Introduction to Resurgence and Non-perturbative Physics

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Kavli Pan Asian Winter School *Strings, Particles and Cosmology*
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GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414, 1603.04924

recent KITP Program: *Resurgent Asymptotics in Physics and Mathematics*, Fall 2017

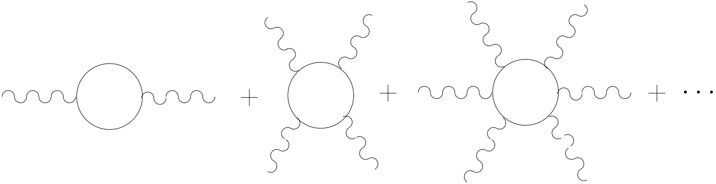
future Isaac Newton Institute Programme: *Universal Resurgence*, 2020/2021

Resurgence and Non-perturbative Physics

1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   - asymptotic series in physics; Borel summation
   - trans-series completions & resurgence
   - examples: linear and nonlinear ODEs
2. Lecture 2: Applications to Quantum Mechanics and QFT
   - instanton gas, saddle solutions and resurgence
   - infrared renormalon problem in QFT
   - Picard-Lefschetz thimbles
3. Lecture 3: Resurgence and Large N
   - Euler-Heisenberg Effective Action
   - resurgence in 2d sigma models
   - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT

Euler-Heisenberg Effective Action (1935)

review: hep-th/0406216



- 1-loop QED effective action in uniform emag field
- the birth of *effective field theory*

$$L = \frac{\vec{E}^2 - \vec{B}^2}{2} + \frac{\alpha}{90\pi}\frac{1}{E_c^2}\left[\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\left(\vec{E}\cdot\vec{B}\right)^2\right] + \dots$$

- encodes nonlinear properties of QED/QCD vacuum

QFT Application: Euler-Heisenberg

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \mathrm{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \mathrm{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$

$$\left(\begin{array}{l}\mathfrak{E}, \mathfrak{B} \quad \text{Kraft auf das Elektron.} \\ |\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} = \text{„Kritische Feldstärke“.}\end{array}\right)$$

- Borel transform of a (doubly) asymptotic series
- resurgent trans-series: analytic continuation $B \longleftrightarrow E$

Euler-Heisenberg Effective Action: Borel summation

- e.g., constant B field:

$$S = -\frac{B^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)\exp\left[-\frac{m^2 s}{B}\right]$$

- perturbative (weak field) expansion:

$$S \sim -\frac{B^2}{2\pi^2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2B}{m^2}\right)^{2n+2}$$

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- characteristic factorial divergence

$$c_n = \frac{(-1)^{n+1}}{8}\sum_{k=1}^{\infty}\frac{\Gamma(2n+2)}{(k\,\pi)^{2n+4}}$$

- instructive exercise: reconstruct Borel transform

$$\sum_{k=1}^{\infty}\frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

Euler-Heisenberg Effective Action: Borel summation

- e.g., constant B field: characteristic factorial divergence

$$c_n = \frac{(-1)^{n+1}}{8} \sum_{k=1}^{\infty} \frac{1}{(k\,\pi)^{2n+4}} \Gamma(2n+2)$$

- recall Borel summation:

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\, \Gamma(\gamma\, n + \delta)$$

$$\to \quad f(g) \sim \frac{1}{\gamma} \int_0^{\infty} \frac{ds}{s} \left(\frac{1}{1+s}\right) \left(\frac{s}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{s}{\beta g}\right)^{1/\gamma}\right]$$

- for each k, reconstruct Borel transform:

$$\sum_{k=1}^{\infty} \frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

Exercise 6:

(i) fill in these steps for the Borel summation of the Euler-Heisenberg effective action

(ii) deduce the imaginary part of the effective action when the background field changes from magnetic to electric

(iii) repeat for the case of *scalar* QED in a background magnetic field, where the Euler-Heisenberg effective action is instead

$$S = \frac{B^2}{16\pi^2} \int_0^\infty \frac{ds}{s^2} \left(\frac{1}{\sinh s} - \frac{1}{s} + \frac{s}{6} \right) \exp\left[-\frac{m^2 s}{B} \right]$$

Euler-Heisenberg Effective Action and Schwinger Effect

B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \mathrm{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

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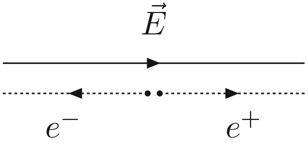
Schwinger effect:



$$2eE\frac{\hbar}{mc} \sim 2mc^2$$

$$\Rightarrow$$

$$E_c \sim \frac{m^2 c^3}{e\hbar} \approx 10^{16} \mathrm{V/cm}$$

WKB tunneling from Dirac sea

$\operatorname{Im} S \to$ physical pair production rate

- Euler-Heisenberg series must be divergent (*cf.* Dyson)

de Sitter/ anti de Sitter effective actions (Das & GD, hep-th/0607168)

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$

$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature: $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in dS_d with d even

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$
- Stark: $c_n \sim (2n)!$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

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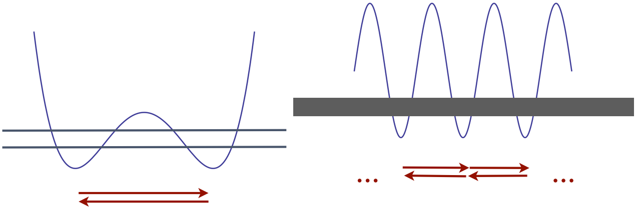
Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...
- level splitting = real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

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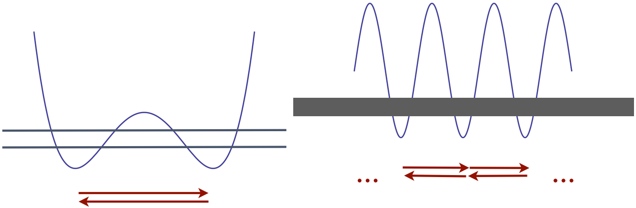
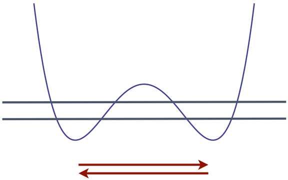


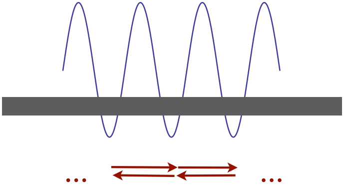
- degenerate vacua: double-well, Sine-Gordon, ...
- level splitting = real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a two-instanton effect

Bogomolny/Zinn-Justin mechanism in QM





- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel summable: ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

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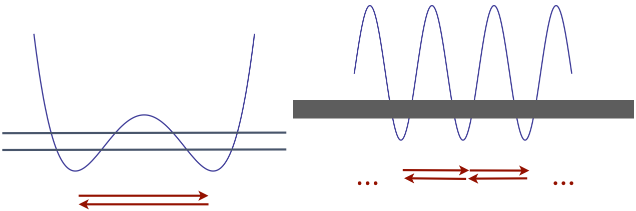

- degenerate vacua: double-well, Sine-Gordon, ...

  1. perturbation theory non-Borel summable:
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  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

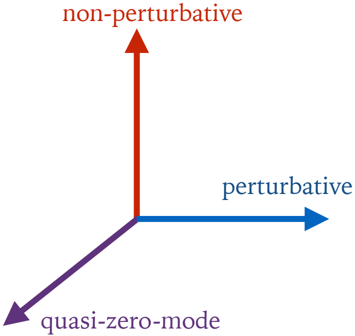
- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

"tip of the (resurgence) iceberg"

Decoding a Resurgent Trans-series

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\ln\left(-\frac{1}{g^2}\right)\right]^q$$



expansions in different directions are <u>quantitatively</u> related

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- perturbative fluctuations about vacuum: $\sum_{n=0}^{\infty} c_{n,0,0}\, g^{2n}$
- divergent (non-Borel-summable): $c_{n,0,0} \sim \alpha \frac{n!}{(2S)^n}$

$\Rightarrow$ ambiguous imaginary non-pert energy $\sim \pm i\, \pi\, \alpha\, e^{-2S/g^2}$

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pert flucs about instanton: $e^{-S/g^2}\ \left(1 + a_1 g^2 + a_2 g^4 + \dots\right)$

divergent:

$a_n \sim \frac{n!}{(2S)^n}\,(a\ \ln n + b) \Rightarrow \pm i\, \pi\, e^{-3S/g^2}\ \left(a \ln \frac{1}{g^2} + b\right)$

Decoding a Resurgent Trans-series

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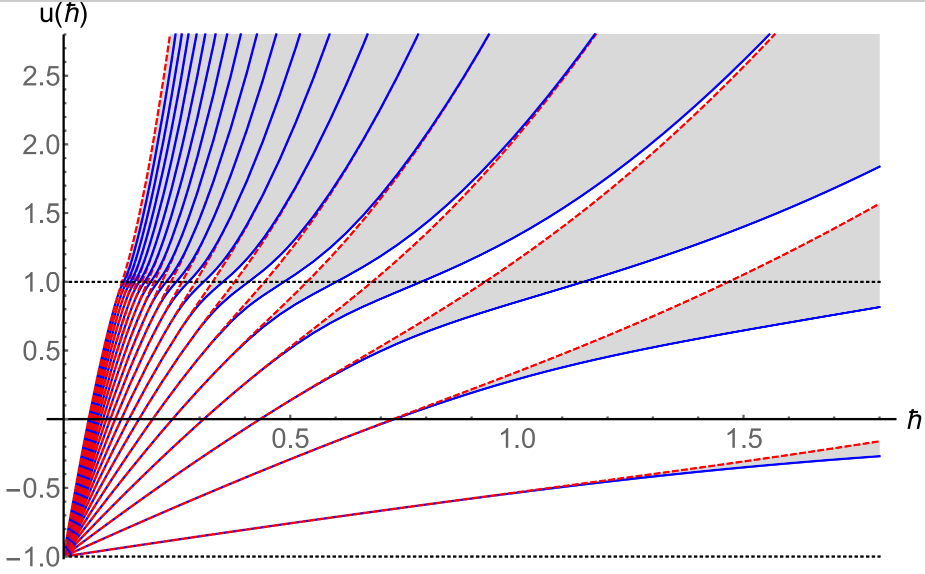
- 3-instanton: $e^{-3S/g^2}\left[\frac{a}{2}\left(\ln\left(-\frac{1}{g^2}\right)\right)^2 + b\ \ln\left(-\frac{1}{g^2}\right) + c\right]$

resurgence: *ad infinitum*, also sub-leading large-order terms in fact, the resurgent structure is much deeper than this ...

Alvarez/Casares (2000, 2003), GD/Unsal (1306.4405, 1401.5202),

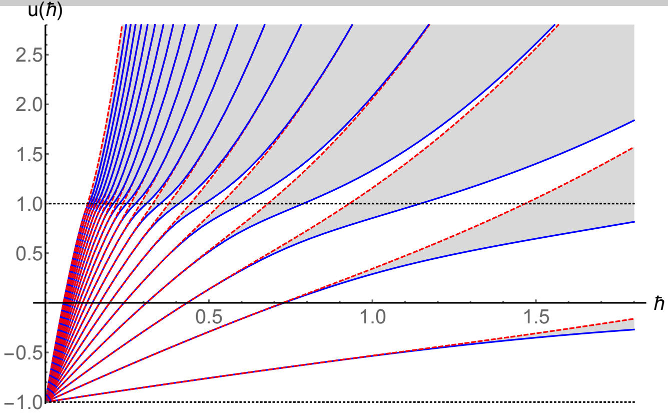
Basar/GD/Unsal (1701.06572)

Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$



classical: stability/instability quantum: bands/gaps

recall: Mathieu Equation: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$

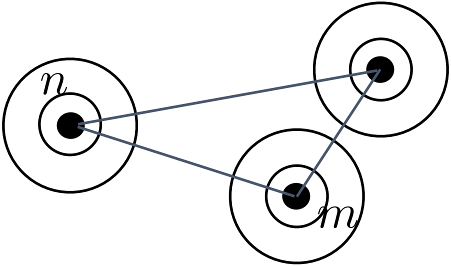


$$u_\pm(\hbar, N) = u_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}}\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right]\mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial u_{\rm pert}(\hbar, N)}{\partial N}\exp\left[S\int_0^\hbar \frac{d\hbar}{\hbar^3}\left(\frac{\partial u_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

GD & Ünsal (2013); Başar & GD (2015): applies to bands & gaps

Perturbation Theory Encodes the Full Trans-series Data



Resurgence of $\mathcal{N}=2$ SUSY SU(2): Mathieu Eqn Spectrum

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}(a, \hbar, \Lambda)$:
- Mathieu equation: (Mironov/Morozov)

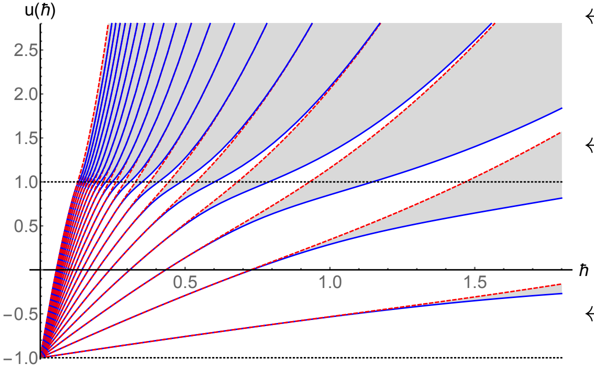
$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2 \cos(x)\, \psi = u\, \psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- Mathieu P/NP relation $\equiv$ (quantum) Matone relation:

$$u(a, \hbar) = \frac{i\pi}{2}\Lambda \frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

- $\mathcal{N} = 2^*$ $\leftrightarrow$ Lamé equation

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



$\longleftarrow$ electric sector (convergent)

$\longleftarrow$ magnetic sector

$\longleftarrow$ dyonic sector (divergent)

- energy: $u=u(N,\hbar)$; 't Hooft coupling: $\lambda\equiv N\,\hbar$
- very different physics for $\lambda\gg 1$, $\lambda\sim 1$, $\lambda\ll 1$

- SUSY: $E^{\text{perturbative}}_{\text{ground state}}(\hbar) = 0$ to all orders !
- how can it encode non-perturbative effects ?

Deconstructing Zero: P/NP Resurgence for SUSY QM

GD & Ünsal: 1609.05770

- SUSY: $E^{\text{perturbative}}_{\text{ground state}}(\hbar) = 0$ to all orders !
- how can it encode non-perturbative effects ?
- broken SUSY: $E^{\text{nonpert.}}_{\text{g.s.}}(\hbar, N) \sim \hbar^{\beta} e^{-S/\hbar} \mathcal{P}_{\text{fluc}}(\hbar, N) > 0$

$$\mathcal{P}_{\text{fluc}}(\hbar, N) = \frac{\partial E^{\text{pert}}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3}\left(\frac{\partial E^{\text{pert}}(\hbar, N)}{\partial N} - \hbar + \frac{N\,\hbar^2}{S}\right)\right]$$

- note that $\left[E^{\text{pert}}\right]_{N=0} = 0$, but $\left[\frac{\partial E^{\text{pert}}}{\partial N}\right]_{N=0} \neq 0$

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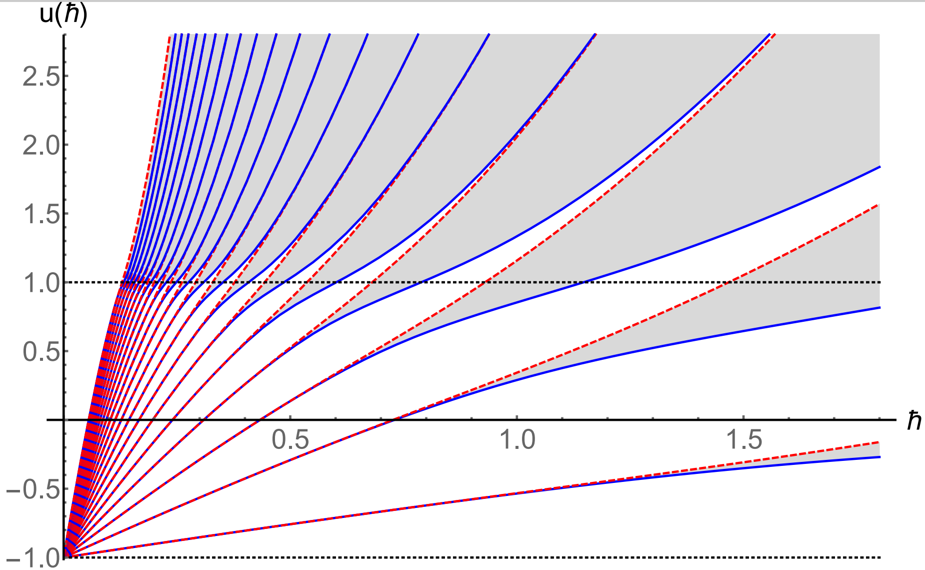
- unbroken SUSY: $E^{\text{non-pert.}}_{\text{g.s.}}(\hbar) = 0$, due to cancellations between two saddles

$\Rightarrow$ resurgence explains SUSY breaking or non-breaking

Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$



classical: stability/instability quantum: bands/gaps

Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$

- small $\hbar$: divergent, non-Borel-summable $\rightarrow$ trans-series

$$
\begin{aligned}
u(N,\hbar) \quad \sim \quad & -1 + \hbar\left[N + \frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2 + \frac{1}{4}\right] \\
& - \frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3 + \frac{3}{4}\left(N+\frac{1}{2}\right)\right] - \dots
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& -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right]-\dots
\end{aligned}
$$

- large $\hbar$: convergent expansion: $\longrightarrow$?? trans-series ??

$$
\begin{aligned}
u(N,\hbar)\sim\frac{\hbar^2}{8}\Bigg(& N^2+\frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4+\frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 \\
& +\frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12}+\dots\Bigg)
\end{aligned}
$$

- note: poles in coefficients

Mathieu Equation Spectrum: Narrow Bands and Gaps

- Narrow Band-Width

[Harrell 1989, Connor/Marcus 1984, Weinstein/Keller 1985]

$$\begin{aligned}\Delta u_{\text{band}} &\sim \frac{2}{\pi}\frac{\partial u}{\partial N}\exp\left[-\frac{1}{\hbar}\,S\right](1+O(\hbar))\\ S &= \int_{\text{turning points}}\sqrt{V(x)-V_{\text{min}}}\,dx\end{aligned}$$

- Narrow Gap-Width

[Dykhne 1961, Avron/Simon 1981, Connor/Marcus 1984, Weinstein/Keller 1987]

$$\begin{aligned}\Delta u_{\text{gap}} &\sim \frac{2}{\pi}\frac{\partial u}{\partial N}\exp\left[-\frac{1}{\hbar}\text{Im}\,\tilde{S}\right](1+O(\hbar))\\ \tilde{S} &= \int_{\text{complex turning points}}\sqrt{V(x)-V_{\text{min}}}\,dx\end{aligned}$$

- note common form: just different turning points

Mathieu Equation Spectrum: Real & Complex Instantons

- narrow bands low in the spectrum: real instantons

$$\Delta u^{\text{band}}(N,\hbar) \quad \sim \quad \frac{32}{\sqrt{\pi}\,N!}\left(\frac{32}{\hbar}\right)^{N-1/2}\exp\left[-\frac{8}{\hbar}\right]$$

- narrow gaps high in the spectrum: complex instantons

$$\Delta u^{\text{gap}}(N,\hbar) \quad \sim \quad \frac{N\,\hbar^2}{2\pi}\left(\frac{e}{N\,\hbar}\right)^{2N}$$

- resurgence $\Rightarrow$ both expressions extend to all orders in terms of perturbative data
- transition at $N\hbar \sim \frac{8}{\pi}$: condensation of instantons

Ionization in Time-Dependent Electric Fields

- Keldysh (1964): atomic ionization in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{\omega\sqrt{2mE_b}}{e\mathcal{E}}$
- WKB $\Rightarrow$ $P_{\text{ionization}} \sim \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\,g(\gamma)\right]$

$$P_{\text{ionization}} \sim \begin{cases} \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\mathcal{E}}{2\omega\sqrt{2mE_b}}\right)^{2E_b/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- semi-classical analysis interpolates between non-perturbative "tunneling ionization" [real instantons] and perturbative "multi-photon ionization" [complex instantons]

Keldysh Approach in QED

Brézin/Itzykson, 1970; Popov, 1971

- Schwinger effect in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{m\,\omega}{\mathcal{E}}$
- WKB $\quad \Rightarrow \quad P_{\rm QED} \sim \exp\left[-\pi\,\frac{m^2}{\hbar\,\mathcal{E}}\,g(\gamma)\right]$

$$
P_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2}{\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \\ \left(\frac{\mathcal{E}}{\omega\,m}\right)^{4m/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}
$$

- semi-classical instanton (saddle) interpolates between non-perturbative ‘tunneling pair-production” and perturbative “multi-photon pair production”

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

$$c_n \sim (\pm 1)^n \frac{n!}{(2S)^n}$$

Towards Resurgence in Asymptotically Free QFT

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

$$c_n \sim (\pm 1)^n \frac{n!}{(2S)^n}$$

QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: "renormalons"

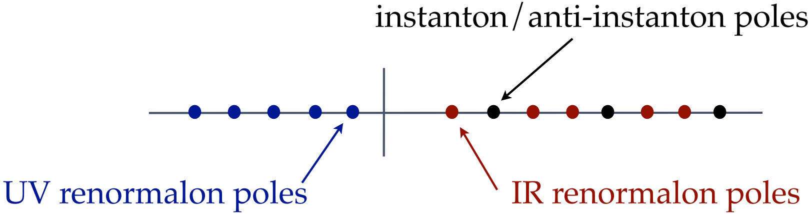
$$c_n \sim (\pm 1)^n \frac{\beta_0^n \, n!}{(2S)^n} = (\pm 1)^n \frac{n!}{(2S/\beta_0)^n}$$

- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow \qquad \pm i\, e^{-\frac{2S}{\beta_0 g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow \qquad \pm i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

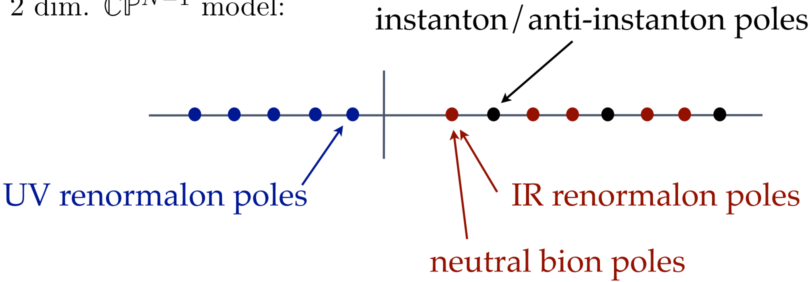
asymptotically free theories remain perturbatively inconsistent

't Hooft, 1980; David, 1981

IR Renormalon Puzzle in Asymptotically Free QFT

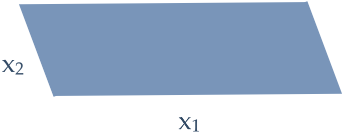
resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

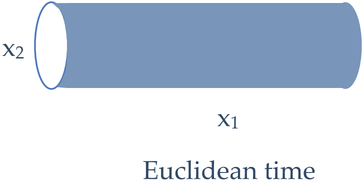
- scale modulus of instantons
- spatial compactification with $\mathbb{Z}_N$ twisted b.c.'s, & principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



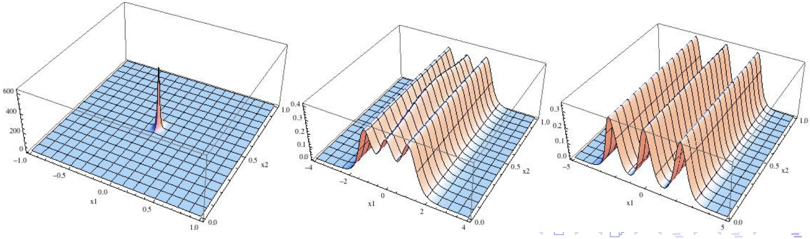
cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification: $\mathbb{R}^2 \to \mathbb{S}^1_L \times \mathbb{R}^1$





$\mathbb{Z}_N$ twist: instantons fractionalize: $S_{\text{inst}} \longrightarrow \frac{S_{\text{inst}}}{N} = \frac{S_{\text{inst}}}{\beta_0}$



Perturbative Analysis

- weak-coupling semi-classical analysis
- perturbative $\rightarrow$ effective QM problem
- perturbation theory diverges & non-Borel summable
- perturbative sector: directional Borel summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

- compare with non-perturbative instanton gas analysis:

$$\left[\mathcal{I}_i \bar{\mathcal{I}}_i\right]_\pm = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right) \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

exact ("BZJ") cancellation !

explicit application of resurgence to nontrivial QFT

- 2d $O(N)$ & principal chiral model have no instantons !
- but they have finite action non-BPS saddles
- Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, principal chiral model, ... all have non-BPS solutions with finite action

(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

Non-perturbative Physics Without Instantons

Dabrowski, GD, 1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127, 1403.1277, GD, Ünsal, 1505.07803

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- "unstable": negative modes of fluctuation operator

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(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

- "unstable": negative modes of fluctuation operator
- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from directional Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$