Emergence of Canonical Ensembles from Pure Quantum States

JC & M. S. Kim, PRL (2010)

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(APCTP)

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Table of Contents

Motivation

PRL **96,** 050403 (2006) PHYSICAL REVIEW LETTERS week ending 10 FEBRUARY 2006

Canonical Typicality

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ARTICLES

Entanglement and the foundations of statistical mechanics

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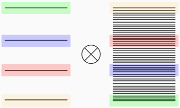
Nature Physics (2006)

- **Microcanonical ensemble**
  - equal *a priori* postulate
  - $\rho(E) \propto I$ in an energy shell $E_0 < E < E_0 + \delta E$
  - ergodicity, chaos, mixing, ...
- **Canonical ensemble**
  - $\rho_S(E_S) \propto \Gamma_B(E - E_S) \propto e^{-E_S/kT}$
  - $k \log \Gamma_B(E - E_S) \simeq S_B(E) - \frac{E_S}{T}$

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$$



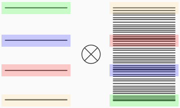
$$\rho_S(E_S) \propto \Gamma_B(E - E_S)$$

- $|\Psi\rangle = \sum_E c_E |\Psi_E\rangle$

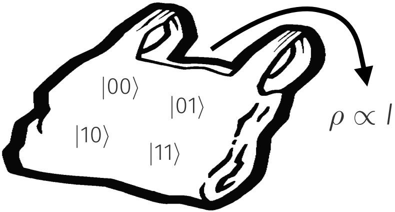
  c_E: uniform random in the energy shell
- $\rho_S = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| \to e^{-E_S/kT}$

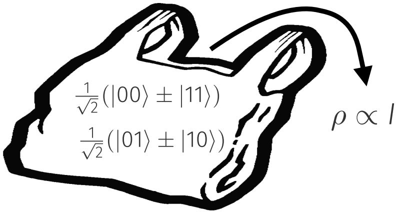
  "Almost always", i.e., "Typical"
- Entanglement
- **Meaning?**

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$$



$$\rho_S(E_S) \propto \Gamma_B(E - E_S)$$





$$\rho_1 = \mathrm{Tr}_2 \rho \qquad\qquad \rho_1 = \mathrm{Tr}_2 \rho$$

$\vert 0\rangle\langle 0\vert$	$\vert 0\rangle\langle 0\vert$ or $\vert 1\rangle\vert 1\rangle$
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$$\rho_1 \propto I \qquad\qquad \rho_1 \propto I$$

Origin of ENTROPY?

Coin Tossing

- (+) $v_j = 1$ (−) $v_j = 1$ $v = \frac{1}{N}\sum_{j=1}^{N} v_j \to$ intensive quantity
- $N = 2$

  (+)(+) : $|\Psi_1\rangle$ $v = 1$

  (+)(−) : $|\Psi_1\rangle$ $v = 0$

  (−)(+) : $|\Psi_1\rangle$ $v = 0$

  (−)(−) : $|\Psi_1\rangle$ $v = -1$

  $\langle v \rangle = 0$ $\to$ ensemble average

  $\langle |\Delta v| \rangle = \frac{1}{2}$ $\to$ thermal fluctuation

- $N \to \infty$ (thermodynamic limit)

  (+)(−)(−)(+)(−)(+) $\cdots$ $v \to 0$

  (−)(+)(−)(+)(+)(−) $\cdots$ $v \to 0$

  $\vdots$

  (+)(+)(+)(+)(+)(+) $\cdots$ $v \neq 0$: measure 0

  $\vdots$

  $\langle v \rangle = 0$

  $\langle |\Delta v| \rangle \to 0$

  "Almost Always"

  $v = 0$

Equilibration

Reimann, PRL (2008); Linden et al., PRE (2009)

$$|\Psi(t)\rangle = \sum_E c_E e^{-iEt}|\Psi_E\rangle$$

$$\rho = \sum_{E,E'} c_E c^*_{E'} e^{-i(E-E')t}|\Psi_E\rangle\langle\Psi_{E'}|$$

$$\langle\rho\rangle_t = \sum_E |c_E|^2|\Psi_E\rangle\langle\Psi_E| \qquad \because \left\langle e^{-i(E-E')t}\right\rangle_t \to 0 \text{ for } E-E' \neq 0$$

- **Time scale**
  - $\tau_H \sim \dfrac{\hbar}{\Delta E}$: Heisenberg time
  - $\Delta E \sim Ne^{-N} \to 0$ ex) $N \sim 10^{23}$
- Note also that the canonical ensemble is NOT to be “derived”.
  - Regular (non-chaotic) systems
  - Dynamical localisation

Bottom Line

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From Quantum Dynamics to the Canonical Distribution: General Picture and a Rigorous Example

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(Received 24 July 1997)

"It is often said that the principles of equilibrium statistical physics have not yet been justified. It is not clear, however, what statement should be regarded as the ultimate justification.
… it seems likely that **there are many independent routes for justification** which can be equally convincing and important."

Bottom line: Different scenarios **shouldn't** be mixed up.

One Possible Scenario

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$$

$$H = H_0 + H_1 = (H_S + H_B) + H_1$$

$$|\Psi(t)\rangle = \sum_j c_j(t)|\Psi_j\rangle$$

$$H_{jk} = \langle \Psi_j | H_1 | \Psi_k \rangle = e^{i\phi_{jk}} |H_{jk}|$$

Assumptions

1. Random phase ensemble
   - ϕ_{jk} : uniform random
   - $H_{jk} \to e^{i(\phi_j - \phi_k)}|H_{jk}| \quad \phi_j$: uniform random

   $\{\phi_1, \phi_2, \phi_3, \cdots\}$
   $\{\phi'_1, \phi'_2, \phi'_3, \cdots\}$
   $\vdots$

2. $|H_{jk}| = |\langle E_S, E_B | H_1 | E'_S, E'_B \rangle| \to$ Quasi-continuous function of E_B, E'_B
   - thermodynamic limit
   - time scale $t \ll \tau_H$
3. Weak interaction

Outline of the Logic (1/2)

- $|j\rangle \equiv |E_S, E_B\rangle, \quad |\Psi(t)\rangle = \sum_j c_j(t)|j\rangle$
- $c_j(t) = \sum_k U_{jk}(t, t_0) c_k(t_0)$ interaction picture
- $p_j(t) \equiv |c_j(t)|^2 = \sum_k |U_{jk}|^2 p_k(t_0) +$ (off-diagonal terms)
- $\langle \cdot \rangle$: ensemble average w.r.t. the phase randomness
- $\left\langle p_j(t) \right\rangle = \sum_k |U_{jk}|^2 \left\langle p_k(t_0) \right\rangle$

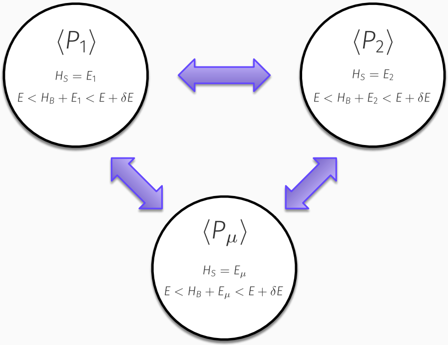
Outline of the Logic (2/2)

- $\left\langle p_j(t) \right\rangle = \sum_k |U_{jk}|^2 \left\langle p_k(t_0) \right\rangle$
- $\frac{d}{dt}|U_{jk}(t,t_0)|^2 \to 2\pi |H_{jk}|^2 \delta(E_j - E_k)$

  Fermi golden rule, wherein $\Gamma_B(E_B)$ appears
- $\mathcal{H} = \bigoplus_{\mu=1}^{d_S} \mathcal{H}_\mu, \quad P_\mu(t) \equiv \sum_{|j\rangle \in \mathcal{H}_\mu} p_j(t)$
- $$\frac{d}{dt}\langle P_\mu(t)\rangle = -\sum_{\nu\neq\mu} W_{\mu\to\nu}\Gamma_B(E - E^S_\nu)\langle P_\mu(t_0)\rangle + \sum_{\nu\neq\mu} W_{\nu\to\mu}\Gamma_B(E - E^S_\mu)\langle P_\nu(t_0)\rangle$$

  $W_{\mu\to\nu} \equiv 2\pi |\langle E^S_\nu, E - E^S_\nu | H_1 | E^S_\mu, E - E^S_\mu\rangle|^2$

Dynamics of Average Populations



- Interaction picture $\rightarrow$ Transition problem
- Fermi golden rule $\rightarrow$ Markov chain of d_S nodes
- Detailed balancing $\rightarrow \langle P_\mu(t) \rangle \propto \Gamma_B(E - E_\mu^S)$ Canonical ensemble

What about the fluctuation?

- $\langle P_\mu(t) \rangle \propto \Gamma_B(E - E_\mu^S)$
- $\left\langle |\Delta P_\mu(t)|^2 \right\rangle \leq 2\sqrt{\frac{1}{d_0}} \langle P_\mu(t) \rangle$

  $d_0 \equiv \frac{1}{\sum_j |c_j|^4}$ effective dimension

$\therefore$ For $d_0 \to \infty$,

Individual dynamics $\to$ Average dynamics

Summary

- Composite system $S + B$ in an **arbitrary pure** state.
- **Typical** interaction in the interaction picture.
- The system part (S) is **almost always** driven to a canonical ensemble under reasonable assumptions.