Topological Strings on Singular Elliptic Calabi-Yau 3-folds and Minimal 6d SCFTs

Minxin Huang

University of Science and Technology of China

East Asian Joint Workshop on Fields and Strings 2018, KIAS

Jie Gu, MH, Amir-Kian Kashani-Poor, Albrecht Klemm, arXiv:1701.00764, JHEP 1705, 130; Michele Del Zotto, Jie Gu, MH, Amir-Kian Kashani-Poor, Albrecht Klemm, Guglielmo Lockhart, arXiv:1712.07017, JHEP 1803, 156.

Introduction

- There have been some resurgent interests in superconformal field theories in 6d, with the amount of supersymmetry $N = (2,0)$ or $(1,0)$, i.e. 16 or 8 real supercharges. They were constructed in string/M theory during 1990's.

- Some salient features: there is no effective Lagrangian due to tensionless strings; falls into discrete families, no continuous marginal deformation; obtain CFTs in lower dimensions by compactification; can be extended to "little string theory"; ADE classification of $(2,0)$ theories.

- What can we learn about these theories? One aspect is to compute the partition functions on $\mathbb{R}^4_\Omega \times T^2$. This is related to topological string partition functions on elliptic Calabi-Yau threefolds.

$(1,0)$ theories

- Classification of $(1,0)$ theories:
  (del Zotto, Heckman, Morrison, Vafa, et al)
  F-theory compactified on a local Calabi-Yau 3-folds X, elliptically fibered over a Kahler base B.
  The theory is characterized by a collection of mutually transversally intersecting holomorphic curves and intersection matrix

$$C_I \subset B\,, \qquad I = 1, \ldots, r + f,$$

  The first r of which are compact, is the rank of the corresponding six-dimensional theory (the tensor branch dimension)
  The remaining f are not compact, representing flavor symmetry.

- We have the intersection pairing

$$A_{IJ} \equiv -C_I \cdot C_J\,, \qquad 1 \leq I, J \leq r \tag{1}$$

  For 6d SCFTs A_{IJ} has to be positive definite,
  While for 6d LSTs A_{IJ} is semi-positive definite, with a single zero eigenvalue, corresponding to the little string charge.

Partition function and Elliptic genus

- The Ramond elliptic genera of the $(0,4)$ worldsheet theory

$$Z_\beta(\tau,\epsilon_1,\epsilon_2,\boldsymbol{m}) \equiv \mathrm{Tr}_R\,(-1)^F q^{H_L}\bar{q}^{H_R} u^{2J_-} v^{2(J_+ + J_r)}\,\mathbf{e}[\boldsymbol{m}]$$

  where $q=\exp(2\pi i\tau), u=\exp(2\pi i\epsilon_-)\ , v=\exp(2\pi i\epsilon_+)\ , \epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$.
  where $\beta \in \Gamma$, the BPS string charge lattice of the theory, from D3-branes wrapped on the curve C_I.

- The 6d Ω-background partition function (= A-model refined topological string partition function on X) has the following expression

$$Z(\epsilon_1,\epsilon_2,\boldsymbol{t}) = Z_0\left(1+\sum_{\beta\in\Gamma}\boldsymbol{Q}_c^\beta Z_\beta(\tau,\epsilon_1,\epsilon_2,\boldsymbol{m})\right),$$

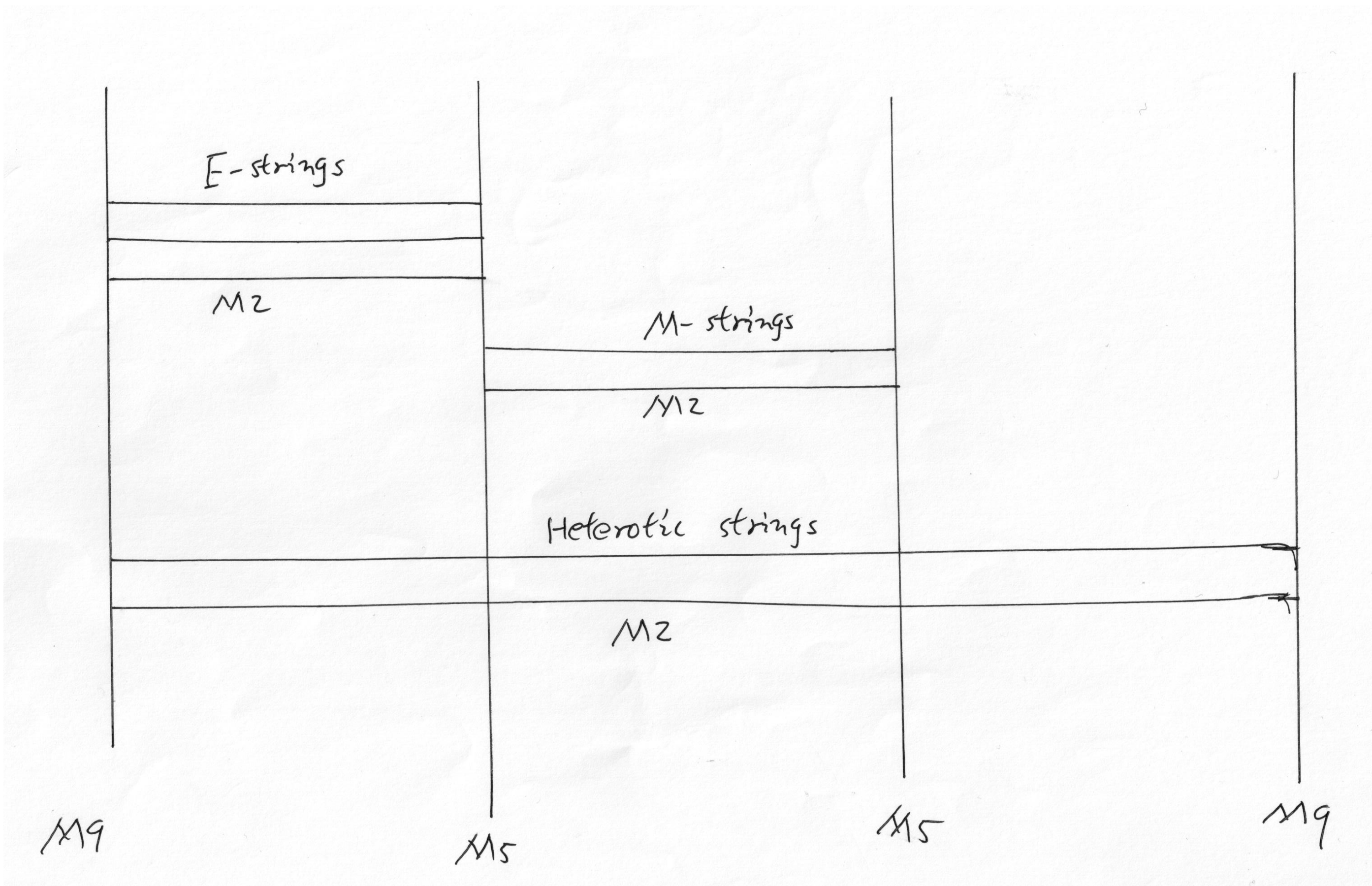
  topological string theory: Q_c^β are exponentials of Kahler parameters of the compact curve classes C_I, $1\le I\le r$; τ Kahler parameter of elliptic fiber; $\boldsymbol{m}$ other Kahler parameters.
  6d partition function: Q_c^β are exponentials of T^2 volume times vev of scalars in 6d $(1,0)$ tensor multiplets; τ complex structure of T^2; $\boldsymbol{m}$ fugacities of global symmetry.

- Minimal 6d SCFT: These are the "atomic" building blocks for $(1,0)$ theories, rank one cases realized as F-theory compactified in elliptic fibration over a base $\mathcal{O}(-n) \to \mathbb{P}^1$, where $n = 1, 2, \cdots, 8$ and $n = 12$. B. Haghighat, A. Klemm, G. Lockhart and C. Vafa, "Strings of Minimal 6d SCFTs," arXiv:1412.3152.

- The singularities in elliptic fibration are classified by Kodaira. In F-theory, we may introduce D7 branes wrapping on singular locus, giving rise to bulk gauge symmetry in 6d theory. This also introduces some extra Kahler moduli parameters in the Calabi-Yau geometries. For $n > 2$ the strings interact with non-Higgsable bulk gauge symmetry.

- Two simplest theories for $n = 1, 2$, with no bulk gauge symmetry. There are dual realizations in M-theory, known as E-strings and M-strings. There are M-theory realizations. E-strings arise as small E_8 instantons in Heterotic string theory, see e.g. O. J. Ganor and A. Hanany, hep-th/9602120.

E, M-strings: brane configuration



	0	1	2	3	4	5	6	7	8	9	10
M9	X	X	X	X	X	X	·	X	X	X	X
M5	X	X	X	X	X	X	·	·	·	·	·
M2	X	X	·	·	·	·	X	·	·	·	·

How to compute elliptic genus

- Use duality with topological strings (refined topological vertex)
  E-strings: topological strings on "half K3" Calabi-Yau.
  M-strings: topological strings on elliptic fibration over $\mathbb{C}^2/A_1$. Haghighat, Iqbal, Kozcaz, Lockhart and Vafa, arXiv:1305.6322

- Localization method. Construct the proper UV world-sheet theory, and use supersymmetric localization to compute the path integral, involving JK residues. e.g. F. Benini, R. Eager, K. Hori and Y. Tachikawa, arXiv:1305.0533, arXiv:1308.4896; Kim, Kim, Lee, Park, Vafa, arXiv:1411.232.

- This talk: modularity ansatz method. Use weak Jacobi forms with the proper modular weight and index, and pole structures. There are only finite number of unknown constants which can be fixed by other methods. Del Zotto, Lockhart arXiv:1609.00310; Jie Gu, MH, Amir-Kian Kashani-Poor, Albrecht Klemm, arXiv:1701.00764

Weak Jacobi Forms

- Consider a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ depend on a modular parameter $\tau \in \mathbb{H}$, an elliptic parameter $z \in \mathbb{C}$. They transform under the modular group as

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}} \varphi(\tau, z), \qquad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})$$

and under translations of the elliptic parameter as

$$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi im(\lambda^2\tau+2\lambda z)} \varphi(\tau, z), \quad \forall \quad \lambda, \mu \in \mathbb{Z} \ .$$

Here $k \in \mathbb{Z}$ is called the *weight* and $m \in \mathbb{Z}$ is called the *index*.

- Due to the periodicity, the function has a Fourier expansion

$$\phi(\tau, z) = \sum_{n,r} c(n,r) q^n y^r, \qquad \text{where} \quad q = e^{2\pi i\tau}, \quad y = e^{2\pi iz}$$

Holomorphic Jacobi form: satisfying $c(n,r) = 0$ unless $4mn \geq r^2$
We use the weak Jacobi form, satisfying $c(n,r) = 0$ unless $n \geq 0$.

- Some weak Jacobi forms can be constructed by theta functions

$$\begin{aligned}\phi_{-2,1}(\tau,z) &= -\frac{\theta_1(z,\tau)^2}{\eta^6(\tau)},\\ \phi_{0,1}(\tau,z) &= 4[\frac{\theta_2(z,\tau)^2}{\theta_2(0,\tau)^2}+\frac{\theta_3(z,\tau)^2}{\theta_3(0,\tau)^2}+\frac{\theta_4(z,\tau)^2}{\theta_4(0,\tau)^2}].\end{aligned}$$

  Here $\phi_{0,1}(\tau,z)$ is the elliptic genus of K3.

- A Theorem (Zagier et al):
  A weak Jacobi form of given index m and even modular weight k is a polynomial of $E_4(\tau)$, $E_6(\tau)$, $\phi_{0,1}(\tau,z)$, $\phi_{-2,1}(\tau,z)$ whose modular weights and indices are $4,6,0,-2$ and $0,0,1,1$ respectively.

The ansatz

- For E-string and M-string

$$Z_\beta = \left(\frac{\sqrt{q}}{\eta(\tau)^{12}}\right)^{-\beta\cdot K_{\breve{B}}} \frac{\phi_{k,n_+,n_-,\beta}(\tau,\boldsymbol{m},\epsilon_+,\epsilon_-)}{\prod_{i=1}^{r}\prod_{s=1}^{\beta_i}\left[\phi_{-1,\frac{1}{2}}(\tau,s\epsilon_1)\phi_{-1,\frac{1}{2}}(\tau,s\epsilon_2)\right]},$$

- Some explanations:

  Here β is the base class. For E-strings and M-strings, only one class in the base $r=1$, and $\beta=n$, the number of strings.

  $\phi_{k,n_+,n_-,\beta}$ a polynomial of appropriate weight and index in $\epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$, and the E_8 Weyl invariant Jacobi forms (E-string) or a Jacobi forms of a mass (M-string) .

  The $\eta(\tau)$ dependent prefactor is argued by a shift of base Kahler class.

  Z_β formally have vanishing weight.

Index from modular anomaly

- Modular forms are generated by Eisenstein series E_4, E_6. The second Eisenstein series E_2 is not exactly a modular form. We usually call the homogeneous polynomials of E_2, E_4, E_6 "quasi-modular" forms, and the dependence on E_2 "modular anomaly".

- The weak Jacobi forms can be expanded in terms of quasi-modular forms

$$\begin{aligned}\phi_{-2,1}(z,\tau) &= -z^2 + \frac{E_2 z^4}{12} + \frac{-5E_2^2 + E_4}{1440} z^6 + \mathcal{O}(z^8),\\ \phi_{0,1}(z,\tau) &= 12 - E_2 z^2 + \frac{E_2^2 + E_4}{24} z^4 + \mathcal{O}(z^6),\end{aligned}$$

  and they satisfy the modular anomaly equation

$$\partial_{E_2}\phi_{-2,1}(z,\tau) = -\frac{z^2}{12}\phi_{-2,1}(z,\tau), \quad \partial_{E_2}\phi_{0,1}(z,\tau) = -\frac{z^2}{12}\phi_{0,1}(z,\tau).$$

- Therefore we can deduce the index of a weak Jacobi form from its modular anomaly.

- E-strings: First proposed by Minahan et al hep-th/9707149. Later generalized to higher genus and refined case S. Hosono, M. H. Saito and A. Takahashi, hep-th/9901151; Huang, Klemm and Poretschkin, arXiv:1308.0619

- M-strings: The refined anomaly equation is derived from refined topological vertex calculations. Haghighat, Iqbal, Kozcaz, Lockhart and Vafa, arXiv:1305.6322

- In both case the anomaly is a quadratic symmetric polynomial of $\epsilon_{1,2}$. So we can determine the modular index for $\epsilon_{\pm}$

$$n_+ = \begin{cases} \frac{n_b}{3}(n_b^2 + 3n_b - 4) & \text{for the E-string,} \\ \frac{n_b}{6}(2n_b^2 + 9n_b - 5) & \text{for the M-string,} \end{cases}$$

and

$$n_- = \begin{cases} \frac{n_b}{3}(n_b^2 - 1) & \text{for the E-string,} \\ \frac{n_b}{6}(2n_b^2 - 3n_b + 1) & \text{for the M-string.} \end{cases}$$

Index from anomaly polynomial

- The anomaly polynomial for 6d SCFT and their 2d world-sheet theories are proposed. Ohmori, Shimizu, Tachikawa, H.-C. Kim, S. Kim, and J. Park et el.

- A dictionary that map the anomaly polynomial to the indices of various elliptic parameters in the elliptic genus. N. Bobev, M. Bullimore, and H.-C. Kim, arXiv:1507.08553

- Kachru et al arXiv:1507.08553 also argues the relation among anomaly coefficients and modular properties of the elliptic genus in a completely different way.

How to fix the ansatz

- A geometric bound (known as Castelnuovo bound): For a given Kahler class, the Gopakumar-Vafa invariants vanish for sufficiently large genus.

- We impose the constrains on the ansatz, and call the remaining ambiguity "restricted ansatz". The upshot: The restricted ansatz has only $A(\tau, z)$ in the denominator while the general ansatz's denominator is $\prod_{k=1}^{n_b} A(\tau, kz)$.

- The modular index in the numerator of the restricted ansatz is actually negative for $n_b > 1$ since $n_- < -1$, so must be zero. In principle we can completely fix the ansatz up to a normalization by these vanishing conditions from Castelnuovo bound.

- A similar argument works for the refined case.

Minimal 6d SCFTs with gauge symmetry

- Our main examples: $n = 3, 4$ in the list of minimal models, with non-Higgsable gauge symmetry $SU(3)$ and $SO(8)$.

- The elliptic genus has been also computed by localization method. $SU(3)$ model, H.-C. Kim, S. Kim, and J. Park, arXiv:1608.03919; $SO(8)$ model, B. Haghighat, A. Klemm, G. Lockhart, and C. Vafa, arXiv:1412.3152.

- The ansatz with Jacobi forms is proposed in M. Del Zotto and G. Lockhart, arXiv:1609.00310. The denominator is more complicated involved a product over simple roots of the Lie algebra. Some simple cases are solved, e.g. with the Kahler parameters corresponding to gauge symmetry set to zero.

- Here we shall tackle the full theory with generic Kahler parameters.

Weyl invariant Jacobi forms

- Let $\mathfrak{g}$ be a simple Lie algebra of rank r with a Cartan sub-algebra $\mathfrak{h}$. Following Wirthmuller 1992, a Weyl-invariant Jacobi form of weight w and index n ($w \in \mathbb{Z}, n \in \mathbb{N}$) is defined to be a holomorphic function $\varphi_{w,n} : \mathbb{H} \times \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ with the various conditions like Modularity, Quasi-periodicity, Weyl symmetry, Fourier expansion. e.g. Modularity, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$

$$\varphi_{w,n}\left(\frac{a\tau+b}{c\tau+d}, \frac{\boldsymbol{z}}{c\tau+d}\right) = (c\tau+d)^w \mathbf{e}\left[\frac{nc(\boldsymbol{z},\boldsymbol{z})_{\mathfrak{g}}}{2(c\tau+d)}\right] \varphi_{w,n}(\tau,\boldsymbol{z}) \ . \qquad (2)$$

The argument $\tau \in \mathbb{H}$ is called the modular parameter, and $\boldsymbol{z} \in \mathfrak{h}_{\mathbb{C}} \cong \mathbb{C}^r$ the elliptic parameter.

- The explicit forms of the ring generators were constructed in Bertola's thesis 2002 for $\mathfrak{g} = \mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{g}_2, \mathfrak{c}_3, \mathfrak{d}_4$.

Affine Weyl invariant Jacobi forms

- The Dynkin diagram is extended to affine Dynkin diagram by an extra node. The modular parameter q is related to the exponential of Kahler parameter Q_0 of the extra node

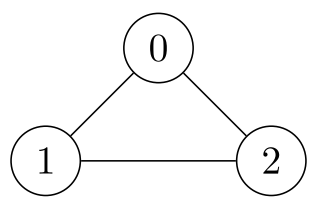
$$q = Q_0 \prod_{i=1}^{r} Q_i^{a_i^\vee}$$

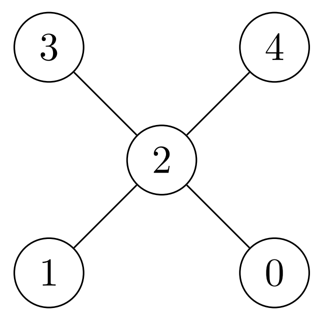
where r is the rank of Dynkin diagram, $a_i^\vee$ and Q_i are the comarks and corresponding exponentials of Kahler parameter of nodes.

- The topological string partitions function should be power series of $Q_i, i = 0, 1, \cdots r$.

- The geometry introduces certain symmetry conditions. The topological string partition for $SU(3)$ model should be symmetric over the permutation of Kahler parameters corresponding to the 3 nodes in the affine $\widehat{A}_2$ Dynkin diagram.

- The topological string partition for $SO(8)$ model should be symmetric over the permutation of Kahler parameters corresponding to the 4 nodes in edges of the affine $\widehat{D}_4$ Dynkin diagram.





- We construct these affine Weyl invariant Jacobi forms from Bertola's basis.

Fixing the ansatz with vanishing BPS conditions

- For model with gauge symmetry, the number of possible terms is generally quite large. However, for the case one-string elliptic genus, or base degree one in topological string setting, we can use the Jacobi forms with $SU(2)_R$ parameter $2\epsilon_+$ instead of ϵ_+. The use of affine invariant Jacobi forms further simplifies the ansatz, which has 5 and 149 coefficients for the $SU(3)$ and $SO(8)$ models respectively

- From previous literature on $SU(3)$ and $SO(8)$ models H.-C. Kim, S. Kim, and J. Park, arXiv:1608.03919; B. Haghighat, A. Klemm, G. Lockhart, and C. Vafa, arXiv:1412.3152, we can compute the BPS invariants and guess the maximal genus with non-vanishing BPS numbers for a given Kahler class.

- There should be some geometric arguments for these vanishing conditions. Assuming the validity of these top genus formulas in the followings, we try to use them to fix the Jacobi form ansatz.

Compare with E-string and M-string

- In E-string and M-string, the base degree one ansatz has only one coefficient which is the overall normalization. Therefore the BPS vanishing conditions give no constrain for base degree one. At the higher base degree, one can argue that the generic BPS vanishing conditions are sufficient to completely fix the higher base degree ansatz. Here *generic* conditions mean we only need to know that the BPS invariants always vanish at sufficiently high genus for a given Kahler class, without the precise top genus formulas.

- On the other hand, for the models with gauge symmetry, one can check that the base degree one ansatz satisfies the generic BPS vanishing conditions, but has more coefficient than just the normalization. So in order to fix the base degree one ansatz, the generic vanishing conditions provide no constrain at all, and we need to use the precise top genus formulas.

- We find that for the $SU(3)$ and $SO(8)$ models, we can completely fix the base degree one ansatz up to a normalization with the vanishing BPS conditions.

- For higher base degree, one can easily find certain sub-family of ansatz which satisfies the generic BPS vanishing conditions, due to the Lie algebra factor in the denominator of the ansatz. So unlike the E-string and M-string, the generic BPS vanishing conditions do not completely fix the higher base degree ansatz for models with gauge symmetry.

- We suspect that the BPS vanishing conditions with precise top genus formulas can completely fix the higher base degree ansatz for all models, although it is computationally much difficult to check than the base degree one case.

E_8 Jacobi forms: some curious observations

- The classification results of Wirthmuller apply to all simply laced Lie algebras except for E_8. For this final case, Sakai has constructed certain holomorphic Jacobi forms, denoted as $A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6$, with the subscript indicating the E_8 elliptic index. The A_n's have modular weight 4 and reduce to the Eisenstein series E_4 in the massless limit, while the B_n's have modular weight 6 and reduce to the Eisenstein series E_6.

- They can be written in terms of Jacobi theta functions, e.g. A_1 is the theta function of E_8 lattice Γ_8

$$A_1(\tau, \vec{m}) = \Theta(\tau, \vec{m}) = \sum_{\vec{w} \in \Gamma_8} \exp(\pi i \tau \vec{w}^2 + 2\pi i \vec{m} \cdot \vec{w}) = \frac{1}{2} \sum_{k=1}^{4} \prod_{j=1}^{8} \theta_k(m_j, \tau)$$

- This set of forms generates a ring over the space of holomorphic modular forms which we shall refer to as R_{Sakai}. It is known that R_{Sakai} does not coincide with the full ring of E_8 Weyl invariant Jacobi forms. Here we construct some examples of weak Jacobi forms.

- Since the leading term in the q-series expansion of the A_n's and B_n's is 1, we can easily construct Jacobi forms which are polynomials which vanish at $q = 0$ for general E_8 mass, which we call *cusp polynomials*. Their formal sums form an ideal in R_{Sakai}. Some examples are the discriminant function $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$, $A_2E_4 - A_1^2$, $B_2E_4 - A_2E_6$. Then the ratio of a cusp polynomial with Δ is holomorphic. It is generally a weak Jacobi form, but not holomorphic Jacobi form.

- After some computer search for cusp polynomials, we conjecture the ideal of cusp polynomials is finitely generated by 43 generators.

- A more curious Jacobi form with E_8 index 5 and modular weight 16:

$$
\begin{aligned}
P \;=\; & 864A_1^3A_2+21E_6^2A_5-770E_6A_3B_2+3825A_1B_2^2 \\
& -840E_6A_2B_3+60E_6A_1B_4.
\end{aligned}
$$

- Our conjecture
  1. The Jacobi form P vanishes at the zero points $\tau=\pm\frac{1}{2}+\frac{\sqrt{3}}{2}i$ of E_4 for general E_8 mass parameters. So far we only check numerically.
  2. Any Jacobi form expressed as a polynomial of A_n's, B_n's and E_6 which vanishes at the zero points of E_4 for general E_8 mass parameters must be divisible by the polynomial P.

- Some later progress by mathematician: (H. Wang, "$W(E_8)$-invariant Jacobi forms," arXiv:1801.08462) The space of holomorphic/weak E_8 Jacobi forms of a given index is a free module of finite rank over $SL(2,\mathbb{Z})$ modular forms. Further, the basis of the space of holomorphic/weak E_8 Jacobi forms for index 2,3,4 are explicitly constructed.

- Questions: Construct the basis for higher index? Is the bigraded ring of E_8 Jacobi forms finitely generated?

Summary and Conclusion

- We compute the elliptic genus of E-strings and M-strings by making a good ansatz using weak Jacobi forms, then fixing the ansatz by geometric constrains. Although the results have been obtained before, our method seems more efficient than previous works in certain aspects, without introducing spurious poles in the ansatz.

- We also consider the models with gauge symmetry $SU(3)$ and $SO(8)$. The computation is much more complicated, requiring more sophisticated tools like Weyl invariant Jacobi forms.

- Our results agree with previous works, by some non-trivial identities of Jacobi theta functions.

- We may study more models, especially those that are not solved by other methods.

Thank you