

# Towards a $C$ -theorem in defect CFT

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# Introduction

We are interested in a monotonic decreasing function under the RG flow.

## C-function

- The C-function counts the effective degrees of freedom.
- The monotonicity provides nonperturbative constraints on the RG dynamics.

## C-theorem

## Even dimensions

C-function = the type A central charge of the conformal anomaly

- $d = 2$  : Zamolodchikov's  $c$ -theorem
- $d = 4$  :  $a$ -theorem

## Odd dimensions (no conformal anomalies)

C-function = the sphere free energy  $F \equiv (-1)^{\frac{d-1}{2}} \log Z[\mathbb{S}^d]$

## Generalized $F$ -theorem

$\tilde{F} \equiv \sin\left(\frac{\pi d}{2}\right) \log Z[\mathbb{S}^d]$  is positive and does not increase along any RG flow

This is the most general  $C$ -theorems proposed in arbitrary dimensions.

## Entanglement entropy

$$S^{(\text{CFT})} = \frac{A_{d-2}}{\epsilon^{d-2}} + \frac{A_{d-4}}{\epsilon^{d-4}} + \dots + \begin{cases} a_{\log} \log\left(\frac{R}{\epsilon}\right), & (d = \text{even}) \\ a_0, & (d = \text{odd}) \end{cases}$$

- Universal terms  $a_{\log}, a_0$  are conjectured to be  $C$ -functions.
- This entropic version of the  $C$ -theorem looks different from the generalized  $F$ -theorem based on the sphere free energy.
- But the two  $C$ -function are the same at the fixed points due to the relation

$$S^{(\text{CFT})} = \log Z^{(\text{CFT})} \quad \text{for spherical entangling surface}$$

# C-theorem in BCFTs & DCFTs

## $g$ -theorem (C-theorem in BCFT<sub>2</sub>)

$$S_{\text{thermal}} = \frac{c\pi L}{3\beta} + \log g \quad g\text{-function}$$

( $c$  : central charge,  $L$  : size,  $\beta$  : inverse temperature)

- $\log g$  monotonically decreases under a boundary RG flow.
- The  $g$ -theorem can also be proved by using the equivalence of the  $g$ -function and the boundary entropy  $S_{\text{bdy}} := S^{(\text{BCFT})} - \frac{1}{2} S^{(\text{CFT})}$ .

## $b$ -theorem (C-theorem in BCFT<sub>3</sub> & DCFT <sub>$d$</sub> with 2-dim defect)

$$\langle t^\mu{}_\mu \rangle = -\frac{1}{24\pi} \left[ b \hat{\mathcal{R}} + d_1 \tilde{\mathcal{K}}_{ab}^{(\alpha)} \tilde{\mathcal{K}}^{(\alpha)ab} + d_2 W_{abcd} \hat{g}^{ac} \hat{g}^{bd} \right] \delta^{d-2}(x_\perp)$$

- When  $d = 3$ , the  $b$ -theorem implies the  $g$ -theorem in BCFT<sub>3</sub>.
- For  $d > 3$  it yields a class of C-theorems in DCFTs.

# Towards a C-theorem in DCFT

We want to establish a C-function under a defect RG flow,

$$I = I_{\text{DCFT}} + \hat{\lambda} \int d^p \hat{x} \sqrt{\hat{g}} \hat{\mathcal{O}}(\hat{x})$$

## Two possibilities

- 1 Defect free energy :  $\log \langle \mathcal{D}^{(p)} \rangle = \log Z^{(\text{DCFT})} - \log Z^{(\text{CFT})}$   
= additional contribution to the sphere free energy from the spherical defect
- 2 Defect entropy :  $S_{\text{defect}} = S^{(\text{DCFT})} - S^{(\text{CFT})}$   
= increment of the EE across a sphere due to the planer defect

## Question

**Which is a C-function?**

# Plan of my talk

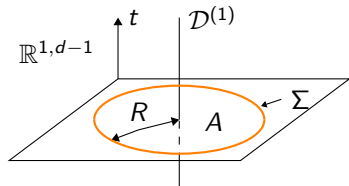
- 1 Introduction
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- 3 Wilson loop as a defect operator
- 4 Conclusion

# Sphere partition function and EE in DCFT

Let us clarify a relation between  $\log \langle \mathcal{D}^{(p)} \rangle$  and  $S_{\text{defect}}$ .

A conformal defect  $\mathcal{D}^{(p)}$  respects  $SO(2, p) \times SO(d - p)$  of the conformal group  $SO(2, d)$ .

$$\mathcal{D}^{(p)} = \{X^p = \dots = X^{d-1} = 0\}$$



## Main result

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d - p - 1) \pi^{d/2+1}}{\sin(\pi p/2) d \Gamma(p/2 + 1) \Gamma((d - p)/2)} a_T$$

- This is a generalization of the result for  $p = 1$  [Lewkowycz-Maldacena '13].
- For  $p = d - 1$ , the defect entropy is given by

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(d-1)} \rangle$$

# Proposal for a $C$ -theorem in DCFT

## Two candidates for a $C$ -function in DCFT

- ① defect free energy  $\log \langle \mathcal{D}^{(p)} \rangle = \log Z^{(\text{DCFT})}[\mathbb{S}^d] / Z^{(\text{CFT})}[\mathbb{S}^d]$

$$\log \langle \mathcal{D}^{(p)} \rangle = \frac{c_p}{\epsilon^p} + \frac{c_{p-2}}{\epsilon^{p-2}} + \cdots + \begin{cases} (-1)^{p/2} B \log \epsilon + \cdots, & (p : \text{even}), \\ (-1)^{(p-1)/2} D, & (p : \text{odd}). \end{cases}$$

- ② defect entropy  $S_{\text{defect}}$

$$S_{\text{defect}} = \frac{c'_{p-2}}{\epsilon^{p-2}} + \frac{c'_{p-4}}{\epsilon^{p-4}} + \cdots + \begin{cases} (-1)^{p/2} B' \log \epsilon + \cdots, & (p : \text{even}), \\ (-1)^{(p-1)/2} D', & (p : \text{odd}). \end{cases}$$

## Our proposal

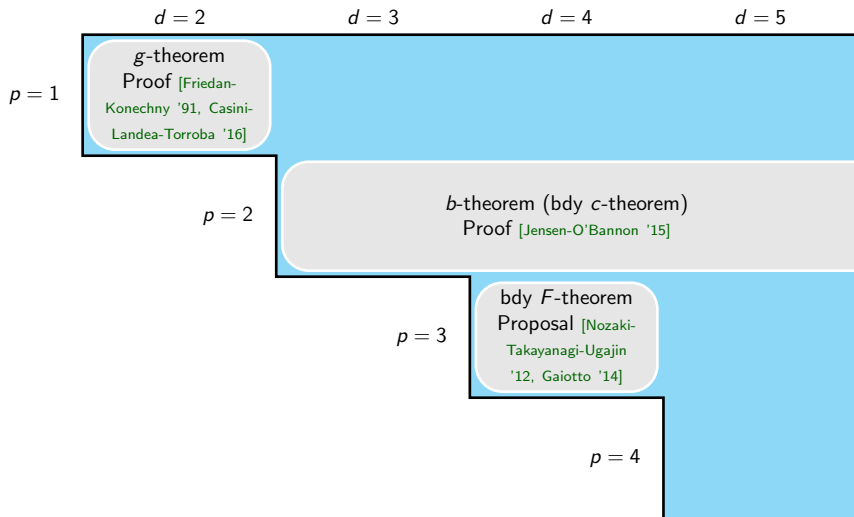
In DCFT $_d$  with a defect of dimension  $p$ , the universal part of the defect free energy

$$\tilde{D} \equiv \sin\left(\frac{\pi p}{2}\right) \log |\langle \mathcal{D}^{(p)} \rangle|$$

does not increase along any defect RG flow  $\tilde{D}_{\text{UV}} \geq \tilde{D}_{\text{IR}}$ .



# Summary of the conjectured and proved $C$ -theorems



Our proposal reduces to the known ones in the shaded regions & provides new ones in the region colored in blue.

# Wilson loop as a defect operator

- We test our proposal for  $p = 1$  using a circular Wilson loop operators

$$W_{\mathfrak{R}}[A] = \text{Tr}_{\mathfrak{R}} \exp \left[ i \int dx^\mu A_\mu \right]$$

- The Wilson loop can be regarded as an action localized on the defect.

[Gomis-Passerini '06, Tong-Wong '14]

$$W_{\mathfrak{R}}[A] = \frac{Z_q[A]}{Z_q[0]} \quad \text{with}$$

$$Z_q[A] \equiv \frac{1}{q!} \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \chi_{a_1}(+\infty) \cdots \chi_{a_q}(+\infty) \chi^{\dagger, a_1}(-\infty) \cdots \chi^{\dagger, a_q}(-\infty) e^{-I_\chi}$$

$$I_\chi = \int dt \chi^\dagger (i \partial_t - A(t)) \chi$$

- The defect theory can flow to the trivial theory without fermions,

$$W_{\mathfrak{R}}[A] \rightarrow 1$$

under the mass deformation

$$I_M = - \int dt M \chi^\dagger \chi, \quad M \rightarrow \infty$$

## U(1) gauge theory in 4d

$$W = \exp \left[ i e \oint dx^\mu A_\mu \right], \quad e \in \mathbb{R}$$

[Lewkowycz-Maldacena '13]

- The defect free energy  $\log \langle W \rangle = e^2/4$
- The defect entropy  $S_{\text{defect}} = 0$
- The Wilson loop becomes trivial under a defect RG flow,  $\log \langle W \rangle \rightarrow 0$ .  
 $\implies$  This is consistent with our conjecture.
- On the other hand, the defect entropy vanishes at both the UV and IR fixed points.  
 $\implies$  The defect entropy doesn't capture degrees of freedom on the defect.

## Free scalar field in 4d

$$W = \exp \left[ \lambda \oint dt \phi(x^\mu(t)) \right], \quad \lambda \in \mathbb{C}$$

- The defect free energy

$$\log \langle W \rangle = 0$$

This result does not contradict with our assertion.

- The defect entropy

$$S_{\text{defect}} = -\frac{\lambda^2}{12}$$

It can be negative for real  $\lambda$  at the UV fixed point.

But it is supposed to be zero at the IR fixed point.

⇒ This is a counterexample for the defect entropy being a  $C$ -function.

# Conclusion

- We examine the defect free energy  $\log \langle \mathcal{D}^{(p)} \rangle$  and the defect entropy  $S_{\text{defect}}$  as a candidate  $C$ -function.
- We find the relation with them

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d-p-1) \pi^{d/2+1}}{\sin(\pi p/2) d \Gamma(p/2+1) \Gamma((d-p)/2)} a_T$$

- We propose a  $C$ -theorem in DCFTs.

The defect free energy does not increase under any defect RG flow.

- We find in Wilson loop examples that
  - the sphere free energy decreases
  - but the EE increases along a certain RG flow triggered by a defect localized perturbation which is assumed to have a trivial IR fixed point without defects.

- We also checked more field theoretic examples,
  - Conformal perturbation theory on defect
  - Wilson loop
    - ① Chern-Simons theory
    - ② 1/2-BPS Wilson loop in 4d  $\mathcal{N} = 4$  SYM
    - ③ 1/6-BPS Wilson loop in ABJM
    - ④  $U(N)$   $\mathcal{N} = 4$  SYM with  $N_f$  hypermultiplets in 3d
- We also provide a proof of our proposal in several holographic models of defect RG flows.
  - ① Domain wall defect RG flow
  - ② Probe brane model
  - ③ Holographic model of defect RG flow [Yamaguchi '02]
  - ④ AdS/BCFT model [Takayanagi '11]

Thank you for your attention!

- Definition :  $T_{\text{DCFT}}^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta \log Z^{(\text{DCFT})}[g_{\mu\nu}]}{\delta g_{\mu\nu}} = T_{\text{CFT}}^{\mu\nu} + t^{\mu\nu}$
- $T_{\text{DCFT}}^{\mu\nu}$  is traceless and partially conserved

$$\partial_\mu T_{\text{DCFT}}^{\mu a} = 0, \quad \partial_\mu T_{\text{DCFT}}^{\mu i} = -\delta_{\mathcal{D}}(x_\perp) D^i, \quad (T_{\text{DCFT}})^\mu{}_\mu = 0$$

- The one-point function of  $T_{\text{CFT}}$

$$\langle T_{\text{CFT}}^{ab}(x) \rangle = \frac{d-p-1}{d} \frac{a_T}{|x_\perp|^d} \delta^{ab}, \quad \langle T_{\text{CFT}}^{ai}(x) \rangle = 0, \quad \langle T_{\text{CFT}}^{ij}(x) \rangle = -\frac{a_T}{|x_\perp|^d} \left( \frac{p+1}{d} \delta^{ij} - \frac{x_\perp^i x_\perp^j}{|x_\perp|^2} \right)$$

Note that  $\langle T_{\text{CFT}}^{\mu\nu} \rangle = 0$  for  $p = d - 1$ .

- $\langle t^{\mu\nu}(x) \rangle = 0$  since  $t^{\mu\nu}$  is localized on the defect,

$$t^{\mu\nu}(x) = \delta_{\mathcal{D}}(x_\perp) \frac{\partial x^\mu}{\partial \hat{x}^a} \frac{\partial x^\nu}{\partial \hat{x}^b} \hat{t}^{ab}(\hat{x})$$

$\hat{t}^{ab}(\hat{x})$  is a defect local operator of dimension  $p$  and invariant under the translation, rotation and scale transformation on the defect.