Near extremal black holes, attractors, and black hole entropy

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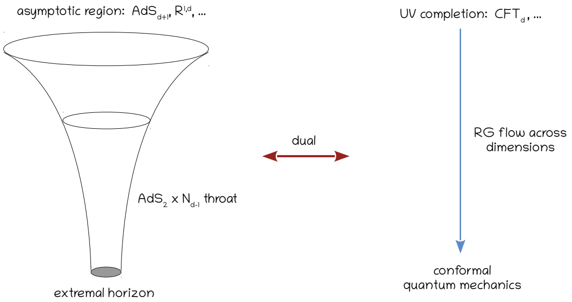
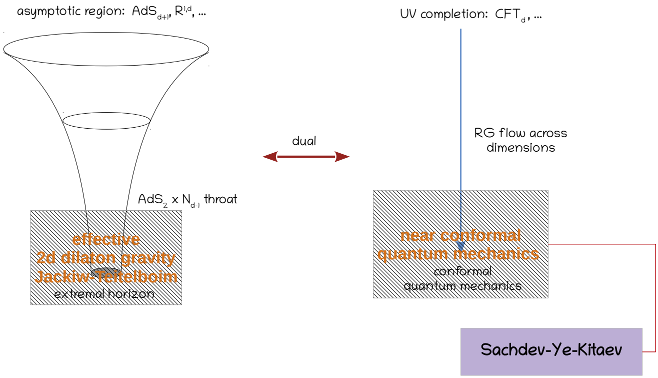
Near extremal black holes

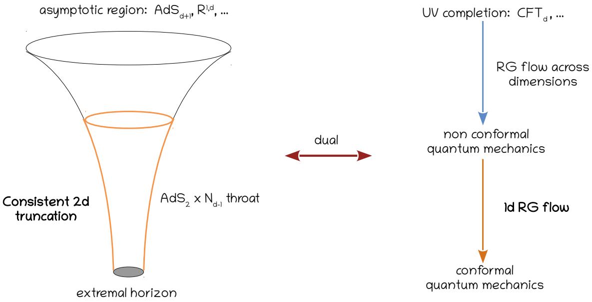


Near extremal black holes



Near extremal black holes

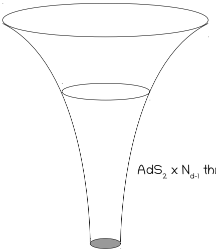
I. Effective 2d truncation



Near extremal black holes

II. Supersymmetry ⟶ RG flow invariants

asymptotic region: AdS_{d+1}, $R^{1,d}$, ...



$AdS_2 \times N_{d-1}$ throat

extremal horizon

dual ⟷

UV completion: CFT_d, ...

RG flow across dimensions

conformal quantum mechanics

quantum entropy ⟷ **(generalized twisted) index**

Outline

Outline

- 3D Einstein-Hilbert gravity

$$S_{3\mathrm{D}} = \frac{1}{2\kappa_3^2}\left(\int \mathrm{d}^3\mathbf{x}\sqrt{-g_3}\,\left(R[g_3] - 2\Lambda_3\right) + \int \mathrm{d}^2x\sqrt{-\gamma_2}\,2K[\gamma_2]\right)$$

- A circle reduction using the Kaluza-Klein ansatz

$$ds_3^2 = e^{-2\psi}\left(dz + A_a dx^a\right)^2 + g_{ab}dx^a dx^b$$

leads to the 2D Einstein-Maxwell-Dilaton model

$$S_{2\mathrm{D}} = \frac{1}{2\kappa_2^2}\left(\int \mathrm{d}^2\mathbf{x}\sqrt{-g}\;e^{-\psi}\left(R[g] + \frac{2}{L^2} - \frac{1}{4}e^{-2\psi}F_{ab}F^{ab}\right) + \int \mathrm{d}t\sqrt{-\gamma}\;e^{-\psi}2K\right)$$

- In the UV this model exhibits a **generalized conformal structure** [Taylor '17]
- Analogous to non-conformal branes [Kanitscheider, Skenderis, Taylor '08], e.g. D4 branes can be uplifted to M5 branes.
- The existence of a UV fixed point allows us to define a notion of a **conformal anomaly** in the lower dimensional non-conformal theory.

- The 4D action

$$S_{4\mathrm{D}} = \frac{1}{2\kappa_4^2} \int_{\mathcal{M}} \mathrm{d}^4\mathbf{x}\sqrt{-g}\Big(R[g] - \frac{3}{2}\partial_\mu\eta\partial^\mu\eta - \frac{3}{2}e^{2\eta}\partial_\mu\chi\partial^\mu\chi - \frac{1}{4}e^{-3\eta}F^0_{\mu\nu}F^{0\mu\nu} - \frac{3}{4}e^{-\eta}(F+\chi F^0)_{\mu\nu}(F+\chi F^0)^{\mu\nu}\Big) - \frac{1}{8\kappa_4^2}\int_{\mathcal{M}} \mathrm{d}^4\mathbf{x}\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\left(\chi^3 F^0_{\mu\nu}F^0_{\rho\sigma} + 3\chi^2 F^0_{\mu\nu}F_{\rho\sigma} + 3\chi F_{\mu\nu}F_{\rho\sigma}\right)$$

  is a consistent truncation of the STU model and admits a class of asymptotically conformally $\mathrm{AdS}_2\times\mathrm{S}^2$ black hole solutions, provided $F_{\mu\nu}$ carries non-zero magnetic flux. Generically they are rotating and electrically charged.

- Such solutions are known as **subtracted geometries** [Cvetič, Larsen '12; Cvetič, Gibbons '12] and have been obtained by various methods from the corresponding asymptotically flat black holes (e.g. Harrison transformations [Virmani '12; M. Cvetič, Guica, Saleem '13]).

Subtracted geometries as a decoupling limit

- In a suitable parameterization, subtracted geometries correspond to turning off certain integration constants in the harmonic functions that enter in the asymptotically flat black hole solutions [Baggio, de Boer, Jottar, Mayerson '13].
- This procedure does not involve any scaling limit and allows for a simpler parameterization of the resulting solutions [An, I.P., Cvetič '16]:

$$
\begin{aligned}
e^{\eta} &= \frac{B^2/\ell^2}{\sqrt{r+\ell^2\omega^2\sin^2\theta}}, \qquad \chi = \frac{\ell^3\omega}{B^2}\cos\theta \\
A^0 &= \frac{B^3/\ell^3}{r+\ell^2\omega^2\sin^2\theta}\left(\sqrt{r_+r_-}\;kdt + \ell^2\omega\sin^2\theta d\phi\right) \\
A &= \frac{B\cos\theta}{r+\ell^2\omega^2\sin^2\theta}\left(-\omega\sqrt{r_+r_-}\;kdt + rd\phi\right) \\
ds^2 &= \sqrt{r+\ell^2\omega^2\sin^2\theta}\left(\frac{\ell^2 dr^2}{(r-r_-)(r-r_+)} - \frac{(r-r_-)(r-r_+)}{r}k^2dt^2 + \ell^2 d\theta^2\right) \\
&\quad + \frac{\ell^2 r\sin^2\theta}{\sqrt{r+\ell^2\omega^2\sin^2\theta}}\left(d\phi - \frac{\omega\sqrt{r_+r_-}}{r}kdt\right)^2
\end{aligned}
$$

Kaluza-Klein reduction ansatz

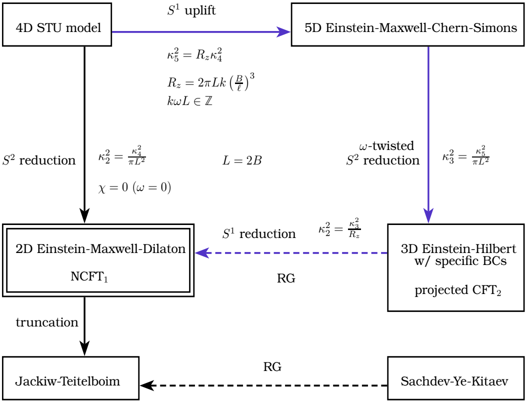
- The 4D truncation of the STU model can be consistently Kaluza-Klein reduced on S^2 by means of the *one-parameter family* of KK ansätze [Cvetič, I.P.:1608.07018]

$$
\begin{aligned}
e^{-2\eta} &= e^{-2\psi} + \lambda^2 B^2 \sin^2\theta, \qquad \chi = \lambda B \cos\theta \\
e^{-2\eta} A^0 &= e^{-2\psi} A^{(2)} + \lambda B^2 \sin^2\theta d\phi, \quad A + \chi A^0 = B\cos\theta d\phi \\
e^{\eta} ds_4^2 &= ds_2^2 + B^2 \left(d\theta^2 + \frac{\sin^2\theta}{1+\lambda^2 B^2 e^{2\psi}\sin^2\theta} (d\phi - \lambda A^{(2)})^2 \right)
\end{aligned}
$$

  where λ is an arbitrary parameter. For any value of λ, the resulting 2D theory is the Einstein-Maxwell-Dilaton theory we considered above – λ drops out in 2D!

- By comparing the KK ansatz with the 4D black hole solutions we see that $\lambda = \omega\ell^3/B^3$, i.e. λ is the angular parameter of the 4D black hole.

- The parameter λ allows any solution of the 2D EMD theory to be uplifted to a *family* of 4D solutions, i.e. it acts as a solution generating mechanism.


4D STU model
S^1 uplift
5D Einstein-Maxwell-Chern-Simons
$\kappa_5^2 = R_z \kappa_4^2$
$R_z = 2\pi L k \left(\frac{B}{\ell}\right)^3$
$k\omega L \in \mathbb{Z}$
S^2 reduction
$\kappa_2^2 = \frac{\kappa_4^2}{\pi L^2}$
$L = 2B$
ω-twisted S^2 reduction
$\kappa_3^2 = \frac{\kappa_5^2}{\pi L^2}$
$\chi = 0$ $(\omega = 0)$
S^1 reduction
$\kappa_2^2 = \frac{\kappa_3^2}{R_z}$
2D Einstein-Maxwell-Dilaton
NCFT_1
RG
3D Einstein-Hilbert w/ specific BCs
projected CFT_2
truncation
RG
Jackiw-Teitelboim
Sachdev-Ye-Kitaev

- 5D Einstein-Hilbert gravity:

$$I_{5\mathrm{D}} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g^{(5)}} \left(\mathcal{R}^{(5)} + \frac{12}{\ell_5^2} \right)$$

  where ℓ_5 is the AdS$_5$ radius

- We include the asymptotically flat case $\ell_5 \to \infty$
- For finite ℓ_5 a holographic description can be provided within $\mathcal{N} = 4$ SYM, but the effective action for near-extremal black hole excitations cannot be obtained analytically, except very near the IR

Kerr-AdS$_5$ with equal angular momenta

- We focus on the Kerr-AdS$_5$ black hole with two equal angular momenta and its Myers-Perry limit ($\ell_5 \to \infty$)
- The rotation breaks the isometry group as

$$SO(4) \cong SU(2)_L \times SU(2)_R \to SU(2)_L \times U(1)_R$$

- The corresponding metric can be written as

$$ds_5^2 = ds_2^2 + e^{-U_1} d\Omega_2^2 + e^{-U_2} \left(\sigma^3 + A\right)^2$$

where

$$ds_2^2 = \frac{r^2 dr^2}{(r^2+a^2)\Delta(r)} - \frac{1}{\Xi}\Delta(r) e^{U_2 - U_1} dt^2$$

$$A = A_t dt = \frac{a}{2\Xi}\left(\frac{r^2+a^2}{\ell_5^2} - \frac{2m}{r^2+a^2}\right) e^{U_2} dt$$

and

$$e^{-U_2} = \tfrac{r^2+a^2}{4\Xi} + \tfrac{ma^2}{2\Xi^2(r^2+a^2)}\,, \quad e^{-U_1} = \frac{r^2+a^2}{4\Xi}$$

$$\Xi = 1 - \tfrac{a^2}{\ell_5^2}\,, \quad \Delta(r) = 1 + \frac{r^2}{\ell_5^2} - \frac{2mr^2}{(r^2+a^2)^2}$$

Kaluza-Klein reduction ansatz

- Our KK ansatz is (note Weyl rescaling of 2D metric)

$$ds^2_{(5)} = e^{\psi+\chi} ds^2_{(2)} + R^2 e^{-2\psi+\chi} d\Omega_2^2 + R^2 e^{-2\chi} \left(\sigma^3 + A\right)^2$$

  where ψ, χ and A depend only on 2D base

- Inserting this in the 5D action leads to the 2D effective theory

$$I_{\rm 2D} = \frac{1}{2\kappa_2^2} \int d^2x \sqrt{-g}\, e^{-2\psi} \Big(\mathcal{R} - \frac{R^2}{4} e^{-3\chi-\psi} F^2 - \frac{3}{2} (\nabla\chi)^2 + \frac{1}{2R^2} \left(4e^{3\psi} - e^{5\psi-3\chi}\right) + \frac{12}{\ell_5^2} e^{\psi+\chi} \Big)$$

  where $\frac{1}{\kappa_2^2} = \frac{16\pi^2 R^3}{\kappa_5^2}$ and R is an arbitrary length parameter

- Have checked that this is a **consistent truncation**! Not a standard sphere reduction – internal manifold is not supported by flux (cf. [Gouteraux, Smolic, Smolic, Skenderis, Taylor '11])

- A holographic understanding of this 2D theory can provide direct insight into the microstates of the Kerr-AdS$_5$ black hole!

Outline

3D AdS gravity: Running dilaton solutions

- The general solution with running dilaton takes the form

$$e^{-\psi} = \beta(t)e^{u/L}\sqrt{\left(1+\frac{m-\beta'^2(t)/\alpha^2(t)}{4\beta^2(t)}L^2e^{-2u/L}\right)^2-\frac{Q^2L^2}{4\beta^4(t)}e^{-4u/L}}$$

$$\sqrt{-\gamma} = \frac{\alpha(t)}{\beta'(t)}\partial_t e^{-\psi}$$

$$A_t = \mu(t)+\frac{\alpha(t)}{2\beta'(t)}\partial_t \log\left(\frac{4L^{-2}e^{2u/L}\beta^2(t)+m-\beta'^2(t)/\alpha^2(t)-2Q/L}{4L^{-2}e^{2u/L}\beta^2(t)+m-\beta'^2(t)/\alpha^2(t)+2Q/L}\right)$$

  where $\alpha(t)$, $\beta(t)$ and $\mu(t)$ are arbitrary functions of time, while m and Q are arbitrary constants.

- This solution is regular provided $m > 0$.

- The leading asymptotic behavior of this solution is

$$\gamma_{tt} = -\alpha^2(t)e^{2u/L}+\mathcal{O}(1), \quad e^{-\psi} \sim \beta(t)e^{u/L}+\mathcal{O}(e^{-u/L}), \quad A_t = \mu(t)+\mathcal{O}(e^{-2u/L})$$

  and so the arbitrary functions $\alpha(t)$, $\beta(t)$ and $\mu(t)$ should be identified with the sources of the corresponding dual operators.

3D AdS gravity: The hairy 2D black hole

- For constant sources α_o, β_o, μ_o and generic $m > 0$ and $|Q| < mL/2$, this is a non-extremal asymptotically AdS_2 black hole. It becomes extremal when $Q = \pm mL/2$.

- The Hawking temperature is

$$T = \frac{\alpha_o \beta_o}{\pi L^{1/2}} \frac{\sqrt{m^2 L^2 - 4Q^2}}{\sqrt{mL + 2Q} + \sqrt{mL - 2Q}}$$

  which indeed vanishes when $m = 2Q/L$.

- The Bekenstein-Hawking entropy is not given by the area law in 2D, but can be computed e.g. using Wald's formula. For 2D black holes with a non trivial dilaton profile ones finds that the entropy is given by the value of the dilaton on the outer horizon [Myers '94; Cadoni, Mignemi '99]:

$$S = \frac{2\pi}{\kappa_2^2} e^{-\psi(u_+)} = \frac{2\pi}{\kappa_2^2} \frac{L^{1/2}}{2} \left(\sqrt{mL + 2Q} + \sqrt{mL - 2Q} \right)$$

3D AdS gravity: Constant dilaton solutions

- Another family of solutions is [Castro, Grumiller, Larsen, McNees '08]

$$
\begin{aligned}
e^{-2\psi} &= LQ \\
\sqrt{-\gamma} &= \widetilde{\alpha}(t)e^{u/\widetilde{L}} + \frac{\widetilde{\beta}(t)}{\sqrt{LQ}}e^{-u/\widetilde{L}} \\
A_t &= \widetilde{\mu}(t) - \frac{1}{\sqrt{LQ}}\left(\widetilde{\alpha}(t)e^{u/\widetilde{L}} - \frac{\widetilde{\beta}(t)}{\sqrt{LQ}}e^{-u/\widetilde{L}}\right)
\end{aligned}
$$

  where $\widetilde{\alpha}(t)$, $\widetilde{\beta}(t)$ and $\widetilde{\mu}(t)$ are arbitrary functions, $Q > 0$ is an arbitrary constant, and $\widetilde{L} = L/2$.

- As above, the functions $\widetilde{\alpha}(t)$ and $\widetilde{\mu}(t)$ are going to be identified with sources of local operators, but we shall see that the function $\widetilde{\beta}(t)$ corresponds to the one-point function of an irrelevant scalar operator of dimension 2.

- Notice that **the gauge field diverges** at the boundary $u \to +\infty$. This is a generic property of rank $p \geq d/2$ antisymmetric tensor fields in AdS$_{d+1}$ and leads to certain subtleties in the holographic dictionary.

3D AdS gravity: The bald black hole

- For constant $\widetilde{\alpha}$, $\widetilde{\mu}$ and $\widetilde{\beta} < 0$ this is a non-extremal asymptotically AdS_2 black hole with

$$T_{2D} = \frac{\sqrt{-\widetilde{\alpha}_o \widetilde{\beta}_o}}{\pi \widetilde{L} (LQ)^{1/4}} \,, \qquad S_{2D} = \frac{2\pi}{\kappa_2^2} \sqrt{LQ} \,, \qquad M_{2D} = 0 \,.$$

- Uplifting to 3D (along a null circle) gives instead

$$T_{3D} = \frac{(LQ)^{1/4} \sqrt{-\widetilde{\alpha}_o \widetilde{\beta}_o}}{\pi \widetilde{L} \left(\sqrt{LQ} + \sqrt{\frac{-2\widetilde{\beta}_o}{LQ}} \right)}, \quad S_{3D} = \frac{2\pi}{\kappa_2^2} \left(\sqrt{LQ} + \sqrt{\frac{-2\widetilde{\beta}_o}{LQ}} \right), \quad M_{3D} = \frac{1}{4\kappa_2^2 \widetilde{L}} \left(LQ - \frac{2\widetilde{\beta}_o}{LQ} \right)$$

- This black hole becomes extremal when $\widetilde{\beta}_o = 0$.
- The two black holes cannot be compared directly since they satisfy different boundary conditions. The hairy black hole is asymptotically AdS_2 with AdS radius L, while the bald solution is asymptotically AdS_2, with AdS radius $\widetilde{L} = L/2$.
- However, they both uplift to the BTZ black hole in 3D, with AdS_3 radius L.

- Since the two classes of solutions have different AdS radii, one expects that there is an RG flow from the running dilaton solution to the constant dilaton solution.
- For the **extremal solutions** this is indeed the case. Setting $m - \beta'^2/\alpha^2 = 2Q/L > 0$ and $\mu = -\alpha/\beta$ and expanding the hairy solution for $u \to -\infty$ gives

$$e^{-\psi} = \sqrt{LQ} + \frac{\beta^2}{2\sqrt{LQ}} e^{2u/L} + \mathcal{O}(e^{4u/L})$$

$$\sqrt{-\gamma} = \frac{\alpha\beta}{\sqrt{LQ}}\, e^{2u/L} \left(1 - \frac{\beta^2}{2LQ} e^{2u/L} + \mathcal{O}(e^{4u/L})\right)$$

$$A_t = -\frac{\alpha\beta}{LQ} e^{2u/L} \left(1 - \frac{\beta^2}{LQ} e^{2u/L} + \mathcal{O}(e^{4u/L})\right)$$

- The limit $\beta \to 0$ keeping $\alpha\beta$ fixed results in an exact bald solution with $\widetilde{\alpha} = \alpha\beta/\sqrt{LQ}$. This limit sets $m = 2Q/L$ and $\mu \to -\infty$, and corresponds to the "Very-Near-Horizon Region" [Strominger '98].

5D AdS gravity: Structure of 2D solutions

- The gauge field can be integrated out:

$$F_{ab} = Qe^{3\psi+3\chi}\epsilon_{ab}, \qquad F^2 = -2Q^2e^{6\psi+6\chi}$$

  where $Q \sim J_5$. Different boundary conditions can be imposed if not integrated out!

- For $\ell_5 \to \infty$, χ can be consistently set to a constant with ψ non-trivial. Such solutions uplift to Taub-NUT in 5D with a 4D Reissner-Nordström base

- The **attractor solutions** are obtained for χ and ψ both constant and $Q \neq 0$. They correspond to the very-near horizon region of extremal Kerr-AdS$_5$ and to the IR fixed point in the dual quantum mechanics

- For finite ℓ_5 the scalar field χ cannot be decoupled and the 2D equations of motion cannot be integrated completely. We will focus on the general near IR solutions

5D AdS gravity: IR fixed point solutions

- The constant values of the scalars are determined by the two equations

$$e^{-2\psi_0} = e^{-3\chi_0} - \frac{R^4Q^2}{2} e^{3\chi_0}$$
$$1 - R^4Q^2e^{6\chi_0} + \frac{2R^2}{\ell_5^2} e^{-2\chi_0} \left(2 - R^4Q^2e^{6\chi_0}\right)^2 = 0$$

- In the radial (Fefferman-Graham) gauge

$$ds^2 = d\rho^2 + \gamma_{tt}(\rho,t)dt^2, \qquad A_\rho = 0$$

the general attractor solution takes the form

$$\begin{aligned} \sqrt{-\gamma_0} &= \alpha(t)e^{\rho/\ell_2} + \beta(t)e^{-\rho/\ell_2} \\ A_t^0 &= \mu(t) - Q\ell_2 e^{3\chi_0+3\psi_0} \left(\alpha(t)e^{\rho/\ell_2} - \beta(t)e^{-\rho/\ell_2}\right) \end{aligned}$$

where $\gamma_{tt} = -(\sqrt{-\gamma})^2$ and

$$\ell_2^{-2} = \frac{1}{R^2} e^{3\psi_0}(1+12q), \qquad q \equiv \frac{1}{8} e^{2\psi_0}(R^4Q^2e^{3\chi_0} - e^{-3\chi_0}) \qquad (q \to 0 \text{ as } \ell_5 \to \infty)$$

5D AdS gravity: Perturbations near IR fixed point

- Small fluctuations on top of the attractor solutions can be parameterized as

$$\mathcal{Y} \equiv e^{-2\psi} - e^{-2\psi_0}\,, \qquad \mathcal{X} \equiv \chi - \chi_0\,, \qquad \sqrt{-\gamma_1} \equiv \sqrt{-\gamma} - \sqrt{-\gamma_0}$$

  and satisfy a system of coupled linear equations that can be solved exactly

- From the linearized equations we read off the AdS_2 masses of the scalar fluctuations, and hence the conformal dimension of the dual operators:

$$\Delta_{\mathcal{Y}} = 2, \qquad \Delta_{\mathcal{X}} = \frac{1}{2}\left(1 + 5\sqrt{\frac{1+\frac{28}{5}q}{1+12q}}\right)$$

  where

$$2 < \frac{1}{6}(3+\sqrt{105}) \leq \Delta_{\mathcal{X}} \leq 3$$

  with $\Delta_{\mathcal{X}} = 3$ corresponding to the Myers-Perry black hole ($\ell_5 \to \infty$)

- In the Fefferman-Graham gauge the general near IR solutions take the form

$$\mathcal{Y} = \nu(t)e^{\rho/\ell_2} + \vartheta(t)e^{-\rho/\ell_2}$$
$$\mathcal{X} = \frac{2q}{1+2q}e^{2\psi_0}\mathcal{Y}$$
$$+ \zeta(t)e^{(\Delta_\chi - 1)\rho/\ell_2}(1+\ldots) - \frac{2\ell_2 e^{2\psi_0} c_1 \zeta(t)^{\frac{\Delta_\chi}{1-\Delta_\chi}} e^{-\Delta_\chi \rho/\ell_2}}{3(\Delta_\chi - 1)(2\Delta_\chi - 1)}(1+\cdots)$$
$$\sqrt{-\gamma_1} = -\frac{(1+10q+8q^2)}{(1+2q)(1+12q)}e^{2\psi_0}\left[\sqrt{-\gamma_0}\mathcal{Y} + 2\ell_2^2\partial_t\left(\frac{\partial_t\nu}{\alpha}\right)\right] + (\zeta(t)\text{ terms})$$

where

$$\beta(t) = -\frac{\ell_2^2}{4}\frac{\alpha}{\partial_t\nu}\partial_t\Big(\frac{1}{\nu}\Big(c_0 + \frac{(\partial_t\nu)^2}{\alpha^2}\Big)\Big)$$
$$\vartheta(t) = -\frac{\ell_2^2}{4\nu}\Big(c_0 + \frac{(\partial_t\nu)^2}{\alpha^2}\Big) - \frac{\ell_2}{2}c_1\zeta^{\frac{1}{1-\Delta_\chi}}$$

and c_0, c_1 are arbitrary constants

- Perturbation theory is valid for

$$|\nu(t)|e^{\rho/\ell_2} << e^{-2\psi_0}, \qquad |\zeta(t)|e^{(\Delta_\chi - 1)\rho/\ell_2} << \chi_0$$

Outline

Radial Hamiltonian formulation of the dynamics

- Inserting the radial ADM decomposition

$$ds^2 = (N^2 + N_t N^t)du^2 + 2N_t du dt + \gamma_{tt} dt^2$$

of the metric in the 2D action gives the radial Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa_2^2} \int \mathrm{d}t \sqrt{-\gamma} N \left(-\frac{2}{N} K(\dot{\psi} - N^t \partial_t \psi) - \frac{1}{2N^2} e^{-2\psi} F_{ut} F_u{}^t + \frac{2}{L^2} - 2\Box_t \right) e^{-\psi}$$

where $K = \gamma^{tt} K_{tt}$ and the extrinsic curvature K_{tt} is given by

$$K_{tt} = \frac{1}{2N} \left(\dot{\gamma}_{tt} - 2D_t N_t \right)$$

with the dot denoting a derivative with respect to the radial coordinate u, and D_t standing for the covariant derivative with respect to the induced metric γ_{tt}.

- The canonical momenta are

$$\pi^{tt} = \frac{\delta \mathcal{L}}{\delta \dot{\gamma}_{tt}} = -\frac{1}{2\kappa_2^2}\sqrt{-\gamma}e^{-\psi}\frac{1}{N}\gamma^{tt}\left(\dot{\psi} - N^t\partial_t\psi\right)$$

$$\pi^t = \frac{\delta \mathcal{L}}{\delta \dot{A}_t} = -\frac{1}{2\kappa_2^2}\sqrt{-\gamma}e^{-3\psi}\frac{1}{N}\gamma^{tt}F_{ut}$$

$$\pi_\psi = \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = -\frac{1}{\kappa_2^2}\sqrt{-\gamma}e^{-\psi}K$$

- The canonical momenta conjugate to N, N_t and A_u vanish identically and, hence, these fields are Lagrange multipliers imposing the first class constrains

$$\mathcal{H} = -\frac{\kappa_2^2}{\sqrt{-\gamma}}e^{\psi}\left(2\pi\pi_\psi + e^{2\psi}\pi^t\pi_t\right) - \frac{\sqrt{-\gamma}}{\kappa_2^2}\left(L^{-2} - \Box_t\right)e^{-\psi} = 0$$

$$\mathcal{H}^t = -2D_t\pi^{tt} + \pi_\psi\partial^t\psi = 0$$

$$\mathcal{F} = -D_t\pi^t = 0$$

- The canonical momenta can alternatively be expressed as gradients of Hamilton's principal function $\mathcal{S}$ as

$$\pi^{tt} = \frac{\delta \mathcal{S}}{\delta \gamma_{tt}}, \quad \pi^{t} = \frac{\delta \mathcal{S}}{\delta A_t}, \quad \pi_{\psi} = \frac{\delta \mathcal{S}}{\delta \psi}$$

  where $\mathcal{S}[\gamma, \psi, A]$ is a functional of the induced fields γ_{tt}, A_t and ψ and their t-derivatives only and coincides with the on-shell action.

- $\mathcal{S}[\gamma, \psi, A]$ coincides with the on-shell action:
  1. Canonical momenta are one-point functions – do not need on-shell action!
  2. Boundary counterterms can be obtained by solving the Hamilton-Jacobi equation for $\mathcal{S}$.

Holographic dictionary for running dilaton solutions

- For the running dilaton solutions the boundary counterterms are

$$S_{\rm ct} = -\frac{1}{\kappa_2^2} \int \mathrm{d}t \sqrt{-\gamma}\, L^{-1} \left(1 - u_o L \Box_t\right) e^{-\psi}$$

- The renormalized one-point functions are given by the renormalized radial canonical momenta:

$$\mathcal{T} = 2\widehat{\pi}_t^t, \quad \mathcal{O}_\psi = -\widehat{\pi}_\psi, \quad \mathcal{J}^t = -\widehat{\pi}^t$$

where

$$\widehat{\pi}_t^t = \frac{1}{2\kappa_2^2} \lim_{u\to\infty} e^{u/L} \left(\partial_u e^{-\psi} - e^{-\psi} L^{-1}\right)$$

$$\widehat{\pi}^t = \lim_{u\to\infty} \frac{e^{u/L}}{\sqrt{-\gamma}} \pi^t$$

$$\widehat{\pi}_\psi = -\frac{1}{\kappa_2^2} \lim_{u\to\infty} e^{u/L} e^{-\psi} \left(K - L^{-1}\right)$$

Holographic dictionary for running dilaton solutions

- Evaluating these expressions using the general solutions with running dilaton gives the one-point functions

$$\mathcal{T} = -\frac{L}{2\kappa_2^2}\left(\frac{m}{\beta} - \frac{\beta'^2}{\beta\alpha^2}\right), \quad \mathcal{J}^t = \frac{1}{\kappa_2^2}\frac{Q}{\alpha}, \quad \mathcal{O}_\psi = \frac{L}{2\kappa_2^2}\left(\frac{m}{\beta} - \frac{\beta'^2}{\beta\alpha^2} - 2\frac{\beta'\alpha'}{\alpha^3} + 2\frac{\beta''}{\alpha^2}\right)$$

- All three operators are crucial to describe the physics. In particular, these one-point functions satisfy the Ward identities

$$\partial_t \mathcal{T} - \mathcal{O}_\psi \partial_t \log\beta = 0, \qquad \mathcal{D}_t \mathcal{J}^t = 0$$

$$\mathcal{T} + \mathcal{O}_\psi = \frac{L}{\kappa_2^2}\left(\frac{\beta''}{\alpha^2} - \frac{\beta'\alpha'}{\alpha^3}\right) = \frac{L}{\kappa_2^2\alpha}\partial_t\left(\frac{\beta'}{\alpha}\right) \equiv \mathcal{A}$$

- From these relations we deduce that the scalar operator $\mathcal{O}_\psi$ is a marginally relevant operator and the theory has a **conformal anomaly** due to the source of the scalar operator.

- The renormalized on-shell action can be obtained (up to a constant that depends on global properties) by integrating the relations

$$\mathcal{T} = \frac{\delta S_{\rm ren}}{\delta \alpha}, \quad \mathcal{O}_\psi = \frac{\beta}{\alpha}\frac{\delta S_{\rm ren}}{\delta \beta}, \quad \mathcal{J}^t = -\frac{1}{\alpha}\frac{\delta S_{\rm ren}}{\delta \mu}$$

  using the above expressions for the one-point functions.

- This gives the exact generating function:

$$S_{\rm ren}[\alpha, \beta, \mu] = -\frac{L}{2\kappa_2^2}\int \mathrm{d}t \left(\frac{m\alpha}{\beta} + \frac{\beta'^2}{\beta\alpha} + \frac{2\mu Q}{L}\right) + S_{\rm global}$$

- $S_{\rm global}$ involves terms evaluated on the horizon and its explicit form is given in [Castro, Larsen, I.P.:1807.06988].

- Under bulk diffeomorphisms and U(1) gauge transformations the non-dynamical components of the bulk fields transform as

$$\delta_\xi g_{uu} = \mathcal{L}_\xi g_{uu} = \dot{\xi}^u, \quad \delta_\xi g_{tt} = \mathcal{L}_\xi g_{ut} = \gamma_{tt}(\dot{\xi}^t + \partial^t \xi^u)$$
$$(\delta_\xi + \delta_\Lambda)A_u = \mathcal{L}_\xi A_u + \delta_\Lambda A_u = \dot{\xi}^t A_t + \dot{\Lambda}$$

  where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector ξ^a.

- To preserve the Fefferman-Graham gauge we must demand

$$\mathcal{L}_\xi g_{uu} = \mathcal{L}_\xi g_{ut} = 0, \quad (\mathcal{L}_\xi + \delta_\Lambda)A_u = 0$$

  which determines the form of the residual local symmetries to be

$$\xi^u = \sigma(t), \quad \xi^t = \varepsilon(t) + \partial_t\sigma(t)\int_u^\infty d\bar{u}\gamma^{tt}(\bar{u},t), \quad \Lambda = \varphi(t) - \sigma'(t)\int_u^\infty d\bar{u}\gamma^{tt}(\bar{u},t)A_t(\bar{u},t)$$

  where $\varepsilon(t)$, $\sigma(t)$ and $\varphi(t)$ are arbitrary functions of time.

- Under these residual symmetries the dynamical fields transform as

$$\delta_\xi\gamma_{tt} = L_\xi\gamma_{tt} + 2K_{tt}\xi^u, \quad (\mathcal{L}_\xi + \delta_\Lambda)A_t = L_\xi A_t + \xi^u\dot{A}_t + \partial_t\Lambda, \quad \delta_\xi\psi = L_\xi\psi + \xi^u\dot{\psi}.$$

- All three sources can be generated by the residual gauge transformations:

  $$\alpha = e^{\sigma}(1+\varepsilon'+\varepsilon\sigma')+\mathcal{O}(\varepsilon^2), \quad \beta = e^{\sigma}(1+\varepsilon\sigma')+\mathcal{O}(\varepsilon^2), \quad \mu = \varphi'+\varepsilon'\varphi'+\varepsilon\varphi''+\mathcal{O}(\varepsilon^2),$$

  where the primes $'$ denote a derivative with respect to t.

- Inserting these expressions in the renormalized action and absorbing total derivative terms in S_{global} we obtain

  $$S_{\text{ren}} = \frac{L}{\kappa_2^2}\int \mathrm{d}t\,(\{\tau,t\} - m/2) + S_{\text{global}}, \qquad \sigma = \log\tau',$$

  where the Schwarzian derivative is given by

  $$\{\tau,t\} = \frac{\tau'''}{\tau'} - \frac{3}{2}\frac{\tau''^2}{\tau'^2}$$

- The Schwarzian derivative action is a manifestation of the **conformal anomaly**!

Holographic dictionary for constant dilaton solutions

- The holographic dictionary for constant dilaton solutions is a bit more subtle, mainly due to the fact that the AdS_2 gauge field diverges close to the boundary:

$$A_t \sim \widetilde{\mu}(t) - \frac{\widetilde{\alpha}(t)}{\sqrt{LQ}} e^{u/\widetilde{L}}$$

- Two different boundary counterterms have been proposed to cancel the corresponding divergences of the on-shell action:
    - [Castro, Grumiller, Larsen, Mc Nees '08]

$$\sim \int \mathrm{d}t \sqrt{-\gamma} A_t A^t$$

    - [Grumiller, McNees, and Salzer '14; Grumiller, Salzer, Vassilevich '15]

$$\sim -\int \mathrm{d}t\ \pi^t A_t + \int \mathrm{d}t \sqrt{-\gamma} \sqrt{1 + \alpha_0 \pi_t \pi^t}$$

- Although both types of counterterms cancel the divergences of the on-shell action, neither respects the **symplectic structure** on the space of solutions, which can lead to inconsistencies at the level of correlation functions.

Holographic renormalization as a canonical transformation

- In order for the variational problem to be well posed the boundary counterterms must correspond to a suitable **canonical transformation** [I. P. '10].
- For the usual gauge field asymptotics the counterterms satisfy

$$\delta\left(S_{\rm reg}+S_{\rm ct}[\gamma,A,\psi]\right)=\int \mathrm{d}t\,\left(\pi^t+\frac{\delta S_{\rm ct}}{\delta A_t}\right)\delta A_t+\cdots$$

so that $S_{\rm ct}[\gamma,A,\psi]$ is the generating function of the canonical transformation

$$\begin{pmatrix}A_t\\ \pi^t\end{pmatrix}\to\begin{pmatrix}A_t\\ \Pi^t\end{pmatrix}=\begin{pmatrix}A_t\\ \pi^t+\frac{\delta S_{\rm ct}}{\delta A_t}\end{pmatrix}$$

- Since the gauge field modes are reversed for constant dilaton solutions, the generating function of the relevant canonical transformation is

$$-\int \mathrm{d}t\ \pi^t A_t + S_{\rm ct}[\gamma,\pi,\psi]$$

where

$$\boxed{S_{\rm ct} = -\frac{1}{2\kappa_2^2 L}\int \mathrm{d}t\left(\sqrt{-\gamma}\,e^{-\psi} + \frac{(L\kappa_2^2)^2}{\sqrt{-\gamma}}\ e^{3\psi}\pi^t\pi_t\right)}$$

- This implements the canonical transformation

$$\begin{pmatrix} A_t \\ \pi^t \end{pmatrix} \to \begin{pmatrix} -\pi^t \\ A_t^{\rm ren} \end{pmatrix} = \begin{pmatrix} -\pi^t \\ A_t - \frac{\delta S_{\rm ct}}{\delta \pi^t} \end{pmatrix}$$

such that

$$\pi^t \sim -\frac{1}{\kappa_2^2}Q, \qquad A_t^{\rm ren} = A_t - \frac{\delta S_{\rm ct}}{\delta\pi^t} \sim A_t + \frac{1}{\sqrt{LQ}}\sqrt{-\gamma} \sim \widetilde{\mu}(t)$$

preserving both the **symplectic structure** and the **gauge symmetries**.

Normalizability and boundary conditions

[O'Bannon,I.P., Probst: 1510.08123]

- Abelian vector field (generically massive)

$$S = -\int d^{d+1}x\sqrt{-g}\left(\frac{1}{4}f_{mn}f^{mn} + \frac{1}{2}m_a^2\, g^{mn}a_m a_n\right)$$

- Equation of motion

$$\nabla^m f_{mn} - m_a^2 a_n = 0$$

- The two linearly independent asymptotic solutions for a_m are of the form $e^{\delta_\pm r}$ with

$$\delta_\pm = -\frac{d-2}{2} \pm \sqrt{\left(\frac{d-2}{2}\right)^2 + m_a^2}.$$

- Dropping a boundary term from the action we define the new norm (cf. [Klebanov, Witten: hep-th/9905104] for scalars)

$$S' = \frac{1}{2}\int d^{d+1}x\sqrt{-g}\, a_p\, g^{pn}\left(\nabla^m f_{mn} - m_a^2 a_n\right)$$

  we find that both modes are normalizable provided

$$-\frac{(d-2)^2}{4} \leq m_a^2 < \frac{d(4-d)}{4}$$

Boundary counterterms and holographic dictionary

- Since Q is constant it does not define a local dual operator, but $\widetilde{\mu}(t)$ does define a local current. The renormalized generating functional in the theory that possesses a local current operator is

$$S_{\text{ren}} = \lim_{u\to\infty}\left(S_{\text{reg}} + S_{\text{ct}} - \int \mathrm{d}t\,\pi^t A_t + \int \mathrm{d}t\,\pi^t A_t^{\text{ren}}\right)$$

- If the finite term that implements the Legendre transformation is omitted one obtains the generating function of a theory without a current operator. This is a choice of boundary conditions.

- The renormalized one-point functions obtained from this renormalized action are

$$\mathcal{T} = 2\widehat{\pi}_t^t = 0, \quad \mathcal{O}_\psi = -\widehat{\pi}_\psi = -\frac{2}{\kappa_2^2 \widetilde{L}}\frac{\widetilde{\beta}}{\widetilde{\alpha}}, \quad \mathcal{J}^t = -\widehat{\pi}^t = \frac{1}{\kappa_2^2}\frac{Q}{\widetilde{\alpha}}$$

- In particular, the non-extremality parameter $\widetilde{\beta}$ of the constant dilaton solutions is identified with the VEV of the (irrelevant) scalar operator $\mathcal{O}_\psi$.

- Besides the current conservation $\mathcal{D}_t \mathcal{J}^t = 0$, the Ward identities are trivially satisfied, but become non-trivial once a perturbative source $\widetilde{\nu}$ for the scalar operator is turned on:

$$\partial_t \mathcal{T} + \mathcal{O}_\psi \partial_t \widetilde{\nu} = \mathcal{O}(\widetilde{\nu}^2), \qquad \mathcal{T} - \widetilde{\nu}\mathcal{O}_\psi = -\frac{\widetilde{L}(LQ)^{1/2}}{\kappa_2^2 \widetilde{\alpha}} \partial_t \left(\frac{\widetilde{\nu}'}{\widetilde{\alpha}} \right) + \mathcal{O}(\widetilde{\nu}^2)$$

- These imply that $\mathcal{O}_\psi$ has dimension 2, while the conformal anomaly matches that of the running dilaton solutions.

- The stress tensor is nonzero if and only if a source for the irrelevant scalar operator is turned on.

Outline

- 2D dilaton gravity captures the effective dynamics of near extremal black holes
- Consistent KK truncations are crucial for studying the RG flow away from IR fixed point and for generating solutions in higher dimensions
- Both modes of gauge fields in AdS_2 are normalizable, which allows for more general boundary conditions

Future directions

- When does the AdS_2 radius depend on AdS_2 Maxwell charge?
- Classification of supersymmetric boundary conditions
- 2D reductions of various supergravities
- Reduction of e.g. $\mathcal{N} = 4$ SYM to 1D using bulk consistent KK reduction