Topological Degeneracy, Chern-Simons Theory, and Anyons

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[Message to Students]: Below are the problems to glimpse the relations between the topological degeneracy, Chern-Simons theory, and anyons. Mathematically, they require only the simple linear algebra. Physically, they require some amounts of arguments and graphical thinking. Due to the length of the arguments, the problems appear to be "long" and "difficult" but they are not really. I hope you can go through these and enjoy them.

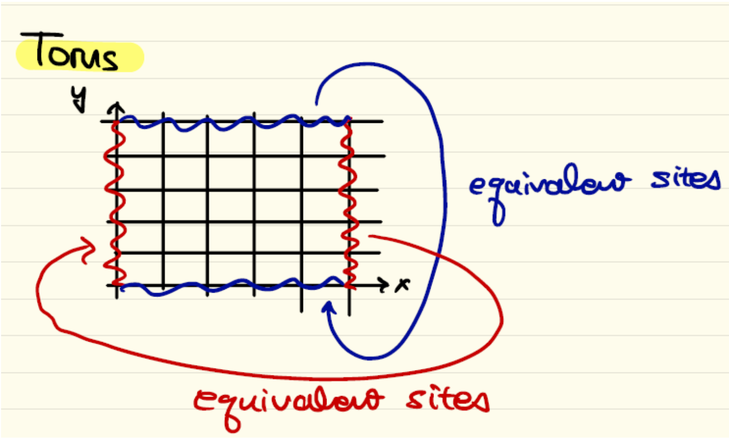
1. <u>Two-fold degeneracy in Spin system</u>: Show generically that the spectrum of any spin-1/2 system is two-fold degenerate if I have the symmetries generated by the following operations:

$$\mathrm{U_x}(\pi) = i\sigma^x, \mathrm{U_y}(\pi) = i\sigma^y$$

in which $\mathrm{U_a}(\pi)$ is the rotation operation on the spin-1/2 around the axis "a" with angle π. Hint: can you choose a simultaneous eigenstate of $\mathrm{U_x}(\pi)$ and $\mathrm{U_y}(\pi)$?

This kind of degeneracy in energy spectrum is often called as "symmetry-protected" degeneracy. Below we will see that the degeneracy can appear in an entirely different way for many particle systems, which is sensitive only to the "topology" of the space and excitation type.

2. <u>Topological Degeneracy</u>: Imagine a many-electron system on a torus:$\mathrm{T}^2 = S^1 \times S^1$:



Furthermore, imagine that we have a set of finite number of states, which are well separated from all the other states (This separation in energy is called as "gap".). For example, the simplest spectrum is in Fig 1 below. In this system, the excited states are

"creating particles" or "annihilating particles = creating of anti-particles"

from the ground state.

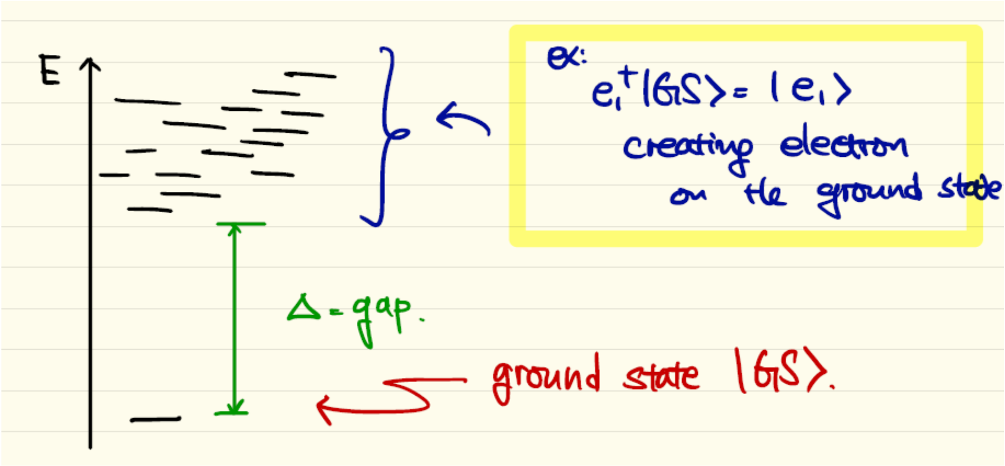
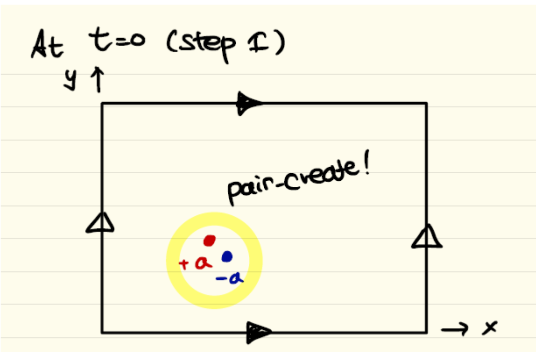


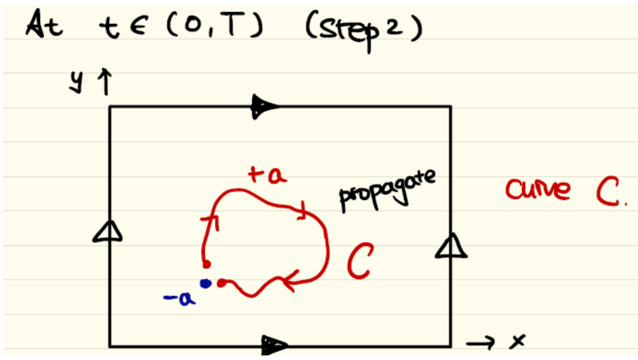
Figure 1. Example of Spectrum

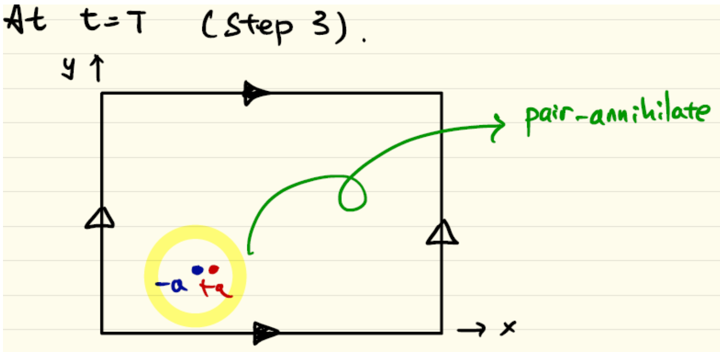
Imagine that we can now send the gap to the infinity (so-called “topological limit”). In these kind of systems, all we are left with is a handful of the degenerate ground states, i.e., <u>a finite number of degenerate ground states, whose number will be determined by a few data without solving any details of Hamiltonian.</u>

Below we assume that there are a few degenerate ground states. Within this set of states, we would like to argue that the only non-trivial information is the statistics of particle-type excitations, from which we deduce the topological structure of this many-body ground state. To show this, follow the arguments below:

Obviously, the single creation or annihilation of particles from the ground state costs huge energy, and thus they are not “good” operators, i.e., they cannot be represented within the ground states. On the other hand, the process [“Pair-creation” followed by “Pair-annihilation” of particle-antiparticle] is a “good” operator and it can be well represented within the ground states. For this, consider the following time-evolution of the states:



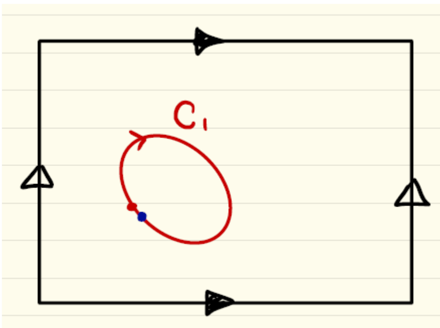


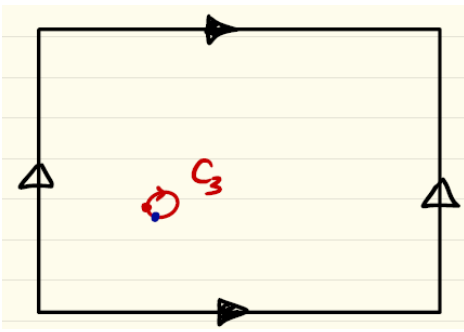


When this operation is acted on the ground states, the classical configurations long before t=0 and long after t=T are the same (no particle excitation above the ground state) and thus the final state must be written as the linear combinations of the ground states. I.E., let's take such an time-evolution operation as: U(a; C)

(1) Show that the above operation is a unitary operator within the ground states. [Hint: Use the standard perturbation theory and show that there is no probability leakage to the excited states. Remember that the gap is infinite.] That is, when the ground state manifold is spanned by $|1\rangle, |2\rangle \cdots |n\rangle$, show that the matrix V such that $V_{ij} = \langle i|U(a;C)|j\rangle$ is an unitary matrix.

(2) Show that the following processes are *topologically equivalent to* each other and "do nothing". That is, one process can be smoothly deformable to another process by deforming the path of the particle propagation.

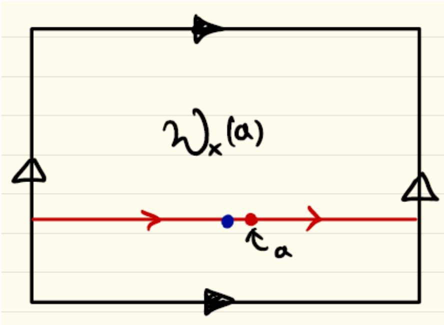


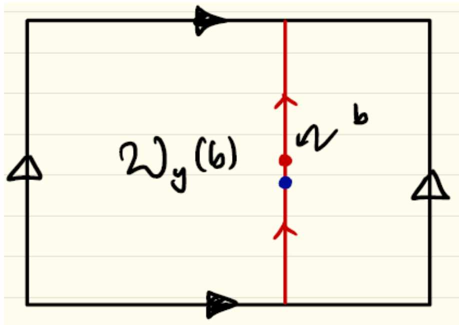


Here the size of the loop C_1 is bigger than the size of the loop C_3. These loops represent the path that the particle propagates during the process illustrated in problem (2).

Note that they are literally the same operator (acting on the ground states) when the gap is infinite because the quantum amplitude of the operations are independent of length of path, speed of particle motions, distances between the particles and so on in that limit.

(3) Show that the following two processes are *topologically inequivalent* from each other and are also inequivalent from "do nothing":





Show that these two are only and the only two non-trivial loops (so-called non-trivial "holonomy") in the torus. Below they are going to be labeled as:

$\mathbf{W_x(\boldsymbol{a})}$**:** particle type "+a" and its anti-particle "-a" are pair-created and dragged "+a" around the torus along x-direction and then pair-annihilated.
$\mathbf{W_y(\boldsymbol{b})}$**:** particle type "+b" and its anti-particle "-b" are pair-created and dragged "+b" around the torus along y-direction and then pair-annihilated.

(4) Now we imagine a successive operation of the followings:

$$S_{ab} = \mathrm{W}_{\mathrm{y}}^{-1}(b) \cdot W_x^{-1}(a) \cdot W_y(b) \cdot W_x(a)$$

Show that $\mathrm{S_{ab}}$ must be a unitary matrix (acting on ground states) and that $\mathrm{S_{ab}} \neq 1$ for any pair $\{a,b\}$ implies the degeneracy of the ground states on the torus.

Now we reach one mathematical conclusion: there are some unitary operators, whose algebra can enforce the degeneracy of the states on the torus. However, what do they mean physically?

(5) Argue that

$$S_{ab} = \mathrm{W}_{\mathrm{y}}^{-1}(b) \cdot W_x^{-1}(a) \cdot W_y(b) \cdot W_x(a)$$

is equivalent to the following physical process:

Step 1. Pair-create particle "+a" and its anti-particle "-a" and move "a" around x-direction and then pair-annihilate them.
Step 2. Pair-create particle "+b" and its anti-particle "-b" and move "b" around y-direction and then pair-annihilate them.
Step 3. Pair-create particle "+a" and its anti-particle "-a" and move "-a" around x-direction and then pair-annihilate them.
Step 4. Pair-create particle "+b" and its anti-particle "-b" and move "-b" around y-direction and then pair-annihilate them.

That is, it is pictorially the following:

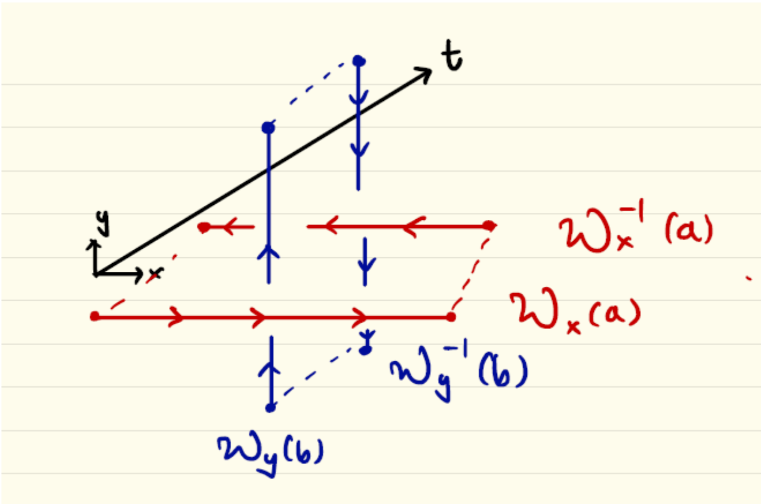
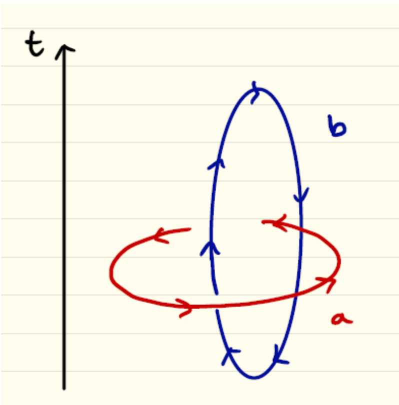


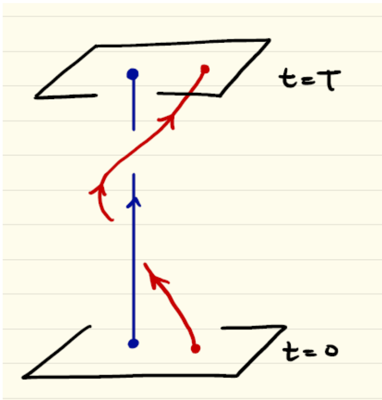
Figure 2. Pictorial Representation of S_{ab}: Here the end points of each lines are identified to each other (Torus geometry).

(6) Argue that

$$S_{ab} = W_y^{-1}(b) \cdot W_x^{-1}(a) \cdot W_y(b) \cdot W_x(a)$$

is equivalent to the below two.





[Note: in the left figure, we interpret "the particle "b" is propagating backward in time" = "the anti-particle of b is propagating in correct direction of the time". In the right figure, we impose the "periodic boundary condition" in time, that is, t=T and t=0 are equivalent.]

(7) From these, show that **the presence of the anyon enforces the degeneracy of the ground states on the torus** by interpreting the right figure of the problem (8) as the braiding phases ("Monodromy") between the particles.

(8) From these, show that the Chern-Simons theory

$$L = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

has non-trivial degeneracy on the torus if k>1 because it has anyons.

We now see a remarkable fact: Without solving any Hamiltonian, we find that there must be a degeneracy for the ground state on the torus if we have anyons. Furthermore, the degeneracy is originating from the topology of the torus. Also, since these degeneracies are independent of specific forms of Hamiltonian, the degeneracy is robust against any smooth deformation of the Hamiltonians or details (I.E., the degeneracy on the torus is the "Topological Invariant" of the Hamiltonian). Because of this robustness, these are known as the "Topological Degeneracy", which is tightly bound with the concept of anyon and topological order.

Any questions/comments to be sent to gilyoungcho@gmail.com