On Large N Solution of linear Quiver
Chern-Simons-matter Matrix Models

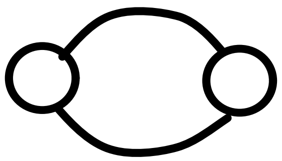
Takao Suyama (KEK)

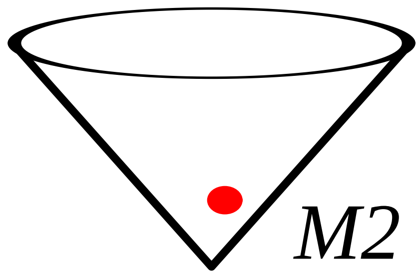
Ref) arXiv:181*.***.**

Introduction

The research on M2-brane was boosted by the discovery of ABJM theory. [BLG][ABJM 08]

This describes N M2 probing $\mathbb{C}^4/\mathbb{Z}_k$ singularity.





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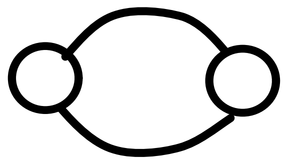

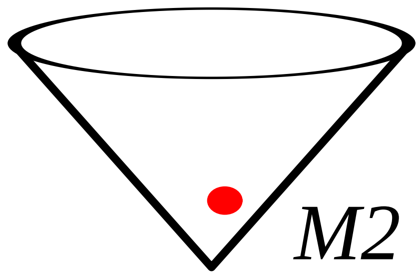

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1. SUSY localization [Kapustin,Willett,Yaakov 09]
2. Matrix model [TS 09][Marino,Putrov 09]
3. Fermi gas [Marino,Putrov 11]

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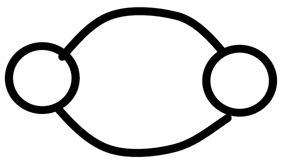

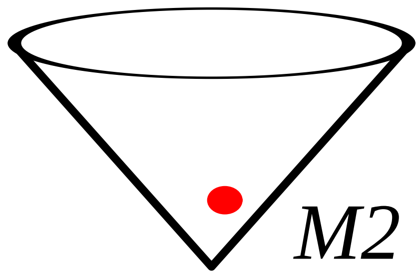

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1. **SUSY localization** **[Kapustin,Willett,Yaakov 09]**
2. **Matrix model** **[TS 09][Marino,Putrov 09]**
3. **Fermi gas** **[Marino,Putrov 11]**

A generalization of ABJM: circular quiver CSMs.

[Hosomichi,Lee^3,Park 08][Imamura,Kimura 08][Jafferis,Tomasiello 08]

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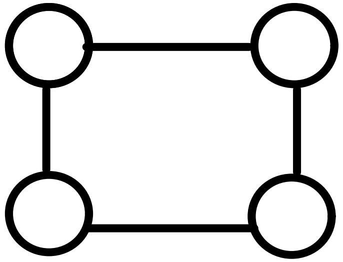
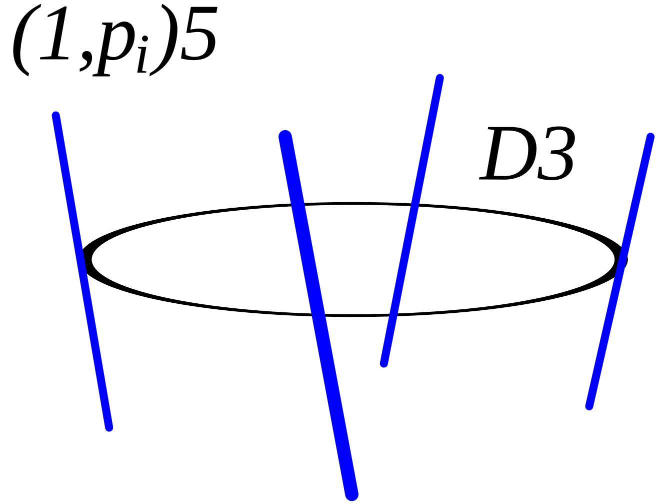
For example,



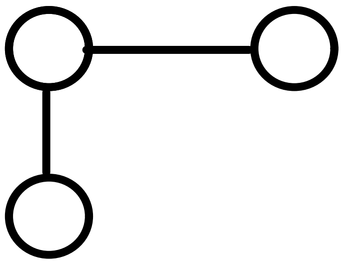


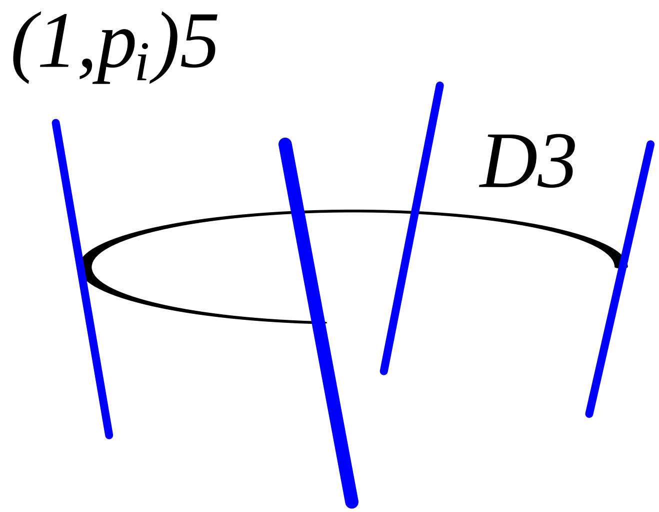
Gauge group: $\mathrm{U}(N_1)_{k_1} \times \mathrm{U}(N_2)_{k_2} \times \cdots \times \mathrm{U}(N_n)_{k_n}$

1 bi-fundamental matter for each $\mathrm{U}(N_a) \times \mathrm{U}(N_{a+1})$**.**

Linear quiver limits:

1. $N_n \to 0$

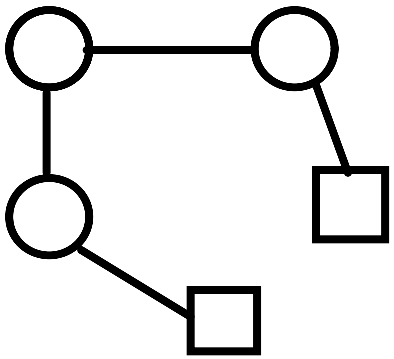


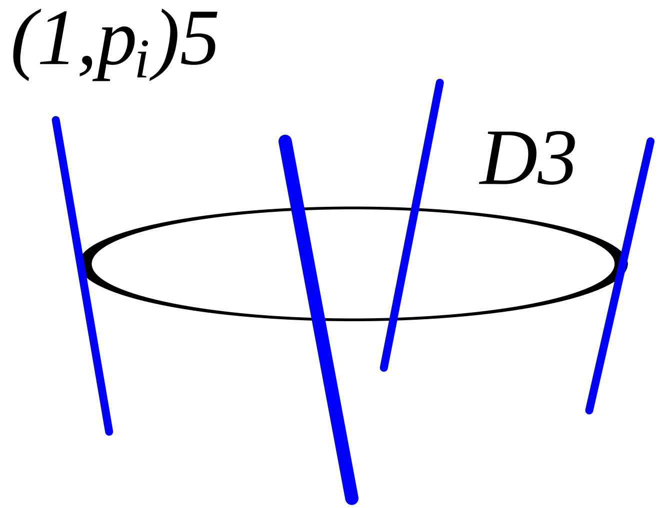


M2 becomes fractional.

Linear quiver limits:

2. $k_n \to \infty$ **(a suitable limit for** p_i**)**





$\mathrm{U}(N_n)$ **becomes global symmetry.**

Some bi-fundamentals become fundamentals.

In both limits, their gravity duals seem to be complicated.

Interestingly, field theory (matrix model) analysis for linear quiver CSM turns out to be quite simple!

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In this talk, we will see how to explicitly obtain resolvents which contain a lot of information. From them, we obtain

- **'t Hooft couplings,**
- **Wilson loops,**
- **free energy, and**
- **moments.**

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Matrix models

Start with the localized partition function:

$$Z = \int \prod_{a=1}^{n} d^{N_a}u^a \, \exp\left[\frac{i}{4\pi}\sum_{a=1}^{n}\sum_{i=1}^{N_a} k_a (u_i^a)^2\right] \frac{\displaystyle\prod_{a=1}^{n}\prod_{i>j}^{N_a}\sinh^2\frac{u_i^a-u_j^a}{2}}{\displaystyle\prod_{a=1}^{n-1}\prod_{i=1}^{N_a}\prod_{j=1}^{N_{a+1}}\cosh\frac{u_i^a-u_j^{a+1}}{2}}.$$

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Planar limit:

$$k \to \infty \quad \textbf{with} \quad \kappa_a := \frac{k_a}{k}, \quad t_a := \frac{2\pi i N_a}{k} \quad \textbf{fixed.}$$

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$\Rightarrow$ **Saddle point equation determines distribution of $\{u_i^a\}$.**

E.g. Wilson loop: $$\langle W_a \rangle = \lim_{k\to\infty} \frac{1}{N_a} \sum_{i=1}^{N_a} e^{u_i^a}$$

The distribution is encoded in the resolvents

$$v_a(z) \;:=\; \lim_{k\to\infty} \frac{t_a}{N_a} \sum_{i=1}^{N_a} \frac{z \pm z_i^a}{z \mp z_i^a}, \qquad z_i^a \;:=\; e^{u_i^a}.$$

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- **'t Hooft coupling:**

$$t_a \;=\; \frac{1}{2}(v_a(\infty) - v_a(0)) \;=\; \frac{1}{2}\int_0^\infty dz\, v_a'(z).$$

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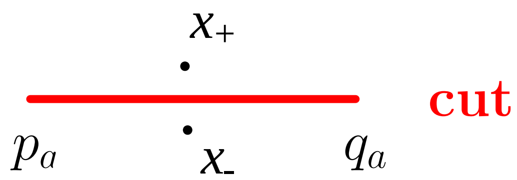
$\Rightarrow$ **Determine** $v_a'(z)$**.**

Introduce $\Omega_a(z)$ $(a = 0, 1, \cdots, n)$, linear combinations of $v'_a(z)$.

The saddle point equations can be written as

$$\Omega_a(x_+) \;=\; \Omega_{a+1}(x_-).$$



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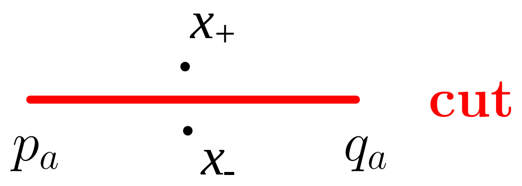
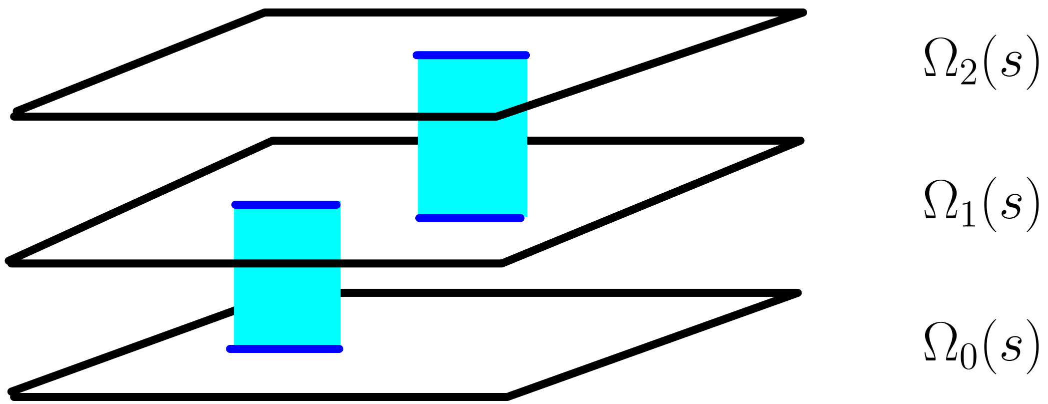


$\Rightarrow$ They define $\Omega(s)$ on $\mathbb{CP}^1$ made by gluing $n+1$ sheets with cuts.



$\Omega(s)$ has simple poles at branch points.

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$\Rightarrow \Omega(s)$ is a rational function:

$$\Omega(s) \; = \; A + \sum_{a=1}^{n} \left[\frac{B_a}{s - \sigma_a} + \frac{C_a}{s - \tau_a} \right].$$

The map from $\mathbb{CP}^1$ (s-plane) to $\mathbb{C}$ (z-plane) is also rational:

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$\Omega(s)$ **and** $z(s)$ **contains all the information.**

$\Rightarrow$ **We only need to fix the parameters** $A, B_a, \sigma_a, \tau_a, \xi_a, \eta_a$**.**

Example: pure Chern-Simons

Pure CS theory corresponds to $n = 1$.

$$\Omega(s) \;=\; A + \frac{B}{s-\sigma} + \frac{C}{s-\tau}, \qquad z(s) \;=\; s\frac{s-\xi}{s-\eta}.$$

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The 7 parameters are fixed by

- $zv'(0) = zv'(\infty) = 0$ **(by construction, fixing 3 parameters),**
- $s = \sigma, \tau$ **are branch points (fixing 2 parameters),**
- $z(\sigma)z(\tau) = 1$ **(by symmetry, fixing 1 parameter).**

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$\Rightarrow$ **The solution is parametrized by one parameter, which is then related to 't Hooft coupling t.**

- **Parameters:**

$$A = -1, \quad B = 2\sigma\frac{\xi - \eta}{\sigma - \tau}, \quad C = -2\tau\frac{\xi - \eta}{\sigma - \tau}, \quad \xi = \frac{2\sigma\tau}{\sigma + \tau}, \quad \eta = \frac{\sigma + \tau}{2},$$

with $\sigma^{-1} + \tau^{-1} = 2$**.**

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- **Wilson loop:**

$$W(C) = \lim_{s\to 0}\frac{\Omega(s)-\Omega(0)}{z(s)} = \frac{e^t-1}{t}.$$

Summary

- **Matrix model analysis of linear quiver CSM is quite simple.**
- **Explicit expressions for physical quantities.**
- **Possibility to give information on gravity duals.**

Open issues

- **Explicit solutions for $n > 1$.**
- **Free energy, moments.**
- **Adding fundamental matters.**
- **Implications to gravity duals.**
- **etc.**