now carries $\tilde{\Delta} = J + (d-1)$ and $\tilde{J} = \Delta - (d-1)$, and double commutator/discontinuity are:

$$\frac{\langle[\mathcal{O}_3,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_4]\rangle}{\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i)} = -2\mathrm{dDisc}_{\mathrm{t}}\left[\mathcal{G}^{(\mathrm{s})}(z,\bar{z})\right]$$
$$= -2\cos\pi\left(\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})}\right)\mathcal{G}^{(\mathrm{s})}(z,\bar{z}) + e^{i\pi(\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})})}\mathcal{G}^{(\mathrm{s}),\mathrm{ccw}}(z,\bar{z}) + e^{-i\pi(\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})})}\mathcal{G}^{(\mathrm{s}),\mathrm{cw}}(z,\bar{z})\,,$$
$$\frac{\langle[\mathcal{O}_4,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_3]\rangle}{\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i)} = -2\mathrm{dDisc}_{\mathrm{u}}\left[\mathcal{G}^{(\mathrm{s})}(z,\bar{z})\right]$$
$$= -2\cos\pi\left(\mathrm{a}^{(\mathrm{s})}-\mathrm{b}^{(\mathrm{s})}\right)\mathcal{G}^{(\mathrm{s})}(z,\bar{z}) + e^{i\pi(\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})})}\mathcal{G}^{(\mathrm{s}),\mathrm{cw}}(z,\bar{z}) + e^{-i\pi(\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})})}\mathcal{G}^{(\mathrm{s}),\mathrm{ccw}}(z,\bar{z})\,,$$

- Now applying to crossing kernel, the double discontinuities for $\hat{\Psi}^{(\mathrm{t})}_{\nu,J}(x_i)$ are [Sleight-Taronna, Liu et. al., Cardona-Sen et a 2018I, Chen-Kyono To Appear]:

$$-2\mathrm{dDisc_t}\left[\frac{\Psi^{(\mathrm{t})}_{\nu,J}(x)}{\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i)}\right] = -4\sum_{\sigma_t=\pm}\sin\pi(\tau_{\sigma_t\nu}-\frac{1}{2}\Delta^+_{14})\sin\pi(\tau_{\sigma_t\nu}-\frac{1}{2}\Delta^+_{23})\mathbf{c}^{(\mathrm{t})}_{h+i\sigma_t\nu,J}G^{(\mathrm{t})}_{h+i\sigma_t\nu,J}(z,\bar{z}),$$

$$-2\mathrm{dDisc_u}\left[\frac{\Psi^{(\mathrm{t})}_{\nu,J}(x)}{\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i)}\right] = 0.$$

They can be computed from the monodromies around $\bar{z}=1$ and $\bar{z}=\infty$ using the explicit basis we found earlier.

- The crossing kernel is now expressed through the following integral:

$$\mathbf{K}^{(\mathrm{ts})}_{J,J'}(\nu,i(\Delta'-h)) = \left(\hat{\Psi}^{(\mathrm{t})}_{\nu,J}(x_i),\hat{\Psi}^{(\mathrm{s})}_{i(\Delta'-h),J'}(x_i)\right)$$
$$= -2\frac{\alpha^{(\mathrm{s})}_{\Delta',J'}(-1)^{J'}}{\mathcal{N}_d}\int_0^1\int_0^1\frac{dzd\bar{z}}{(z\bar{z})^d}|z-\bar{z}|^{d-2}[(1-z)(1-\bar{z})]^{\mathrm{a}^{(\mathrm{s})}+\mathrm{b}^{(\mathrm{s})}}G^{(\mathrm{s})}_{\tilde{\Delta}',\tilde{J}'}(z,\bar{z})\mathrm{dDisc_t}\left[\frac{\Psi^{(\mathrm{t})}_{\nu,J}(x)}{\mathcal{T}^{(\mathrm{s})}_{\Delta_i}(x_i)}\right]$$

where we have also used the shadow identity for s-channel block.

- For fixed (n_s, n_t), using Mellin-Barnes representation for ${}_2\mathbf{F}_1$, the $(z, \bar{z})$ integration is given by Selberg integral:

$$S_2(\alpha,\beta,\gamma) = \int_0^1 dz \int_0^1 d\bar{z}(z\bar{z})^{\alpha-1}[(1-z)(1-\bar{z})]^{\beta-1}|z-\bar{z}|^{2\gamma} = \frac{\Gamma(\alpha)\Gamma(\alpha+\gamma)\Gamma(\beta)\Gamma(\beta+\gamma)\Gamma(1+2\gamma)}{\Gamma(\alpha+\beta+\gamma)\Gamma(\alpha+\beta+2\gamma)\Gamma(1+\gamma)}$$

- The remaining s and t integrations reduce to picking up residues at $s, t = 0, -1, -2, \ldots$ as before:

$$\mathbf{K}^{(\mathrm{ts})}_{J,J'}(\nu, i(\Delta'-h)) = -4\frac{\alpha^{(\mathrm{s})}_{\Delta',J'}(-1)^{J'}}{\mathcal{N}_d \mathbf{c}^{(\mathrm{s})}_{\tilde{\Delta}',\tilde{J}'}} \sum_{\sigma_t=\pm} \sin\pi(\tau_{\sigma_t\nu} - \frac{1}{2}\Delta^+_{14}) \sin\pi(\tau_{\sigma_t\nu} - \frac{1}{2}\Delta^+_{23})$$
$$\times \widetilde{\sum_{r,k}\sum_{r',k'}} \prod_{i=1}^{4} \left(1-\gamma_i^{(\mathrm{t})}\right)_{J-r-\sum_j k_{ji}} \left(1-\tilde{\gamma}_i^{(\mathrm{s})}\right)_{\tilde{J}'-r'-\sum_j k'_{ji}} \sum_{n_s,n_t=0}^{\infty} \frac{(-1)^{n_t+n_s}}{n_t! n_s!} \Gamma(-i\sigma_t\nu - n_t)\Gamma(\tilde{\Delta}' - h - n_s)$$
$$\times \sum_{(ij)=\{12,34\}} \sum_{(i'j')=\{14,23\}} \mathcal{I}^{(\mathrm{ts})(kr;k'r')}_{(ij)(i'j')}(\sigma_t\nu, i(\tilde{\Delta}'-h); n_t, n_s)$$

- It turn out the total residues are given in terms of a special case of *Kampẽ de Fẽriet* function:

$$\mathcal{I}^{(\mathrm{ts})(kr;k'r')}_{(ij)(i'j')}(\sigma_t\nu, i(\tilde{\Delta}'-h); n_t, n_s) = \frac{\pi^2}{\sin\pi(1+\varpi^{(\mathrm{t})}_{ij})\sin\pi(1+\varpi^{(\mathrm{s})}_{i'j'})}\frac{\Gamma(2h-1)}{\Gamma(h)}$$
$$\times\tilde{\mathbb{F}}^{0,4}_{2,1}\left[\begin{matrix}\cdot:\ 1+\tilde{\zeta}_{ij;n_sr'}, h+\tilde{\zeta}_{ij;n_sr'}, \kappa^{i(\mathrm{t})}_{\sigma_t\nu}+n_t+r, \kappa^{j(\mathrm{t})}_{\sigma_t\nu}+n_t+r; 1+\eta^{\sigma_t\nu}_{i'j';n_tr}, h+\eta^{\sigma_t\nu}_{i'j';n_tr}, \kappa^{i'(\mathrm{s})}(\tilde{\tau}')+n_s+r', \kappa^{j'(\mathrm{s})}(\tilde{\tau}')+n_s+r' \\ 1+h+\tilde{\zeta}_{ij;n_sr'}+\eta^{\sigma_t\nu}_{i'j';n_tr}, 2h+\tilde{\zeta}_{ij;n_sr'}+\eta^{\sigma_t\nu}_{i'j';n_tr}\ :\ 1+\varpi^{(\mathrm{t})}_{ij}; 1+\varpi^{(\mathrm{s})}_{i'j'}\end{matrix}; 1, 1\right]$$

$$\tilde{\zeta}_{ij;n_sr'} = \tilde{\tau}'+k_{ij}-\frac{1}{2}\bar{\Delta}^{+}_{ij}+n_s+r', \quad \eta^{\sigma_t\nu}_{i'j';n_tr} = \tau_{\sigma_t\nu}+k'_{i'j'}-\frac{1}{2}\hat{\Delta}^{+}_{i'j'}+n_t+r.$$

which are further generalization of Appell's hypergeometric function:

$$\mathbb{F}^{p,q}_{r,s}\left[\begin{matrix}a_1,\ldots,a_p:\ b_1,\ldots,b_q;b'_1,\ldots,b'_q \\ c_1,\ldots,c_r:\ d_1,\ldots,d_s;d'_1,\ldots,d'_s\end{matrix}; x, y\right] = \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a_1)_{m+n}\cdots(a_p)_{m+n}}{(c_1)_{m+n}\cdots(c_r)_{m+n}}\frac{(b_1)_m\cdots(b_q)_m(b'_1)_n\cdots(b'_q)_n}{(d_1)_m\cdots(d_s)_m(d'_1)_n\cdots(d'_s)_n}\frac{x^my^n}{m!n!},$$

$$\tilde{\mathbb{F}}^{p,q}_{r,s}\left[\begin{matrix}a_1,\ldots,a_p:\ b_1,\ldots,b_q;b'_1,\ldots,b'_q \\ c_1,\ldots,c_r:\ d_1,\ldots,d_s;d'_1,\ldots,d'_s\end{matrix}; x, y\right] = \frac{\prod_{i=1}^{p}\Gamma(a_i)\prod_{j=1}^{p}\Gamma(b_j)\Gamma(b'_j)}{\prod_{k=1}^{r}\Gamma(c_k)\prod_{l=1}^{s}\Gamma(d_l)\Gamma(d'_l)}\mathbb{F}^{p,q}_{r,s}\left[\begin{matrix}a_1,\ldots,a_p:\ b_1,\ldots,b_q;b'_1,\ldots,b'_q \\ c_1,\ldots,c_r:\ d_1,\ldots,d_s;d'_1,\ldots,d'_s\end{matrix}; x, y\right],$$

The pole structures are given by the Γ-function pre-factors, in particular they contain double trace operators.

- Applications to AdS/CFT to directly extract explicit AdS predictions to compare with CFT?
- Testing crossing kernel with few existing simple examples, where spectra in certain channel are available? Ising model?
- Application of crossing kernel to Mellin bootstrap by relating the Mellin amplitudes? [Gopakumar-Sinha 2018]