Holography, Weyl Anomaly and Induced Current in BCFT

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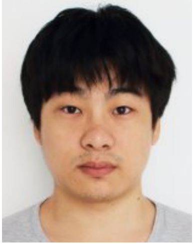

Wu-Zhong Guo

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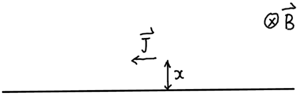
1706.09652 with Miao,
1803.03068 with Miao,
1804.01648 with Miao,
plus work in progress hep-th/1811....

Outline

Goals

- Casimir effect arises from energetic response of the vacuum to the presence of boundary. In this talk, I want to talk about a new kind of response of the vacuum to the boundary:



$$\vec{J} = c\,\frac{\vec{n}\times\vec{B}}{x}$$

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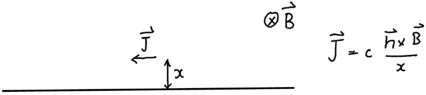
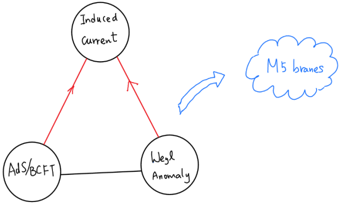
- Casimir effect arises from energetic response of the vacuum to the presence of boundary. In this talk, I want to talk about a new kind of response of the vacuum to the boundary:



- This can be derived from 1. exact analysis of BCFT or from 2. AdS/BCFT
- 3. We will apply it to boundary M5-branes system and learn something about it.



Outline

1.1.1. Casmir effect in BCFT

- In general, for a d-dimensional BQFT, vev of renormalized stress tensor has the asymptotic behaviour near boundary:

$$\langle T_{ij}\rangle = x^{-d} T^{(d)}_{ij} ... + x^{-1} T^{(1)}_{ij} + \cdots , \quad x \sim 0,$$

x is proper distance from boundary, and

$$T^{(d)}_{ij} = \alpha_0 h_{ij}, \qquad T^{(d-1)}_{ij} = 2\alpha_1 \bar{k}_{ij},$$
$$T^{(d-2)}_{ij} = \frac{-4\alpha_1}{d-1} n_{(i} h^l_{j)} \nabla_l k - \frac{4\alpha_1}{d-2} n_{(i} h^l_{j)} n^p R_{lp} + \frac{2\alpha_1}{d-2}(n_i n_j - \frac{h_{ij}}{d-1})\mathrm{Tr}\bar{k}^2 + t_{ij},$$
$$t_{ij} := \lceil \beta_1 C_{ikjl} n^k n^l + \beta_2 \mathcal{R}_{ij} + \beta_3 k k_{ij} + \beta_4 k^l_i k_{lj} \rceil,$$

where n_i, h_{ij} and $\bar{k}_{ij}$ are respectively the normal vector, induced metric and the traceless part of extrinsic curvature of the boundary P.

1.1.1. Casmir effect in BCFT

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  where n_i, h_{ij} and $\bar{k}_{ij}$ are respectively the normal vector, induced metric and the traceless part of extrinsic curvature of the boundary P.
- The Casimir coefficient $(\alpha_0, \alpha_1, \beta_i)$ fixes the shape dependence of the leading Casimir effects of BCFT.
- For BCFT, conformal symmetry requires $\alpha_0 = 0$.

1.1.2. Casimir effects from Weyl Anomaly

Boundary Weyl anomaly

- Weyl anomaly

$$\mathcal{A} := \partial_\sigma W[e^{2\sigma} g_{ij}]|_{\sigma=0} = \int_M \langle T_i^i \rangle,$$

where

$$\left\langle T_i^i \right\rangle = \left\langle T_i^i \right\rangle_M + \delta(x_\perp) \langle T_a^a \rangle_P \, .$$

$\langle T_i^i \rangle_M =$ bulk Weyl anomaly, $\langle T_a^a \rangle_P =$ boundary Weyl anomaly.

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$\left\langle T^i_i \right\rangle_M$ = bulk Weyl anomaly, $\langle T^a_a \rangle_P$ = boundary Weyl anomaly.
- Weyl anomaly are classified in terms of curvature invariants.
3d: $\left\langle T^i_i \right\rangle = \delta(x)[b_1 \mathcal{R} + b_2 \mathrm{Tr}\bar{k}^2]$
4d: $\left\langle T^i_i \right\rangle = \frac{c}{8}\mathrm{Tr}C^2 - \frac{a}{16\pi^2} E_4 + \delta(x)[\frac{a}{16\pi^2} E_4^{\mathrm{bdy}} + b_3 \mathrm{Tr}\bar{k}^3 + b_4 C^{ac}{}_{bc} \bar{k}^b{}_a]$
- Bulk central charges c do not depend on BC. Boundary central charges b_i depend on BC in general.

Claim: the energy momentum tensor $T_{\mu\nu}$ has universal behaviour near the boundary. (Chu, Miao 17, 18)

- Consider a BCFT with a well defined effective action. The Weyl anomaly $\mathcal{A}$ can be obtained as the logarithmic UV divergent term of the effective action,

$$I = \cdots + \mathcal{A}\log(\frac{1}{\epsilon}) + I_{\text{finite}}, \quad \epsilon = \text{UV cutoff.}$$

- From this one obtain the "integrability condition":

$$\boxed{(\delta\mathcal{A})_{\partial M} = \left(\frac{1}{2}\int_{x\geq\epsilon}\sqrt{g}\,T^{ij}\delta g_{ij}\right)_{\log(1/\epsilon)}}$$

where $(\delta\mathcal{A})_{\partial M}$ is the boundary terms in the variations of Weyl anomaly and T^{ij} is the renormalized bulk stress tensor.

- Note that the right hand side must give an exact variation, this imposes strong constraints on the possible form of the stress tensor near the boundary.

- E.g. For 3d BCFT, Weyl anomaly has only boundary contributions

$$\mathcal{A} = \int_{\partial M} \sqrt{h}(b_1 \mathcal{R} + b_2 \mathrm{Tr}\bar{k}^2),$$

$$LHS = (\delta\mathcal{A})_{\partial M} = b_2 \int_{\partial M} \sqrt{h}\Big[(\frac{\mathrm{Tr}\bar{k}^2}{2} h^{ab} - 2\bar{k}^a_c k^{cb})\delta h_{ab} + 2\bar{k}^{ab}\delta k_{ab}\Big].$$

On the other hand, we can use the near-boundary expression of $T_{\mu\nu}$, integrate over x and pick up the log divergent term

$$\begin{aligned} RHS = & - \alpha_1 \int_P \sqrt{h}[(\frac{\mathrm{Tr}\bar{k}^2}{2} h^{ab} - 2\bar{k}^a_c k^{cb})\delta h_{ab} + 2\bar{k}^{ab}\delta k_{ab}] \\ & + \int_P \sqrt{h}[(\frac{\beta_3}{2} - \alpha_1) k\bar{k}^{ab}\delta h_{ab} + \frac{\beta_4}{2}\lceil k^a_c k^{cb}\rceil \delta h_{ab}]. \end{aligned}$$

The integrability condition then give

$$\boxed{\alpha_1 = -b_2, \quad \beta_3 = -2b_2, \quad \beta_4 = 0.}$$

- Similar analysis for 4d.

- It is remarkable that the Casimir coefficients are completely determined by the boundary central charges.
- The relations between them are universal and independent of BC and theory.

1.2.1. Chiral Anomaly and Transport

Two famous Anomaly induced transport of charges:

- The chiral magnetic effect (CME) which refers to the generation of currents parallel to an external magnetic field **B**. (Vilenkin 80; Giovannini Shaposhnikov 98; Froehlich etal 98)

$$\mathbf{J}_V = \sigma_{(\mathcal{B})V}\mathbf{B} \quad \mathbf{J}_A = \sigma_{(\mathcal{B})A}\mathbf{B},$$

where the chiral magnetic conductivities are

$$\sigma_{(\mathcal{B})V} = \frac{e\mu_A}{2\pi^2}, \quad \sigma_{(\mathcal{B})A} = \frac{e\mu_V}{2\pi^2}$$

- The chiral vortical effect (CVE) refers to the generation of a current due to rotational motion in the charged fluid. (Kharzeev, Zhitnitsky 07; Erfmemger etal 09; Son etal 09; Landsteiner etal 11)

$$\mathbf{J}_V = \sigma_{(\mathcal{V})V}\omega, \quad \mathbf{J}_A = \sigma_{(\mathcal{V})A}\omega,$$

where the chiral vortical conductivities are

$$\sigma_{(\mathcal{V})V} = \frac{\mu_V\mu_A}{\pi^2}, \quad \sigma_{(\mathcal{V})A} = \frac{\mu_V^2+\mu_A^2}{2\pi^2} + \frac{T^2}{6}$$

- Note that these anomalous transport occurs only in a material system where the chemical potentials are non-vanishing.

Q. In the presence of boundary, can the phenomena of anomalous transport occur in vacuum like Casimir effect? if so, how is it possible?

1.2.2. Weyl Anomaly and induced current

- Just as energy momentum tensor has an universal expansion near boundary, the renormalized current is also divergent near the boundary:

$$\langle J_i \rangle = x^{-3} J_i^{(3)} + x^{-2} J_i^{(2)} + x^{-1} J_i^{(1)}, \quad x \sim 0,$$

  where x is the proper distance from the boundary and $J_i^{(n)}$ depend only on the background geometry and the background vector field strength.
- Imposing current conservation

$$\nabla_i \langle J^i \rangle = 0,$$

  we obtain the gauge invariant solutions

$$J_\mu^{(3)} = 0, \qquad J_\mu^{(2)} = 0,$$
$$J_\mu^{(1)} = \alpha_1 F_{\mu\nu} n^\nu + \alpha_2 \mathcal{D}_\mu k + \alpha_3 \mathcal{D}_\nu k_\mu^\nu + \alpha_4 \star F_{\mu\nu}\, n^\nu$$

- Like the case of $T_{\mu\nu}$, these current coefficients can also be determined in terms of central charge

- One has similarly the integrability condition

$$\boxed{(\delta\mathcal{A})_{\partial M} = \left(\int_M \sqrt{g} J^\mu \delta A_\mu\right)_{\log\frac{1}{\epsilon}}}$$

- Consider background $U(1)$ gauge field, e.g. QED

$$\mathcal{A} = \int_M \sqrt{g}[b_1 F_{\mu\nu}F^{\mu\nu} + \text{metric part}], \quad b_1 = -\beta(e)/(2e^3), \text{ centeral charge.}$$

This implies that

$$(\delta\mathcal{A})_{\partial M} = -4b_1 \int_{\partial M} \sqrt{h} F^b{}_n \delta a_b.$$

Matching it with the RHS of the integrability condition, we obtain

$$\alpha_1 = 4b_1, \quad \alpha_2 = \alpha_3 = \alpha_4 = 0$$

and for the expectation value of the current

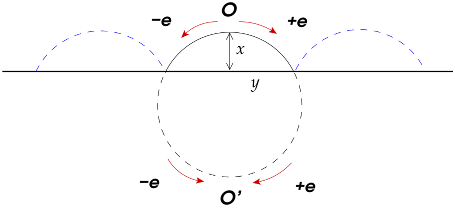
$$\boxed{J_b = \frac{4b_1 F_{bn}}{x}, \quad x \sim 0}$$

- Spelling it out, we obtain the induced current near the boundary of a BQFT:

$$\boxed{\mathbf{J} = \frac{e^2 c}{\hbar}\frac{4 b_1 \mathbf{n} \times \mathbf{B}}{x}, \quad x \sim 0}$$

Q. What is the physics of this current?

- The current is a result of charge separation due to vacuum fluctuation in the presence of external B-field.



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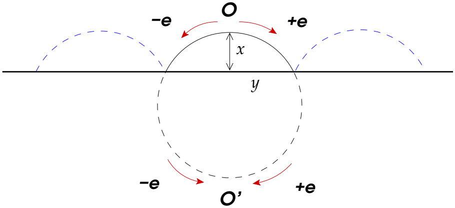
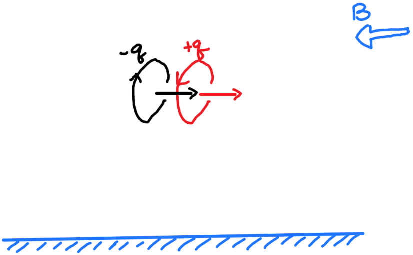


- Equivalently, it can also be seen as arising from the magnetization of the vacuum due to presence of boundary: $\mathbf{J} = \nabla \times \mathbf{M}$.



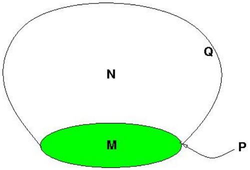
Outline

2.1. Statement of AdS/BCFT

- Consider d dimensional CFT defined on $R^{1,d-1}$. There is a $SO(2,d)$ conformal symmetry. This is realized in holography as isometries of the bulk of AdS$_{d+1}$ space (Takayanagi 11)

$$ds^2 = \frac{dz^2 + dx_i^2}{z^2}, \quad z \geq 0.$$

- When a boundary is introduced, the full conformal symmetry is reduced at the boundary. Takayanagi proposed to extend the d dimensional manifold M to a $d+1$ dimensional manifold N so that $\partial N = M \cup Q$, where Q is a d dimensional manifold which satisfies $\partial Q = \partial M = P$.



- The bulk grav action is given by

$$I = \int_N \sqrt{G}(R - 2\Lambda) + 2\int_M \sqrt{g}K + 2\int_Q \sqrt{h}(K - T) + 2\int_P \sqrt{\sigma}\theta,$$

T measures the boundary degrees of freedom (g-function).
differential equation for Q.

- The central issue is the determination of the location of Q in the bulk. Takayanagi proposed to impose Neumann boundary condition on Q to fix its position:

$$\text{EOM of } Q: \quad K_{\alpha\beta} - (K - T)h_{\alpha\beta} = 0, \qquad (*1)$$

  This gives a second order

- However since Q is of co-dimension 1,t the location of Q is determined by a single embedding function:

$$z = z(x^i), \quad \text{here } x^i = \text{coordinates of } M$$

  The embedding equation

$$K_{\alpha\beta} - (K - T)h_{\alpha\beta} = 0, \qquad (*1)$$

  generally imposes too many constraints and (*1) does not has solution for general shape P of BCFT.

(Chu, Guo, Miao 17; Chu, Miao 17)

- Alternatively, we proposed to impose on Q a mixed BC so that the trace part of Takayanagi's proposal is effective:

$$\boxed{(1-d)K + dT = 0} \qquad (*2)$$

  (*2) is natural as there is only one embedding function for Q and we expect one condition for it.

- This is a consistent proposal and describes the duals for a wide class of BCFTs, with the original proposal of Takayanagi as a special case.

Non FG (Fefferman-Graham) expansion

- In the standard AdS/CFT, FG expansion of the bulk metric is assumed.

$$ds^2 = \frac{dz^2 + g_{ij} dx^i dx^j}{z^2},$$

where $g_{ij} = g^{(0)}_{ij} + z^2 g^{(1)}_{ij} + \cdots$. $g^{(0)}_{ij}$ is the metric of BCFT on M. $g^{(1)}_{ij}$ is fixed by the PBH (Penrose-Brown-Henneuax) transformation:

$$g^{(1)}_{ij} = -\frac{1}{d-2}\left(R^{(0)}_{ij} - \frac{R^{(0)}}{2(d-1)} g^{(0)}_{ij}\right).$$

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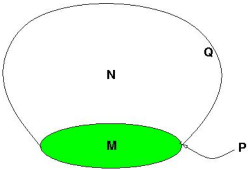
- However in the case of AdS/BCFT, the dual manifold N has discontinuity at the corner P where Q and M meet.



- We have to give up the assumption that the dual space has a metric that can be FG expanded.

- The problem is to solve Einstein equation for the dual metric, but without the assumption that the me tric can be expanded in small z near M. Hard!
- Convenient to use the Gauss normal coordinates. The metric $g_{ij}^{(0)}$ of the BCFT takes the form

$$ds_0^2 = dx^2 + (\sigma_{ab} + 2xk_{ab} + x^2 q_{ab} + \cdots)dy^a dy^b,$$

  where P is located at $x = 0$.
- We found a new systematic construction of non-FG expanded metric by employing k_{ab}, q_{ab} etc as expansion parameter, but keeping both the z and x dependence as exact. In this way, we are able to construct a perturbative solution to the bulk Einstein equation: (Chu, Miao 2017)

$$ds^2 = \frac{dz^2 + dx^2 + \left(\delta_{ab} - 2x\bar{k}_{ab} f\left(\frac{z}{x}\right)\right) dy^a dy^b}{z^2} + \cdots .$$

- At the order $O(k)$, the Einstein equation has the solution

$$f(s) = 1 - \lambda_1 \frac{s^d \, {}_2F_1\left(\frac{d-1}{2}, \frac{d}{2}; \frac{d+2}{2}; -s^2\right)}{d},$$

where λ_1 is free parameter.

- $\lambda_1 = 0$ gives FG. More free constants appear in higher order terms of the solution.

2.2 Application: Induced current near boundary

- To investigate the renormalized current in holographic models of BCFT, we consider the following gauge invariant action for holographic BCFT

$$I = \int_N \sqrt{G}[R - 2\Lambda - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] + 2\int_Q \sqrt{\gamma}[K - T]$$

- Consider 4d and planar boundary, one can solve for the bulk Maxwell equation and BC, and obtain

$$\mathcal{A}_a = F_{xa}\sqrt{x^2 + z^2}, \tag{2}$$

where F_{xa} is the field strength at the boundary.

- The holographic current is

$$\langle J^a \rangle = \lim_{z \to 0} \frac{\delta I}{\delta A_a} = \lim_{z \to 0} \sqrt{G}\mathcal{F}^{za} = -\frac{F_{ax}}{x} + \cdots \tag{3}$$

Same as before!

Outline

3.1 Boundary string current from holography

- Consider a charged particle moving on the worldline C: $x^\mu = x^\mu(\tau)$. The motion gives a current

$$J^\mu(x) = \delta^{(d-1)}(x - x(\tau))\frac{dx^\mu(\tau)}{d\tau}.$$

It couples to the gauge field as

$$\int_M J_\mu A^\mu = \int_C A_\mu dx^\mu$$

- Similarly, movement of strings gives the higher 2-form current

$$J_{\mu\nu} = \delta^{(d-2)}(x - x(\sigma,\tau))\epsilon^{\alpha\beta}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}.$$

It couples to the 2-form potential $B_{\mu\nu}$ as

$$\int_M J_{\mu\nu}B^{\mu\nu} = \int_\Sigma B_{\mu\nu}dx^\mu dx^n$$

over the worldsheet.

Q. Any implication of knowing the existence of such a coupling?

- Consider a BCFT in 6d and denote the Weyl anomaly as $\mathcal{A}$. The gravational part is well understood.
  Q. What about the contribution from background gauge field?
- One can similarly establish the relation

$$\boxed{(\delta \mathcal{A})_{\partial M} = \Big(\int_{M_\epsilon} J_{\mu\nu} \delta B^{\mu\nu} \Big)_{\log 1/\epsilon}.}$$

  Thus, knowing the current, or vice versa, would allow us to learn something about the anomaly structure of the 6d CFT.

- Consider a 6d BCFT dual to the the bulk action with an H-field

$$I = \int d^7x\sqrt{G}(R - 2\Lambda - \frac{1}{12}H^2_{\mu\nu\lambda})$$

Using our BCFT holography, one finds a string current parallel to the boundary when a H-field strength is turned on

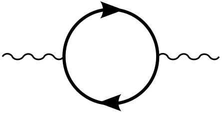
$$J_{ab} = b_1 \frac{H_{abx}}{x}.$$

3.2. Prediction for the Weyl anomaly in 6d

- The relation $(\delta \mathcal{A})_{\partial M} = \left(\int_{M_\epsilon} J_{\mu\nu} \delta B^{\mu\nu} \right)_{\log 1/\epsilon}$ predicts the Weyl anomaly in the 6d CFT:

$$\boxed{\mathcal{A} = \int_M \frac{b_1}{12} H^2}$$

- It is interesting to understand how matter fields would couple to the $B_{\mu\nu}$ field (covariant derivatives?) and give rises to the Weyl anomaly.



- For a system of N M5-branes, the field strength is self dual and the Weyl anomaly cannot be given by H^2.
  Perry-Schwarz/Henneaux-Teitelboim (also Pasti-Sorokin-Tonin) has shown that a action formulation for a single M5-brane can be written down if one give up manifest Lorentz invariance.

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  Perry-Schwarz/Henneaux-Teitelboim (also Pasti-Sorokin-Tonin) has shown that a action formulation for a single M5-brane can be written down if one give up manifest Lorentz invariance.
- In the Perry-Schwarz formulation, x_5 is singled out and the self-dual tensor gauge field is represented by a 5×5 antisymmetric tensor field $B_{\mu\nu}$. i.e. $B_{\mu 5}$ never appear.
- Denote the 5d and 6d coordinates by x^μ and $x^\mu = (x^a, x^5)$. The Perry-Schwarz action is

$$S_{PS}(B) = \frac{1}{2}\int d^6x \left(-\tilde{H}^{ab}\tilde{H}_{ab} + \tilde{H}^{ab}\partial_5 B_{ab}\right)$$

where

$$\tilde{H}^{ab} := \frac{1}{6}\epsilon^{abcde}H_{cde}, \qquad H^{abc} = -\frac{1}{2}\epsilon^{abcde}\tilde{H}_{de}.$$

- Using the PS variables, one can show that the expression

$$\boxed{\mathcal{A} = b_1 \int_M \tilde{H}_{ab} \partial_x B^{ab},} \quad x^\mu = (x^a, x).$$

satisfies the relation $(\delta \mathcal{A})_{\partial M} = \left(\int_{M_\epsilon} J_{\mu\nu} \delta B^{\mu\nu} \right)_{\log 1/\epsilon}$ for the current predicts by holography. We conjecture that this is the contribution of the self-dual field strength to the Weyl anomaly of M5-branes.

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- Restoring the units, b_1 is given by

$$b_1 = \frac{R^5}{16\pi G_7} = \frac{N^3}{3\pi^3},$$

  where $G^{(7)} = G^{(11)}/R_S^4$, $R_S = l_P(\pi N)^{1/3}$ is the 4-sphere radius and $R = 2R_S$ is the AdS_7 radius.
- Therefore for a system of N M5-branes with boundary, one finds for the singlet current

$$\boxed{J_{ab} = \frac{N^3}{3\pi^3}\frac{H_{abx}}{x}.}$$

  This is another way to see that there is N^3 degrees of freedom in the theory.

Outline

Conclusions and Discussions

1. We found new universal relations for near boundary behaviour of stress tensor and electric current.
   - The induced current in the presence of a magnetic field is a pure quantum effect without classical analogy.
2. Based on the original work of Takayanagai, we have provided complete proposal of holographic BCFT.
3. Applying the BCFT holography to 6d, we find an induced string current. This predicts a Weyl anomaly for the M5-branes system.

Open questions:

- What is the origin of the Weyl anomaly of M5-brane? Implicition on the partition function?
- It is known that a constant magnetic field is related to a noncommutative gauge theory with space-space noncommutativity

$$[x^i, x^j] = i\theta^{ij}.$$

  Can understand the magnetization of the vacuum in terms of NCG
- Similarly, our result of the induced string current may help us to understand what kind of higher NCG on the M5-brane worldvolume

$$[x^i, x^j, x^k] = i\theta^{ijk}$$