Lecture on Cosmology Homework and Solution

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Noted by

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1 HomeWork

2 For energy momentum conservation, derive the equation $d(\rho a^3) = -pda^3$

Solution)

The stress-energy tensor and metric tensor are

$$
\begin{aligned}
T^{\mu}_{\nu} &= diag(\rho, -p, -p, -p) \\
g_{\mu\nu} &= diag(-1, a^2, a^2, a^2) \\
T^{\mu\nu} &= diag(-\rho, -a^{-2}p, -a^{-2}p, -a^{-2}p)
\end{aligned}
$$

and energy momentum conservation is

$$T^{\mu\nu}{}_{;\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\mu}{}_{m\nu} T^{m\nu} + \Gamma^{\nu}{}_{m\nu} T^{\mu m} = 0.$$

For the $\mu = 0$,

$$T^{0\nu}{}_{;\nu} = \partial_\nu T^{0\nu} + \Gamma^{0}{}_{m\nu} T^{m\nu} + \Gamma^{\nu}{}_{m\nu} T^{0m}.$$

We need to calculate the christoffel symbol

$$
\begin{aligned}
\Gamma^{i}{}_{nl} &= \frac{1}{2} g^{lm} \left(\frac{\partial g_{mn}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^n} - \frac{\partial g_{nl}}{\partial x^m} \right) \\
\Gamma^{0}{}_{ij} &= \frac{1}{2}(-1)\delta_{ij} \left(-\frac{\partial a^2}{\partial x^0} \right) = a\dot{a} \\
\Gamma^{i}{}_{j0} = \Gamma^{i}{}_{0j} &= \frac{1}{2} a^{-2} \delta_{ij} \left(2a\dot{a} \right) = \frac{\dot{a}}{a} \\
\text{Otherwises} &= 0
\end{aligned}
$$

The result is

$$
\begin{aligned}
T^{0\nu}{}_{;\nu} &= \partial_\nu T^{0\nu} + \Gamma^{0}{}_{m\nu} T^{m\nu} + \Gamma^{\nu}{}_{m\nu} T^{0m} \\
&= \frac{\partial}{\partial t} (-\rho) + 3a\dot{a} \left(-a^{-2} p \right) + 3 \left(-\rho \right) \frac{\dot{a}}{a} = 0 \\
&\Rightarrow \frac{\partial \rho}{\partial t} + 3\rho \frac{\dot{a}}{a} + 3p \frac{\dot{a}}{a} = 0 \qquad \times a^3 \\
&\Rightarrow \frac{\partial \left(\rho a^3 \right)}{\partial t} + p \frac{\partial a^3}{\partial t} = 0 \\
&\therefore d(\rho a^3) = -pda^3
\end{aligned}
$$

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3 Derive the $\rho \propto a^{-3(1+w)}$ for equation of state $p = w\rho$

Solution)

The Continuity equation is

$$
\begin{aligned}
\dot{\rho} + 3\ \frac{\dot{a}}{a}\ (\rho + p) = 0 \\
\dot{\rho} + 3\ \frac{\dot{a}}{a}\ (1 + w)\ \rho = 0
\end{aligned}
$$

by equation of state $p = w\rho$.

And we solve the differential equation,

$$\begin{aligned}
\frac{\dot\rho}{\rho} &= -\ 3\ \frac{\dot a}{a}\ (1+w) \\
\ln\rho &= -\ 3\ (1+w)\ \ \ln a + C_0 \\
\rho &= C_1\ a^{-3\ (1+w)}
\end{aligned}$$

then, we can write

$$\rho\ \propto\ a^{-\ 3\ (1+w)}.$$

And we consider the $\rho \propto H^2$ (H is Hubble constant),

$$\begin{aligned}
\frac{\dot a}{a}\ &\propto\ a^{-\frac{3}{2}(1+w)} \\
a^{\frac{3}{2}(1+w)}\ &\propto\ t \\
a\ &\propto\ t^{\frac{2}{3(1+w)}}.
\end{aligned}$$

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4 In thermal equilibrium, derive the number density n, energy density ρ and pressure p. And Calculate the n, ρ and p for occupation small number by using the Mathematica program.

Solution)

Phase-space distribution

$$f(E) = \frac{1}{e^{(E-\mu)/T} \pm 1}$$

where μ is chemical potential. And + case is Fermi-Dirac distribution, - case is Bose-Einstein distribution.

We consider the n, ρ and p

$$\begin{aligned}
n &= g\int \frac{d^3q}{(2\pi)^3}\ f(E) \\
\rho &= g\int \frac{d^3q}{(2\pi)^3}\ f(E)\ E \\
p &= g\int \frac{d^3q}{(2\pi)^3}\ f(E)\ \frac{|q|^2}{3E}
\end{aligned}$$

For occupation small number, we consider the boltzmann distribution $f(E) \sim e^{-E/T}$ and represent momentum q to energy E

$$\begin{aligned}
\frac{d^3q}{(2\pi)^3} &= \frac{g^2 dq d\Omega}{(2\pi)^3} = \frac{1}{2\pi^2}q^2 dq \qquad (\because \text{all solid angle is } 4\pi) \\
&= \frac{1}{2\pi^2}E\sqrt{E^2 - m^2}dE \qquad (\because q^2 = E^2 - m^2).
\end{aligned}$$

We rewrite the n, ρ and p

$$
\begin{aligned}
n &= \frac{g}{2\pi^2}\int_m^\infty E(E^2-m^2)^{1/2}e^{-E/T}dE \\
\rho &= \frac{g}{2\pi^2}\int_m^\infty E^2(E^2-m^2)^{1/2}e^{-E/T}dE \\
&= \frac{g}{2\pi^2}\left[\int_m^\infty (E^2-m^2)^{3/2}e^{-E/T}dE + \int_m^\infty m^2(E^2-m^2)^{1/2}e^{-E/T}dE\right] \\
p &= \frac{g}{2\pi^2}\int_m^\infty \frac{1}{3}(E^2-m^2)^{3/2}e^{-E/T}dE
\end{aligned}
$$

And we can calculate the n, ρ and p by using Mathematica program.

The result are

$$
\begin{aligned}
n &= \frac{g}{2\pi^2}m^2TK_2\left(\frac{m}{T}\right) \\
\rho &= \frac{g}{2\pi^2}\left(m^3TK_1\left(\frac{m}{T}\right)+3m^2T^2K_2\left(\frac{m}{T}\right)\right) \\
p &= \frac{g}{2\pi^2}m^2T^2K_2\left(\frac{m}{T}\right)
\end{aligned}
$$

where $K_\alpha(x)$ is second modified Bessel function.

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5 For $T > v_{EW}$ case, calculate the effective degree of freedom g_*

Solution)

The effective degree of freedom is

$$
g_* = \sum_{i=boson} g_i \;+\; \frac{7}{8}\sum_{i=fermion} g_i.
$$

For $T > v_{EW}$ case, all the particles in Standard Model is possible to production. So, we can count degree of freedom for all the particles

$$
\begin{aligned}
Spin-1: \quad & Gluon & g_g &= 8(\#\ of\ glouns)\times 2(polarization) \\
& W/Z-boson & g_{W/Z} &= 3(W^\pm,\ Z^0)\times 2(polarization) \\
& Photon & g_\gamma &= 2(polarization) \\
Spin-0: \quad & Higgs & g_H &= 2(doublet)\times 2(real-/complex-) \\
Spin-1/2: \quad & Quark & g_q &= 6(\#\ of\ quarks)\times 3(colors) \\
& & & \quad \times 2(L-/R-)\times 2(particle/anti-) \\
& Leptons(e,\ \mu,\ \tau) & g_l &= 3(\#\ of\ leptons) \\
& & & \quad \times 2(L-/R-)\times 2(particle/anti-) \\
& Neutrinos & g_\nu &= 3(\#\ of\ neutrinos)\times 2(particle/anti-)
\end{aligned}
$$

For the results, we can calculate the effective degree of freedom

$$
\begin{aligned}
g_* &= 28_{(i=boson)} \;+\; \frac{7}{8}\;\times\; 90_{(i=fermion)} \\
&= 106.75
\end{aligned}
$$

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6 For $T < 100$ MeV case, calculate the effective degree of freedom g_*

Solution)

The effective degree of freedom is

$$g_* = \sum_{i=boson} g_i \; + \; \frac{7}{8} \sum_{i=fermion} g_i.$$

For $T < 100$ MeV case, we only consider photon, neutrinos, electron and positron ($m_e = 0.511$ MeV, $m_\mu = 105$ MeV)

$$\begin{aligned} Spin-1: \quad & \quad Photon \quad & g_\gamma = 2(polarization) \\ Spin-1/2: \quad & \quad e^-, e^+ \quad & g_l = 2(L-/R-) \times 2(particle/anti-) \\ & \quad Neutrinos \quad & g_\nu = 3(\#\ of\ neutrinos) \times 2(particle/anti-) \end{aligned}$$

For the results, we can calculate the effective degree of freedom

$$\begin{aligned} g_* &= 2_{(i=boson)} \; + \; \frac{7}{8} \; \times \; 10_{(i=fermion)} \\ &= 10.75 \end{aligned}$$

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