Discrete Painlevé system and the double scaling limit of the matrix model for irregular conformal block and gauge theory

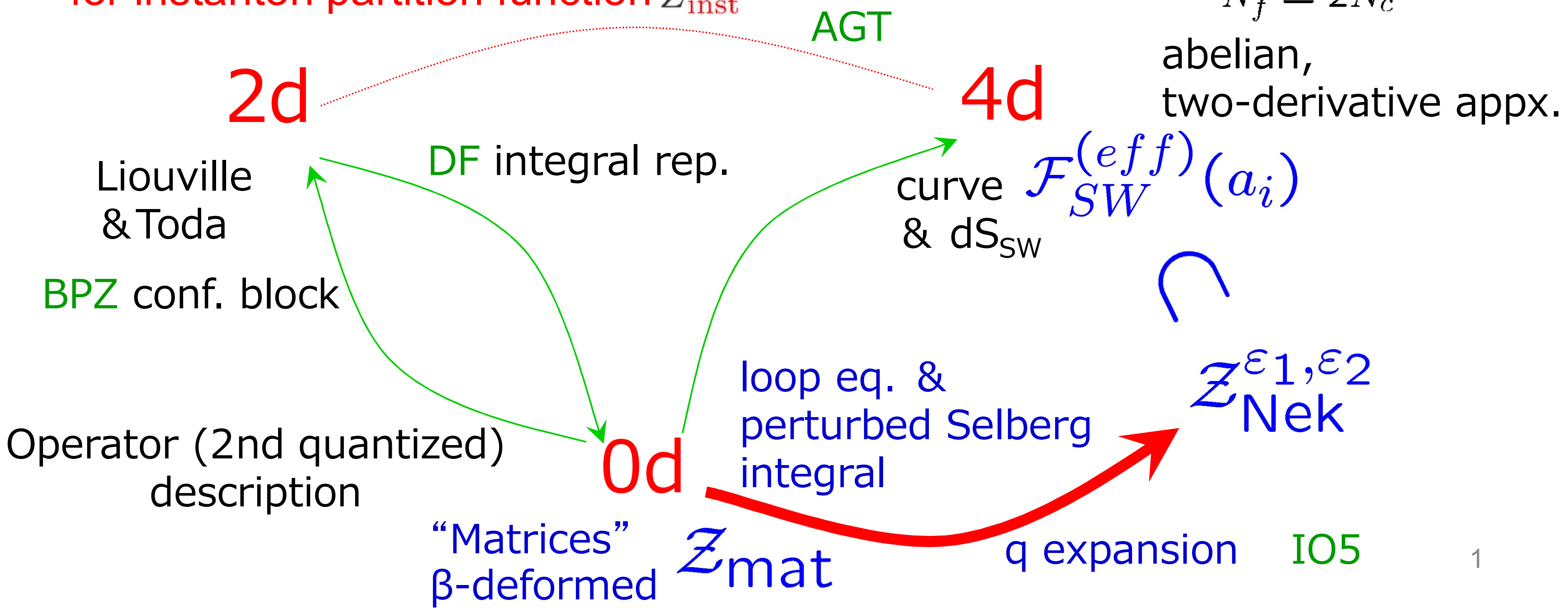
H.I., T. Oota, Katsuya Yano, arXiv:1805.05057 to appear in PLB
and the one in preparation

also IO5: I-Oota, arXiv:1003.2929, Nucl. Phys. B
IOYone: I-Oota-Yonezawa, arXiv:1008.1861, Phys. Rev. D

I) Introduction & punchline:

- 2d-4d conformal connection through matrices
  for instanton partition function Z_{inst}

$\mathcal{N} = 2, \quad SU(N_c)$
$N_f = 2N_c$



- In this talk, we start out from the simplest prototypical example of the one-matrix model representing the SU(2) $N_f=2$ instanton partition function Z_{inst} and the irregular conformal block as the limiting cases of the above picture.

$$m_3,\ m_4 \to \infty$$

- Our goal is the derivation of discrete Painlevé system associated with this matrix model and the double scaling limit via the method of orthogonal polynomial, but for this to be true, we find it necessary to work on another partition function $\underline{Z}$ with the same contour for all integrations and having one less parameters (hence ignoring the filling fraction)

- The list:

$$P_{\mathrm{I}}: \quad y'' = 6y^2 + t,$$

$$P_{\mathrm{II}}: \quad y'' = 2y^3 + ty + \alpha,$$

$$P_{\mathrm{III}}: \quad y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},$$

$$P_{\mathrm{IV}}: \quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y},$$

$$P_{\mathrm{V}}: \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1},$$

$$P_{\mathrm{VI}}: \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).$$

- Painlevé history in Physics
  - Ising correlation fn. Barouch-McCoy-Wu ('73) …
  - matrix model for 2d gravity Brezin-Kazakov, Douglas-Shenker, Gross-Migdal ('90)
  - matrix model for 2d-4d (non)conformal connection

    Nf=4, CFT Gamayun-Iorgov-Lisovyy ('12)

    …

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III) $N_f = 4 \rightarrow 3 \rightarrow 2$ and from Z_{inst} to $\underline{Z} = Z_{U(N)}$ with log potential

IV) method of orthogonal polynomial and discrete Painlevé system

V) double scaling limit and PII

II) Quite generically, the partition function of the β-deformed one-matrix model corresponds to 4d, $\mathcal{N}=2$, SU(2) gauge theories with N_f hypermultiplets is

$$Z_{\rm inst}^{(N_f)} = \mathcal{N}_{(N_f)} \left(\prod_{I=1}^{N} \int_{\mathcal{C}_I^{(N_f)}} dw_I \right) \Delta(w)^{2\beta} \exp\left(\sqrt{\beta} \sum_{I=1}^{N} W^{(N_f)}(w_I) \right).$$

For N_f=4, choose as

$$W^{(4)}(w) = \alpha_1 \log(w) + \alpha_2 \log(w-q_0) + \alpha_3 \log(w-1).$$

In order for this to represent the instanton sum $Z_{\rm inst}$ from the BPZ 4pt conformal block, we need to specify "the filling fraction", i. e.

$N = N_L + N_R,$ $\quad N_L$ of $\mathcal{C}_I^{(4)}$ paths are $[0, q_0]$

N_R of $\mathcal{C}_J^{(4)}$ paths are $[1, \infty]$

$$c = 1 - 6Q_E^2, \quad Q_E = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}$$

seven parameters $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \beta,\ N_L,\ N_R$ with $\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\sqrt{\beta}N = 2\,Q_E.$

gauge theory side: $\dfrac{\epsilon_1}{g_s}, \dfrac{a}{g_s}, \dfrac{m_1}{g_s}, \dfrac{m_2}{g_s}, \dfrac{m_3}{g_s}, \dfrac{m_4}{g_s},$

0d-4d dictionary from the lowest order in the q_0 expansion: IO5

$$\sqrt{\beta}N_L = \frac{a-m_2}{g_s}, \qquad \alpha_1 = \frac{1}{g_s}(m_2 - m_4 + \epsilon), \quad \alpha_2 = \frac{1}{g_s}(m_2+m_4),$$

$$\sqrt{\beta}N_R = -\frac{a+m_1}{g_s}, \quad \alpha_3 = \frac{1}{g_s}(m_1+m_3), \qquad \alpha_4 = \frac{1}{g_s}(m_1 - m_3 + \epsilon).$$

- **The “right” object**

to study in this work is not Z_{inst}, but

$$\underline{Z}_N(\mu_L, \mu_R) \equiv \sum_{\substack{N_L, N_R \geq 0 \\ N_L + N_R = N}} (\mu_L)^{N_L} (\mu_R)^{N_R} \frac{Z_{\text{inst}}(N_L, N_R)}{N_L! N_R!}$$

$$= (\text{const}) \frac{1}{N!} \int_C \cdots \int_C (\text{the same integrand})$$

Here

$$\int_C = \mu_L \int_{C_L} + \mu_R \int_{C_R},$$

namely the same contour C for all integrations.

This is the recipe suggested in the recent Painlevé ref.

Gamayun-Iorgov-Lisovyy, Bershtein-Shchechkin, Gavrylenko-Lisovyy, ...,

in particular, the argument in

Mironov-Morozov, 1707.02443

- Furthermore, one can set, for instance,

$$\mu_L = 1, \mu_R = 0 \quad \Leftrightarrow \quad N_L = N, N_R = 0,$$

which amounts to ignoring the filling fraction, and working with one less parameter.

III)

Start from $N_f = 4$ Z_{inst}

IOYone

- $N_f = 3$ limit: $m_4 \to \infty$ with $\Lambda_3 \equiv 4q_0 m_4$ fixed,

  namely, the $q_0 \to 0$ limit with $2q_{03} \equiv q_0(-\alpha_1 + \alpha_2) = \frac{\Lambda_3}{2g_s}$

  and $\alpha_{1+2} \equiv \alpha_1 + \alpha_2$ fixed:

$$\alpha_{1+2} + \alpha_3 + \alpha_4 + 2\sqrt{\beta}N = 2Q_E,$$

$$W^{(3)}(w) = \alpha_{1+2} \log w + \alpha_3 \log(w-1) - \frac{q_{03}}{w}.$$

- $N_f = 2$ limit: $m_3 \to \infty$ with $\Lambda_2 \equiv (m_3 \Lambda_3)^{1/2}$ fixed,

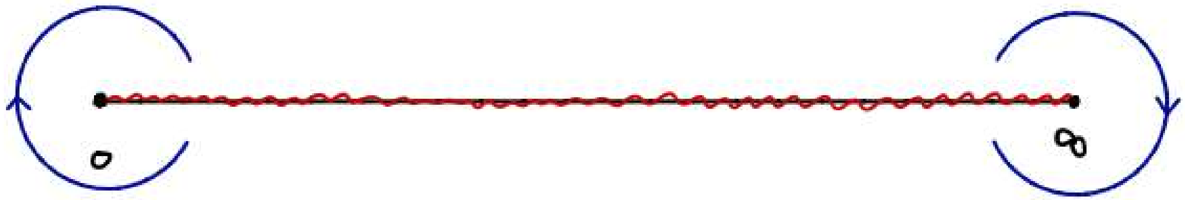
  namely, the $q_{03} \to 0$ limit with $q_{02}^2 \equiv \frac{1}{2} q_{03}(\alpha_3 - \alpha_4) = (\frac{\Lambda_2}{2g_s})^2$

  and $\alpha_{3+4} \equiv \alpha_3 + \alpha_4$ fixed:

$$\alpha_{1+2} + \alpha_{3+4} + 2\sqrt{\beta}N = 2Q_E,$$

$$W^{(2)}(w) = \alpha_{3+4} \log w - q_{02}\left(w + \frac{1}{w}\right).$$

- $N_f = 2$ contour: omitting the argument, IOYone



The integration contours of $N_f = 2$ matrix model. The contour C_R reduces to C_∞.

- parameter counting
  - originally 6 net parameters at N_f=4, letting q_0 aside
  - now 4, aside q_{02} or Λ_2
  - $\beta = 1$ & ignoring the filling fraction, $\therefore$ 2

These are 2d or m. m. 4d gauge

$$N = -(m_1 + m_2)/g_s$$

$$M \equiv \alpha_{3+4} + N = (m_1 - m_2)/g_s$$

From $Z_{\rm inst}$ to $\underline{Z}$ = unitary m. m. with log potential

From now on, we set $\beta=1$, $N_f=2$, and

M to be an integer

The recipe at $N_f = 4$ with $\mu_L = 1$, $\mu_R = 0$ $\Longrightarrow$

amounts to ignoring the filling fraction and working in the same contour $C=C_L$=unit circle for all integrations and having one-less parameters:

$$\underline{Z}_{U(N)} = \frac{(-1)^{(1/2)N(N-1)}}{N!}\left(\prod_{I=1}^{N}\oint_C \frac{\mathrm{d}w_I}{2\pi\mathrm{i}}\right)\Delta(w)^2\,\exp\left(\sum_{I=1}^{N} W(w_I)\right)$$

$$= \frac{1}{N!}\left(\prod_{I=1}^{N}\oint_C \frac{\mathrm{d}w_I}{2\pi\mathrm{i}\,w_I}\right)\Delta(w)\Delta(w^{-1})\exp\left(\sum_{I=1}^{N} W_U(w_I)\right),$$

$$W_U(w) = W^{(2)}(w) + N\log w = -q_{02}\left(w+\frac{1}{w}\right) + M\log w$$

IV)

Let $$\mathrm{d}\mu(z) := \frac{\mathrm{d}z}{2\pi\mathrm{i}\, z}\exp\Big(W_U(z)\Big).$$

Then $$\underline{Z}_{U(N)} = \frac{1}{N!}\int\prod_{i=1}^{N}\mathrm{d}\mu(z_i)\,\Delta(z)\Delta(z^{-1}).$$

- **monic orthogonal polynomials**
  - **Definitions and properties**

$$\int \mathrm{d}\mu(z)p_n(z)\tilde{p}_m(1/z) = h_n\delta_{n,m},$$

where
$$p_n(z) = z^n + \sum_{k=0}^{n-1}A_k^{(n)}z^k, \qquad \tilde{p}_n(1/z) = z^{-n} + \sum_{k=0}^{n-1}B_k^{(n)}z^{-k}.$$

Introduce the moments μ_n for the measure
$$\mu_n := \int \mathrm{d}\mu(z)z^n, \qquad (n\in\mathbb{Z}).$$

Define $\mathcal{K}_k^{(n)}$ by $\mathcal{K}_k^{(n)} := \det\big(\mu_{j-i+k}\big)_{1\le i,j\le n}, \qquad (n\ge 0, k\in\mathbb{Z}).$

Then
$$p_n(z) = \frac{1}{\tau_n}\begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \mu_{-n+3} & \cdots & \mu_1 \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix}, \qquad \tilde{p}_n(1/z) = \frac{1}{\tau_n}\begin{vmatrix} \mu_0 & \mu_{-1} & \mu_{-2} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \mu_{-1} & \cdots & \mu_{-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \cdots & \mu_{-1} \\ 1 & z^{-1} & z^{-2} & \cdots & z^{-n} \end{vmatrix},$$

where $\tau_n := \mathcal{K}_0^{(n)} = \det\big(\mu_{j-i}\big)_{1\le i,j\le n}.$

The normalization constants h_n are given by $h_n = \frac{\tau_{n+1}}{\tau_n} = \frac{\mathcal{K}_0^{(n+1)}}{\mathcal{K}_0^{(n)}}.$

The constant terms of these polynomials will play important roles.

$$A_n := p_n(0) = A_0^{(n)} = (-1)^n \frac{\mathcal{K}_1^{(n)}}{\mathcal{K}_0^{(n)}}, \qquad B_n := \tilde{p}_n(0) = B_0^{(n)} = (-1)^n \frac{\mathcal{K}_{-1}^{(n)}}{\mathcal{K}_0^{(n)}}.$$

Using $\tau_n^2 - \tau_{n+1}\tau_{n-1} = \left(\mathcal{K}_0^{(n)}\right)^2 - \mathcal{K}_0^{(n+1)}\mathcal{K}_0^{(n-1)} = \mathcal{K}_1^{(n)}\mathcal{K}_{-1}^{(n)},$

we obtain $\frac{h_n}{h_{n-1}} = 1 - A_n B_n.$

- **Partition function and the orthogonal polynomials**

The partition function is evaluated as

$$\underline{Z}_{U(N)} = \frac{1}{N!}\int \prod_{i=1}^{N} \mathrm{d}\mu(z_i)\,\Delta(z)\Delta(z^{-1}) = \prod_{k=0}^{N-1} h_k = \prod_{k=0}^{N-1} \frac{\tau_{k+1}}{\tau_k} = \tau_N.$$

Also $\underline{Z}_{U(N)} = h_0^N \prod_{j=1}^{N-1} (1 - A_j B_j)^{N-j}.$

- **Recursion relations for orthogonal polynomials**

The orthogonal polynomials $p_n(z)$ obey the following relations:

$$z\, p_n(z) = p_{n+1}(z) + \sum_{k=0}^{n} C_k^{(n)}\, p_k(z),$$

where $C_k^{(n)} = (-1)^{n-k}\frac{\mathcal{K}_1^{(n+1)}\mathcal{K}_{-1}^{(k)}}{\mathcal{K}_0^{(n)}\mathcal{K}_0^{(k+1)}} = -\frac{h_n}{h_k}A_{n+1}B_k, \qquad (0 \le k \le n).$

- **String equations**

  Start from for $k \in \mathbb{Z}$ and $\ell, m \geq 0$,

$$0 = \int \mathrm{d}z \frac{\partial}{\partial z} \left[\frac{z^k}{2\pi\mathrm{i}} \exp\Big(W_U(z)\Big) p_\ell(z) \tilde{p}_m(1/z) \right]$$

$$= \int \mathrm{d}\mu(z)\, z^{k+1}\, W_U'(z) p_\ell(z) \tilde{p}_m(1/z) + \int \mathrm{d}\mu(z)\, z \frac{\partial}{\partial z} \big(p_\ell(z) z^k \tilde{p}_m(1/z)\big),$$

  Consider the following three cases: (i) $(k, \ell, m) = (-1, n, n-1)$, (ii) $(k, \ell, m) = (0, n, n)$ and (iii) $(k, \ell, m) = (1, n-1, n)$.

$$\int \mathrm{d}\mu(z) W_U'(z) p_n(z) \tilde{p}_{n-1}(1/z) = n(h_n - h_{n-1}),$$

$$\int \mathrm{d}\mu(z)\, z\, W_U'(z) p_n(z) \tilde{p}_n(1/z) = 0,$$

$$\int \mathrm{d}\mu(z)\, z^2\, W_U'(z) p_{n-1}(z) \tilde{p}_n(1/z) = -n(h_n - h_{n-1}).$$

- **Unitary matrix model with logarithmic potential**

$$W_U(z) = -\frac{1}{2\,\underline{g}_s} \left(z + \frac{1}{z} \right) + M \log z. \qquad \underline{g}_s = g_s/\Lambda_2, \quad M \text{ integer.}$$

- **Moments and related quantities** We can evaluate

$$\mu_n = (-1)^{M+n} I_{|M+n|}(1/\underline{g}_s),$$

  where $I_\nu(z)$ is the modified Bessel function of the first kind:

  Note that $\mathcal{K}_k^{(n)} = (-1)^{n(M+k)} K_{M+k}^{(n)}$,

  where $K_\nu^{(n)} := \det\Big(I_{j-i+\nu}(1/\underline{g}_s)\Big)_{1 \leq i,j \leq n}, \qquad (\nu \in \mathbb{C}; n = 0, 1, 2, \cdots).$

$$A_n = p_n(0) = (-1)^n \frac{\mathcal{K}_1^{(n)}}{\mathcal{K}_0^{(n)}} = \frac{K_{M+1}^{(n)}}{K_M^{(n)}},$$

$$B_n = \tilde{p}_n(0) = (-1)^n \frac{\mathcal{K}_{-1}^{(n)}}{\mathcal{K}_0^{(n)}} = \frac{K_{M-1}^{(n)}}{K_M^{(n)}}.$$

- **String equations**

$$W_U'(z) = -\frac{1}{2\,\underline{g_s}} \left(1 - \frac{1}{z^2}\right) + \frac{M}{z},$$

the string equations for this potential

$$\frac{1}{2\,\underline{g_s}} \left(\widetilde{C}_n^{(n)} + \widetilde{C}_{n-1}^{(n-1)}\right) + M = n \left(1 - \frac{h_{n-1}}{h_n}\right),$$

$$-\frac{1}{2\,\underline{g_s}} \left(C_n^{(n)} - \widetilde{C}_n^{(n)}\right) + M = 0,$$

$$-\frac{1}{2\,\underline{g_s}} \left(C_n^{(n)} + C_{n-1}^{(n-1)}\right) + M = -n \left(1 - \frac{h_{n-1}}{h_n}\right),$$

Using $\frac{h_n}{h_{n-1}} = 1 - A_n B_n, \qquad C_n^{(n)} = -A_{n+1} B_n, \qquad \widetilde{C}_n^{(n)} = -A_n B_{n+1},$

we obtain $A_{n+1} = -A_{n-1} + \frac{2\,n\underline{g_s}\,A_n}{1 - A_n\,B_n}, \qquad B_{n+1} = -B_{n-1} + \frac{2\,n\underline{g_s}\,B_n}{1 - A_n\,B_n},$

$$A_n B_{n+1} - A_{n+1} B_n = 2\,M\,\underline{g_s}.$$

$$A_n(M) = \frac{K_{M+1}^{(n)}}{K_M^{(n)}}, \qquad B_n(M) = \frac{K_{M-1}^{(n)}}{K_M^{(n)}}, \qquad (M \in \mathbb{C})$$

indeed solve the string equations

- **Equations for $R_n^2 = A_n B_n$**

Let $A_n = R_n D_n$ and $B_n = R_n / D_n$.

$$\underline{Z}_{U(N)} = h_0^N \prod_{j=1}^{N-1} (1 - R_j^2)^{N-j}.$$

The 3rd string eq. turns into $R_n R_{n+1} \left(\frac{D_n}{D_{n+1}} - \frac{D_{n+1}}{D_n} \right) = 2\, M\, \underline{g}_s.$

This leads to

$$\frac{D_n}{D_{n+1}} = \frac{M\, \underline{g}_s + \sqrt{R_n^2\, R_{n+1}^2 + M^2\, \underline{g}_s^2}}{R_n\, R_{n+1}},$$

$$\frac{D_{n+1}}{D_n} = \frac{-M\, \underline{g}_s + \sqrt{R_n^2\, R_{n+1}^2 + M^2\, \underline{g}_s^2}}{R_n\, R_{n+1}}.$$

Substituting these relations into the remaining string eqs, we obtain

$$(1 - R_n^2)\left(\sqrt{R_n^2\, R_{n+1}^2 + M^2\, \underline{g}_s^2} + \sqrt{R_n^2\, R_{n-1}^2 + M^2\, \underline{g}_s^2} \right) = 2\, n\, \underline{g}_s\, R_n^2.$$

This is equivalent to

$$\begin{aligned} 0 =& \eta_n^2 \left[\xi_n^2 (1 - \xi_n)^2 - \eta_n^2\, \xi_n^2 + \zeta^2 (1 - \xi_n)^2 \right] \\ & + \frac{1}{2}\, \eta_n^2\, \xi_n\, (1 - \xi_n)^2 (\xi_{n+1} - 2\, \xi_n + \xi_{n-1}) - \frac{1}{16} (1 - \xi_n)^4 (\xi_{n+1} - \xi_{n-1})^2, \end{aligned}$$

where $\xi_n \equiv R_n^2$, $\eta_n \equiv n\, \underline{g}_s$, $\zeta \equiv M\, \underline{g}_s$.

This deserves the name “discrete Painlevé” as we will see.

- the planar continuum limit

$(\xi_n, \eta_n, \zeta) \to (\xi, \eta, \zeta)$, the 2nd line of the last eq. ignored

3 out of 4 roots in ξ degenerate to zero at $\eta = \pm 1,\ \zeta = 0.$

In fact, $\xi = a^2 U,\ \eta = \pm 1 - (1/2)a^2 t,\ \zeta = \pm a^3 z$. We obtain

$$\pm t = 2U - \frac{z^2}{u^2} \qquad \text{at } \mathcal{O}(a^6)$$

- With the introduction of the homogeneous coordinates

$$(\mathcal{X} : \mathcal{Y} : \mathcal{Z} : \mathcal{W}) = (\xi : \eta : \zeta : 1) \text{ of } \mathbb{P}^3,$$

union of $\mathcal{Y} = 0$ and

$$-\mathcal{Y}^2 \mathcal{X}^2 + \mathcal{X}^2(\mathcal{X} - \mathcal{W})^2 + (\mathcal{X} - \mathcal{W})^2 \mathcal{Z}^2 = 0.$$

singular K3 surface

whose meaning is still not known to us.

- <u>Susceptibility</u>

Look at the critical behavior of free energy:

$$F \equiv -\lim_{N\to\infty} \frac{Z_{U(N)}}{N^2} = -\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N-1} (N-n)\log(1-\xi_n)$$
$$\sim -\int_0^1 (1-x)\log\left(1-\xi(x)\right)\, dx,$$

where $x \equiv n/N$. $\eta = \widetilde{S}x = 1 - a^2 t,\ \widetilde{S} \equiv N\underline{g}_s,$

$$F \sim -\frac{a^2}{\widetilde{S}^2}(1-\widetilde{S}) \int_{a^{-2}}^{t_R} \left(1 - \frac{t}{t_R}\right) \log\left(1 - a^2 U(t;z)\right) dt,$$

$t_R \equiv (1-\widetilde{S})/a^2$ is fixed in the limit $\widetilde{S} \to 1,\ a \to 0.$

$$F \sim \frac{(1-\widetilde{S})^3}{{t_R}^2} \int^{t_R} \left(1 - \frac{t}{t_R}\right) U(t;z)dt = C(t_R;z)(1-\widetilde{S})^3 = C(t_R;z)(1-\widetilde{S})^{2-\gamma},$$

hence $\gamma = -1$ independent of z.

relation to alt-dPII & PIII for generic $N_f = 2$

With the help of referee (anonymous), we could present the following:

Let

$$x_n = A_{n+1}/A_n, \quad y_n = B_{n+1}/B_n$$

- Our recursion relations become known eq called "alt-dPII"

$$\frac{2(n+1)\underline{g}_s}{1+x_n x_{n+1}} + \frac{2n\,\underline{g}_s}{1+x_n x_{n-1}} = -x_n + \frac{1}{x_n} + 2\,n\underline{g}_s - 2M\underline{g}_s,$$
$$\frac{2(n+1)\underline{g}_s}{1+y_n y_{n+1}} + \frac{2n\,\underline{g}_s}{1+y_n y_{n-1}} = -y_n + \frac{1}{y_n} + 2\,n\underline{g}_s + 2M\underline{g}_s.$$

Fokas et al. (1993)

Nijhoff et al. (1996)

Forrester, Witte (2002)

- “alt-dPII” is closely related to (differential) PIII.

$$t = 1/\underline{g}_s^2$$

$$\sigma(t) := -t\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathrm{e}^{-t/4}t^{M^2/4}K_M^{(N)}\right).$$

$\sigma(t)$ satisfies a variant of the Painlevé III equation

$$(t\sigma'')^2 - v_1v_2(\sigma')^2 + \sigma'(4\sigma' - 1)(\sigma - t\sigma') - \frac{1}{64}(v_1 - v_2)^2 = 0,$$

with

$$v_1 = -M + N = -\frac{2m_1}{\underline{g}_s}, \qquad v_2 = M + N = -\frac{2m_2}{\underline{g}_s}.$$

V)

Let $x \equiv n/N$, $a^3 \equiv 1/N$,

$$\eta_n = \widetilde{S}\, x = 1 - (1/2)a^2\, t, \qquad \zeta = a^3\, \widetilde{S}\, M,$$

$$\xi(x) = \xi(n/N) = \xi_n = a^2\, u(t).$$

These define the double scaling limit.

The dressed coupling constant is

$$\kappa \equiv \frac{1}{N} \frac{1}{(1-\widetilde{S})^{1-\frac{\gamma}{2}}}, \qquad \gamma = -1,$$

γ being the susceptibility.

In the d.s.l., the string eq. at the discrete level becomes PII:

$$u'' = \frac{(u')^2}{2\,u} + u^2 - \frac{1}{2}\,t\,u - \frac{M^2}{2\,u},$$

which can be written as a Hamiltonian system:

$p_u = -u'/u$ and

$$H_{\mathrm{II}}(u, p_u; t) = -\frac{1}{2}\,p_u^2\,u + \frac{1}{2}\,u^2 - \frac{1}{2}\,t\,u + \frac{M^2}{2\,u}.$$

By a canonical transformation $(u, p_u) \to (v, p_v)$ with $u = -p_v$ and $p_u = v + (M/p_v)$,

$$H_{\mathrm{II}} = \frac{1}{2}\,p_v^2 + \frac{1}{2}\big(v^2 + t\big)p_v + M\,v,$$

$$v'' = \frac{1}{2}\,v^3 + \frac{1}{2}\,t\,v + \left(\frac{1}{2} - M\right).$$