Notes on 3d supersymmetric gauge theories with massive matter fields

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Based on [KS, arxiv:1809.03679]

Introduction and Motivation

- With the help of SUSY localization methods, we can calculate some quantities exactly because a path integral reduces to the finite dimensional integrals.

$\Rightarrow$ For 3d theories, the finite dimensional integrals are tractable. [Kapstin-Willet-Yaakov '09] [Jafferis '10] [Hama-Hosomichi-Lee '10]

- The S^3 partition functions are divergent when the numbers of the matter multiplets are less than the threshold.

$\Rightarrow$ This threshold is important since it is related to the bound where the IR physic drastically changes. [Gaiotto-Witten `08]

- Of particular interest are cases in which the mass-deformation naively leads to such a 'divergent' theory due to decoupling of the massive fields.

Decoupling of matter fields

$$\underbrace{Z^{S^3}(m=0)}_{\text{convergent}} \Rightarrow \text{divergent ?}$$

The key point is decoupling of the matter fields $(m \to \infty)$

- Which matter fields will decouple depends on a choice of a vacuum since Higgs mechanism gives a mass to matter fields.

→ <u>A choice of vacuum is need to argue decoupling of matter fields.</u>

- The S^3 partition function is represented by integrations over the Coulomb branch (SUSY vacua) parameters. This implies that there are no choice of vacua.

→ <u>we cannot immediately know which matter fields decouple from the theory.</u>
(It is ambiguous whether we can argue decoupling of matter fields on three-sphere or not.)

What happens to the partition function when $m \to \infty$

Summary of our result

- We investigate $\mathcal{N}=4$ $\mathrm{U}(N)$ SQCD with massive matter fields. (two types of deformation)
- A specific point of the Coulomb branch dominantly contributes to the three-sphere partition function Z .
  (here we focus on the leading part in the mass infinite limit)

$$Z^{S^3}_{\mathrm{U}(N)} = \frac{1}{N!}\int\left(\prod_i^N d\sigma_i\right) e^{-S[\sigma]} \underset{(m\to\infty)}{\sim} e^{-S[\sigma_0]}. \quad \sigma_i\text{: Coulomb branch (CB) pamameter.}$$

- In this limit, the three-sphere partition function becomes

$$\boxed{Z^{S^3}_{\text{massive U}(N)} \to Z^{S^3}_{\text{free massive}}(m) Z^{S^3}_{\text{Higgsed Theory}}}$$

- These means that in the infinite mass limit, the massive U(N) theory higgses to the effective theory on the dominant point $\boldsymbol{\sigma_0}$.

Set up

Set up ($\mathcal{N}=4$ U(N) SQCD with N_f massive flavors)

- Model 1 $(N_f \geq 2N-2)$

Vector fields part

$$Z^{N_f}_{\text{Massive U}(N)} = \frac{1}{N!}\int \prod_{i=1}^{N} d\sigma_j \frac{\prod_{i<j} 4\sinh^2\left(\pi(\sigma_i-\sigma_j)\right)}{\prod_i \left(\underline{2\cosh\pi(\sigma_i+m)2\cosh\pi(\sigma_i-m)}\right)^{\frac{N_f}{2}}}$$

⇒ No massless fields

Flavors part

- Model 2 $(2N > \frac{N_f}{3} > \frac{2N-2}{3})$ or $(\frac{N_f}{3} \geq 2N)$

$$Z^{N_f}_{\text{massive U}(N)} = \frac{1}{N!}\int \prod_{i=1}^{N} dx_i \frac{\prod_{i<j} 4\sinh^2\left(\pi(\sigma_i-\sigma_j)\right)}{\prod_i \left(2\cosh\pi(\sigma_i+m)2\cosh\pi(\sigma_i-m)2\cosh\pi(\sigma_i)\right)^{\frac{N_f}{3}}}.$$

⇒ $\frac{N_f}{3}$ massless fields

Naïve consideration of decoupling

- Model 1 $(N_f \geq 2N-2)$

If decouple

$$Z^{N_f}_{\text{Massive U}(N)} = \frac{1}{N!}\int \prod_{i=1}^{N} d\sigma_j \frac{\prod_{i<j} 4\sinh^2\left(\pi(\sigma_i-\sigma_j)\right)}{\cancel{\prod_i \left(2\cosh\pi(\sigma_i+m)2\cosh\pi(\sigma_i-m)\right)^{\frac{N_f}{2}}}}. \Rightarrow \frac{1}{N!}\int \prod_{i=1}^{N} d\sigma_j \prod_{i<j} 4\sinh^2\left(\pi(\sigma_i-\sigma_j)\right)$$

Divergent

Naïve consideration of decoupling

- Model 2

If decouple

$$Z^{N_f}_{\text{Massive U}(N)} = \frac{1}{N!}\int \prod_{i=1}^{N} dx_i \frac{\prod_{i<j} 4\sinh^2\left(\pi(x_i-x_j)\right)}{\cancel{\prod_i \Big(2\cosh\pi(x_i+m)2\cosh\pi(x_i-m)}2\cosh\pi(x_i)\Big)^{\frac{N_f}{3}}}. \Rightarrow \frac{1}{N!}\int \prod_{i=1}^{N} dx_i \frac{\prod_{i<j} 4\sinh^2\left(\pi(x_i-x_j)\right)}{\left(\prod_i 2\cosh\pi(x_i)\right)^{\frac{N_f}{3}}}.$$

$$\Rightarrow \begin{cases} \text{Divergent} & (2N > \frac{N_f}{3} > \frac{2N-2}{3}) \\ \text{Convergent} & (\frac{N_f}{3} \geq 2N) \end{cases}$$

From the large N limit to finite N calculation

- To investigate the dominant point, we consider the large N limit.

- The partition function can be evaluated by the solution of the saddle point equation in the large N limit

$$\frac{\partial S[\sigma]}{\partial \sigma_i} = 0$$

- We focus on the solution of the saddle point equation since this corresponds to a point of the CB $(\sigma_{01}, \cdots, \sigma_{0N})$.

⇒ We can regard the corresponding point as the dominant one in the large N limit and large mass limit.

- Then we solve the saddle point equation in large N limit (by the resolvent methods)

- Moreover, we expect that the point of the CB is the dominant in the large mass limit even for finite N case.

- We check this expectation from actual calculation of the partition function for sufficiently small rank N

The result of the model 1

The result of the Model1

- We can calculate the leading part of the partition function.
  The leading part of the partition function becomes in the infinite mass limit

$$\boxed{Z^{N_f}_{\text{massive U}(N)} \to Z^{\frac{N_f}{2}}_{\mathrm{U}(\frac{N}{2})} Z^{\frac{N_f}{2}}_{\mathrm{U}(\frac{N}{2})} Z_{\text{massive free}}}$$

- The result suggests that the dominant point of the CB corresponds to

$$\sigma_0 = \left(\begin{array}{c|c} -m\mathbf{1}_{\frac{\mathbf{N}}{\mathbf{2}}\times\frac{\mathbf{N}}{\mathbf{2}}} & \mathbf{0} \\ \hline \mathbf{0} & m\mathbf{1}_{\frac{\mathbf{N}}{\mathbf{2}}\times\frac{\mathbf{N}}{\mathbf{2}}} \end{array}\right) \qquad \text{since } \mathrm{U}(N) \to \mathrm{U}(\tfrac{N}{2})\times \mathrm{U}(\tfrac{N}{2})$$

The result of the model 2

The result of the Model 2 for $(2N > \frac{N_f}{3} > \frac{2N-2}{3})$ (less massless matter fields)

- The partition function becomes factorized.

$$\boxed{Z^{N_f}_{\text{massive U}(N)} \to Z^{\frac{N_f}{3}}_{\mathrm{U}(\frac{6-\xi}{12}N)} Z^{\frac{N_f}{3}}_{\mathrm{U}(\frac{\xi}{6}N)} Z^{\frac{N_f}{3}}_{\mathrm{U}(\frac{6-\xi}{12}N)} Z_{\text{massive free}}}$$

- The gauge group higgses to

(When $\frac{\xi}{6}N$ and $\frac{6-\xi}{12}N$ are integers.)

$$\boxed{\mathrm{U}(N) \to \mathrm{U}(\tfrac{6-\xi}{12}N) \times \mathrm{U}(\tfrac{\xi}{6}N) \times \mathrm{U}(\tfrac{6-\xi}{12}N)}$$

- In this case, the dominant point of the CB is $\sigma_0 = \left(\begin{array}{c|c|c} -m\mathbf{1}_{\mathbf{N_2}\times\mathbf{N_2}} & & \\ \hline & \mathbf{0}_{\mathbf{N_1}\times\mathbf{N_1}} & \\ \hline & & m\mathbf{1}_{\mathbf{N_2}\times\mathbf{N_2}} \end{array}\right).$

$$N_1 = \frac{\xi}{6}N, \quad N_2 = \frac{6-\xi}{12}N, \quad \frac{N_f}{N} \equiv \xi,$$

The result of the Model 2 for $(\frac{N_f}{3} \geq 2N)$ (Enough massless matter fields!)

$$\boxed{Z^{N_f}_{\text{massive U}(N)} \to Z^{\frac{N_f}{3}}_{\text{U}(N)} Z_{\frac{2N_f}{3}\text{massive free hyper}}}$$

the massive hyper multipelts decouple from the theory because the partition function is still convergent.

$$Z \sim \frac{1}{N!} \int \prod_{i=1}^{N} dx_i \frac{\prod_{i<j} 4\sinh^2\left(\pi(x_i - x_j)\right)}{\left(\prod_i 2\cosh \pi(x_i)\right)^{\frac{N_f}{3}}}.$$

- This implies that the origin of the CB is dominant in the large mass limit.

Summary

Summary

- Model 1
- Model 2 for $\left(2N > \frac{N_f}{3} > \frac{2N-2}{3}\right)$

In the infinite mass limit, a non-trivial effective theory appears, not divergent theory.

Summary

- A specific point of the CB is chosen in this limit. The dominant point can depend on the number of the flavors.

$$Z^{S^3}_{\mathrm{U}(N)} = \frac{1}{N!}\int \left(\prod_i^N d\sigma_i\right) e^{-S[\sigma]} \sim e^{-S[\sigma_0]}.$$

- In the mass infinite limit, the partition function becomes factorized into two parts,

$$\boxed{Z^{S^3}_{\text{massive U}(N)} \to Z^{S^3}_{\text{free massive}}(m) Z^{S^3}_{\text{Higgsed Theory}}}$$

Discussion

- It is interesting to proof the F-theorem directly by the properties of S^3 partition function.

- We can counter the $Z^{S^3}_{\text{free massive}}(m) \sim e^{\#\pi m}$ by introducing Einstein-Hilbelrt term $\Lambda \int_{S^3} d^3x \sqrt{g} R + \dots$ and tuning Λ.

  This is reason why we ignore the decouple free massive sector when we consider F-theorem.

- In some cases, the sphere partition function may not corresponds to a IR fixed

  (ex) when *N/2* is not integer for model 1

- Model 1 $(N_f \geq 2N-2)$

$$Z=\frac{1}{N!}\int_{-\infty}^{\infty}\prod_{i=1}^{N}d\lambda_j\frac{\prod_{i<j}4\sinh^2\left(\pi(x_i-x_j)\right)}{\prod_i\left(2\cosh\pi(x_i+m)2\cosh\pi(x_i-m)\right)^{\frac{N_f}{2}}}.$$

$\xrightarrow{m \to \infty}$?

Naively, Pure Yang-Mills theory appear ? $\Rightarrow$ **No**

(That is divergent Integral)

- It is expected that there appear effectively massless multiplets on the dominant point of the CB .

$\Rightarrow$ <u>A non-trivial effective theory will appear in this limit,</u>
(not pure Yang-Mills)

- Summary of analysis of model 1

From large N analysis, we anticipate that the partition function becomes when $\frac{N}{2}$ is an integer

$$\boxed{Z^{S^3}_{\text{massive U}(N)} \to Z_{\mathrm{U}(\frac{N}{2})} Z_{\mathrm{U}(\frac{N}{2})} Z_{\text{massive free}}}$$

- We can confirm this assumption for finite N (sufficiently small) when $\frac{N}{2}$ is an integer (This calculation can be done by mathematica!!)

- The solution is given by the resolvent methods

$$\rho(X) = \frac{\xi}{2}\left[\frac{M\sqrt{(a-X)(X-\frac{1}{a})}}{(X+M)\sqrt{(M+a)(M+\frac{1}{a})}} + \frac{M^{-1}\sqrt{(a-X)(X-\frac{1}{a})}}{(X+M^{-1})\sqrt{(M^{-1}+a)(M^{-1}+\frac{1}{a})}}\right].M \equiv e^{2\pi m}, X \equiv e^{2\pi x}$$

$$-4 = 2\xi\left(-1 + \frac{1}{\sqrt{(M+a)(M+\frac{1}{a})}} + \frac{1}{\sqrt{(M^{-1}+a)(M^{-1}+\frac{1}{a})}}\right)$$

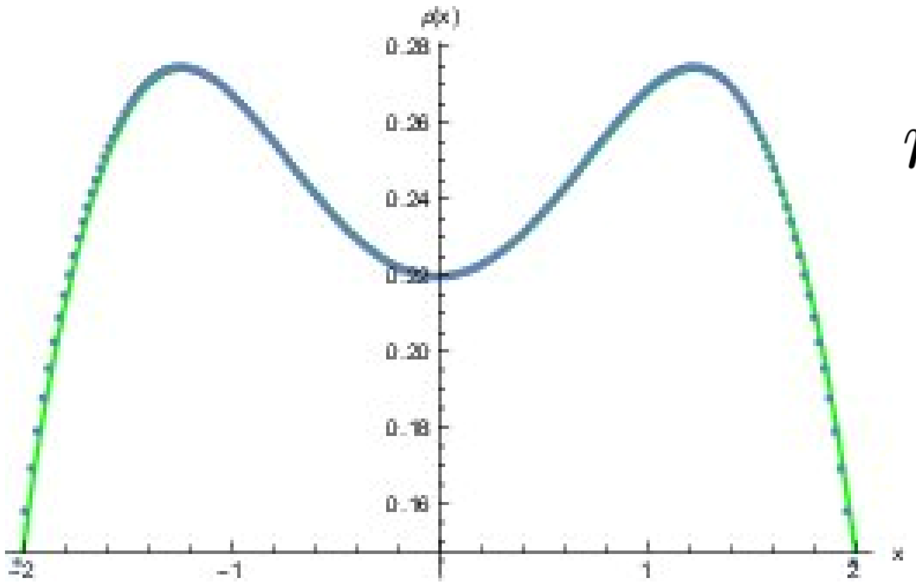
We can solve the saddle point equation with the resolvent methods.

From the exact solution, σ_i are concentrated around $\pm m$

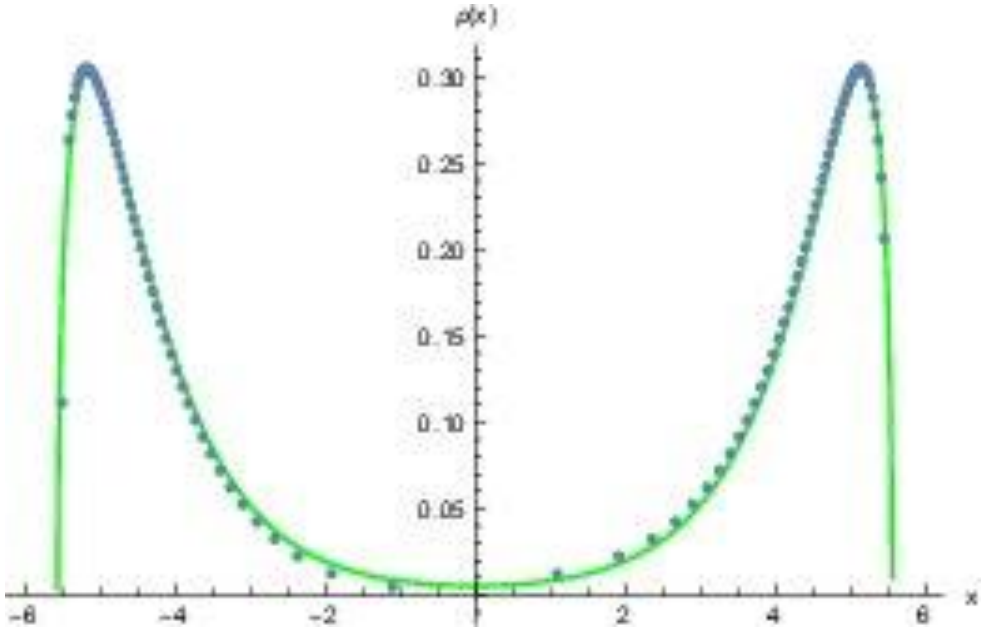
assume

$$\sigma_i = \begin{cases} m - \lambda_i & (i = 1, \ldots \frac{N}{2}), \\ -m - \widetilde{\lambda}_i & (i = \frac{N}{2} + 1 \ldots N), \end{cases}$$

Pictures of $\rho(\sigma)$



$m \to \text{large}$



Form large N analysis, we expect that the gauge group is broken in large mass limit

When $\frac{\xi}{6}N$ and $\frac{6-\xi}{12}N$ are integers.

$$\boxed{\mathrm{U}(N) \to \mathrm{U}(\tfrac{6-\xi}{12}N) \times \mathrm{U}(\tfrac{\xi}{6}N) \times \mathrm{U}(\tfrac{6-\xi}{12}N)}$$

$$\boxed{Z^{S^3}_{\text{massive U}(N)} \to Z_{\mathrm{U}(\frac{6-\xi}{12}N)} Z_{\mathrm{U}(\frac{\xi}{6}N)} Z_{\mathrm{U}(\frac{6-\xi}{12}N)} Z_{\text{massive free}}}$$

$$\Rightarrow \quad \sigma_i = \begin{cases} -m - \lambda_i^1, & (i = 1, \ldots, N_2), \\ \lambda_i^2, & (i = N_2 + 1, \ldots N_1 + N_2), \\ m - \lambda_i^3 & (i = N_1 + N_2 + 1, \ldots, N). \end{cases} \sim \sigma = \left(\begin{array}{c|c|c} -m\mathbf{1}_{\mathbf{N_2}\times\mathbf{N_2}} & & \\ \hline & \mathbf{0}_{\mathbf{N_1}\times\mathbf{N_1}} & \\ \hline & & m\mathbf{1}_{\mathbf{N_2}\times\mathbf{N_2}} \end{array}\right).$$

- Actually, it can be confirmed by the sufficiently small N analysis.

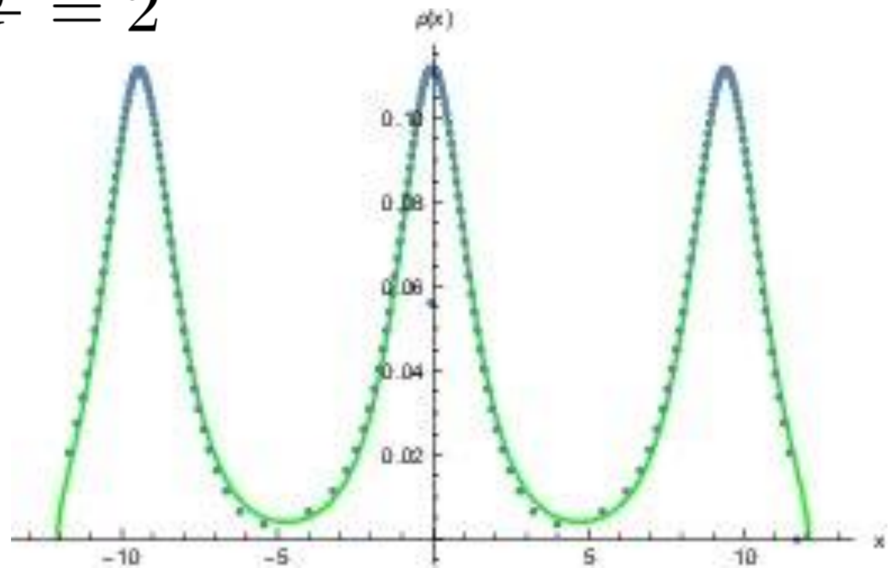
(i) $(2N > \frac{N_f}{3})$

- There are three peaks around $\pm m$ and 0
- The number of the eigenvalues around each peaks is different!

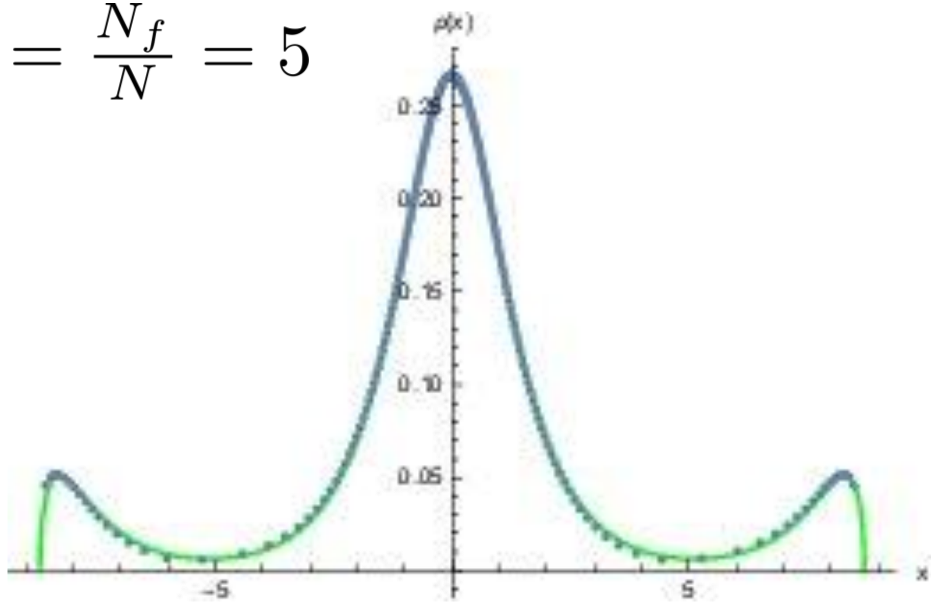
$$\mathrm{U}(N) \rightarrow \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3)$$

- the rank of each gauge group is determined by the number of the eigenvalues around each peaks.

$\xi = \frac{N_f}{N} = 2$



$\xi = \frac{N_f}{N} = 5$



(ex) Model 1 for $N = 2$ and $N_f = 4$

$$Z = \frac{1}{2!}\int_{-\infty}^{\infty} dxdy \frac{4\sinh^2\pi(x-y)}{(2\cosh\pi(x+m)2\cosh\pi(x-m)\cdot 2\cosh\pi(y+m)2\cosh\pi(y-m))^{\frac{N_f}{2}}}$$

$$= \frac{M\left(M\left(M^2-1\right)^2 - 4M^3\log^2(M)\right)}{4\pi^2\left(M^2-1\right)^4} \qquad (M = e^{2\pi m})$$

- Summary of the analysis of model 2

(i) $\left(2N > \frac{N_f}{3} \geq \frac{2N-2}{3}\right)$ When $\frac{\xi}{6}N$ and $\frac{6-\xi}{12}N$ are integers.

$$\boxed{Z^{S^3}_{\text{massive U}(N)} \to Z_{\mathrm{U}(\frac{6-\xi}{12}N)} Z_{\mathrm{U}(\frac{\xi}{6}N)} Z_{\mathrm{U}(\frac{6-\xi}{12}N)} Z_{\text{massive free}}}$$

<u>Non-trivial point of CB is dominant.</u>

(ii) $\left(\frac{N_f}{3} \geq 2N\right)$

$$\boxed{Z^{S^3}_{\text{massive U}(N)} \to Z_{\mathrm{U}(N)} Z_{\frac{N_f}{2}\text{massive free hyper}}}$$

<u>Origin of CB is dominant.</u>