Elementary Introduction to Information Theory

1 Quantum Channel

In order to describe the most general physically sensible evolution of a density matrix, we learned in the class the notion of a quantum channel which involves the so-called *Kraus* operator. The quantum channel also plays a key role showing the monotonicity of relative entropy. Let us work out some exercises below to familiarize oneself with the notion of quantum channel.

(1) Let ψ be any pure state of a given system. Find Kraus operators of a quantum channel mapping any density matrix ρ to $|\psi\rangle\langle\psi|$. Describe a physical realization of this process.

(2) Find Kraus operators of a quantum channel that maps a given system with any density matrix ρ to a system in a maximally mixed state where a density matrix is proportional to the identity.

(3) Suppose that there exists a cavity probed by atoms. Denote by $|n\rangle$ the state of the cavity when it contains n photons. Assume that the number of photons n remains unchanged when an atom passes through the cavity. In other words, the probability finding the cavity in a state $\{|n\rangle\}$ is then unchanged by the interaction with an atom passed by. This implies that the diagonal elements of the initial density matrix in the basis $|n\rangle$ are constant. After making a number of observation of the atoms passing through the cavity, it will end up with high probability in an eigenstate of the photon number operator regardless of what state the cavity begins in. It suggests that the final density matrix of the cavity is diagonal in the basis $\{|n\rangle\}$.

We learn from the above physical process that there must be Kraus operators of a quantum channel that maps any k-dimensional density matrix ρ into the corresponding diagonal density matrix $\rho_D = \mathrm{diag}(\rho_{11}, \rho_{22}, .., \rho_{kk})$. Find them.

2 Physical Meaning of Quantum Relative Entropy

Let us consider a system X with the k-dimensional Hilbert space $\mathcal{H}$. If our initial hypothesis is that X has a density matrix σ. However the actual density matrix is ρ. After N trials with an optimal measurement to test the initial hypothesis, we learn in the class that the confidence that the initial hypothesis was wrong can be controlled by

$$2^{-NS(\rho||\sigma)}, \tag{2.1}$$

where $S(\rho||\sigma)$ is the quantum relative entropy.

As explained in the class, the only way to learn that two density matrices ρ and σ are different is by observing corresponding classical probability distributions R and S are different, i,e, a process controlled by

$$2^{-NS_{\rm cl}(R||S)}, \tag{2.2}$$

where $S_{\rm cl}$ is the classical relative entropy. We can then show from the monotonicity of quantum relative entropy that one cannot do better than (2.1). More precisely, $S(\rho||\sigma)$ gives an upper bound on how well we can do:

$$2^{-NS_{\rm cl}(R||S)} \geq 2^{-NS(\rho||\sigma)} \ . \tag{2.3}$$

Notice that a measurement projecting on one-dimensional eigenspaces of σ leads to probability distribution R and S satisfying

$$S(\rho_D||\sigma) = S_{\rm cl}(R||S) \ . \tag{2.4}$$

Here one can construct a diagonal matrix ρ_D, in some basis where σ becomes diagonal, by dropping from ρ the off-diagonal matrix elements. In the limit of large N, it is actually possible to saturate the bound.

(1) First replace the Hilbert space $\mathcal{H}$ of a given system X by $\mathcal{H}^{\otimes M}$, and the density matrices ρ, ρ_D and σ by $\rho^{\otimes M}, r_D^{\otimes M}$ and $\sigma^{\otimes M}$. Show that

$$\begin{aligned} S(\rho||\sigma) - S(\rho_D||\sigma) &= \frac{1}{M}\Big[S(\rho^{\otimes M}||\sigma^{\otimes M}) - S(\rho_D^{\otimes M}||\sigma^{\otimes M})\Big] \\ &= \frac{1}{M}\Big[\mathrm{Tr}\big(\rho^{\otimes M}\log\rho^{\otimes M}\big) - \mathrm{Tr}\big(\rho_D^{\otimes M}\log\rho_D^{\otimes M}\big)\Big]. \end{aligned} \tag{2.5}$$

(2) Verify that, for large M, the difference between two relative entropy $S(\rho^{\otimes M}||\sigma^{\otimes M})$ and $S(\rho_D^{\otimes M}||\sigma^{\otimes M})$ scales as $\log M$,

$$S(\rho^{\otimes M}||\sigma^{\otimes M}) - S(\rho_D^{\otimes M}||\sigma^{\otimes M}) = \text{const.} \times \log M, \tag{2.6}$$

which is negligible compared to M for large M. (Use the Schur-Weyl duality.)

In the class, we learn that (2.6) is crucial to saturate the upper bound (2.1). This confirms that quantum relative entropy has the same physical interpretation as classical one. It is noteworthy that the measurement to accomplish this depends on σ (the initial hypothesis) and not on ρ (the unknown answer).