Introduction to Resurgence and Non-perturbative Physics

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GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414, 1603.04924

recent KITP Program: Resurgent Asymptotics in Physics and Mathematics, Fall 2017
future Isaac Newton Institute Programme: Universal Resurgence, 2020/2021
1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   - asymptotic series in physics; Borel summation
   - trans-series completions & resurgence
   - examples: linear and nonlinear ODEs

2. Lecture 2: Applications to Quantum Mechanics and QFT
   - instanton gas, saddle solutions and resurgence
   - infrared renormalon problem in QFT
   - Picard-Lefschetz thimbles

3. Lecture 3: Resurgence and Large $N$
   - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT

4. Lecture 4: Resurgence and Phase Transitions
   - Gross-Witten-Wadia Matrix Model
recall: Mathieu Equation: \[-\frac{\hbar^2}{2} \frac{d^2 \psi}{dx^2} + \cos(x) \psi = u \psi\]

\[u(\hbar)\]

\[u_{\pm}(\hbar, N) = u_{\text{pert}}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp \left[-\frac{8}{\hbar}\right] \mathcal{P}_{\text{inst}}(\hbar, N) + \ldots\]

\[\mathcal{P}_{\text{inst}}(\hbar, N) = \frac{\partial u_{\text{pert}}(\hbar, N)}{\partial N} \exp \left[S \int_0^\hbar d\hbar' \frac{1}{\hbar'^3} \left(\frac{\partial u_{\text{pert}}(\hbar', N)}{\partial N} - \hbar + \frac{(N + \frac{1}{2}) \hbar^2}{S}\right)\right]\]

GD & Ünsal (2013); Başar & GD (2015): applies to bands & gaps
recall: Resurgence of $\mathcal{N} = 2$ SUSY SU(2)

- moduli parameter: $u = \langle \text{tr } \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial W}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}(a, \hbar, \Lambda)$:
- Mathieu equation: (Mironov/Morozov)

$$- \frac{\hbar^2}{2} \frac{d^2 \psi}{dx^2} + \Lambda^2 \cos(x) \psi = u \psi, \quad a \equiv \frac{N \hbar}{2}$$

- Mathieu P/NP relation $\equiv$ (quantum) Matone relation:

$$u(a, \hbar) = i\pi \frac{\Lambda}{2} \frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

- $\mathcal{N} = 2^*$ $\Leftrightarrow$ Lamé equation
Classical Genus 1 Structure

- energy-momentum relation defines a Riemann surface

\[ u = \frac{p^2}{2} + V(x) \]

\[ p^2 = 2u - 2V(x) \]
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- quartic \( V \) (or less) \( \Rightarrow \) genus 1: torus

- two independent cycles: \( \alpha = "\text{well}" \ , \ \beta = "\text{barrier}" \)

\[ a_0(u) = \sqrt{2} \int_{\alpha} dx \sqrt{u - V(x)} \ , \ \omega_0(u) = \frac{1}{\sqrt{2}} \int_{\alpha} \frac{dx}{\sqrt{u - V(x)}} \]
\[ a_0^D(u) = \sqrt{2} \int_{\beta} dx \sqrt{u - V(x)} \ , \ \omega_0^D(u) = \frac{1}{\sqrt{2}} \int_{\beta} \frac{dx}{\sqrt{u - V(x)}} \]
Classical Genus 1 Structure

- energy-momentum relation defines a Riemann surface
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- quartic \( V \) (or less) \( \Rightarrow \) genus 1: torus
- two independent cycles: \( \alpha = "well" \) , \( \beta = "barrier" \)
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\begin{align*}
a_0(u) &= \sqrt{2} \int_\alpha dx \sqrt{u - V(x)} \quad , \quad \omega_0(u) = \frac{1}{\sqrt{2}} \int_\alpha \frac{dx}{\sqrt{u - V(x)}} \\
a_0^D(u) &= \sqrt{2} \int_\beta dx \sqrt{u - V(x)} \quad , \quad \omega_0^D(u) = \frac{1}{\sqrt{2}} \int_\beta \frac{dx}{\sqrt{u - V(x)}}
\end{align*}
\]
- periods and actions are elliptic functions: \( \wp, \sigma, \zeta \)
- periods satisfy 2nd order ODE with respect to \( u \)
- actions satisfy 2nd/3rd order ODE (Picard-Fuchs) w.r.t. \( u \)
Quantization: "All-orders WKB", "Exact WKB"

- formal expansion in $\hbar^2$

\[
\begin{align*}
  a(u, \hbar) &= \sum_{n=0}^{\infty} \hbar^{2n} a_n(u), \\
  a^D(u, \hbar) &= \sum_{n=0}^{\infty} \hbar^{2n} a^D_n(u)
\end{align*}
\]

- explicit expansion (Dunham, 1932)

\[
\begin{align*}
  a(u, \hbar) &= \sqrt{2} \left( \int_{\alpha} \sqrt{u-V} \, dx - \frac{\hbar^2}{2^6} \int_{\alpha} \frac{(V')^2}{(u-V)^{5/2}} \, dx \\
  & \quad \quad - \frac{\hbar^4}{2^{13}} \int_{\alpha} \left( \frac{49(V')^4}{(u-V)^{11/2}} - \frac{16V'V'''}{(u-V)^{7/2}} \right) \, dx - \ldots \right) \\
  a^D(u, \hbar) &= \sqrt{2} \left( \int_{\beta} \sqrt{u-V} \, dx - \frac{\hbar^2}{2^6} \int_{\beta} \frac{(V')^2}{(u-V)^{5/2}} \, dx \\
  & \quad \quad - \frac{\hbar^4}{2^{13}} \int_{\beta} \left( \frac{49(V')^4}{(u-V)^{11/2}} - \frac{16V'V'''}{(u-V)^{7/2}} \right) \, dx - \ldots \right)
\end{align*}
\]

- identical integrands!
1. classical geometry (Riemann): $a_0(u)$ determines $a_0^D(u)$
Origin of Perturbative/Non-Perturbative Relation for Genus 1

1. classical geometry (Riemann): \( a_0(u) \) determines \( a_0^D(u) \)

2. perturbation theory is equivalent to inversion of all-orders Bohr-Sommerfeld (here, a monodromy condition):

\[
a(u, \hbar) = 2\pi \hbar \left( N + \frac{1}{2} \right), \quad N = 0, 1, 2, \ldots
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3. all higher order terms, $a_n(u)$ and $a^D_n(u)$, are generated by action of differential operators on $a_0(u)$ and $a_0^D(u)$

$$a_n(u) = \mathcal{D}_u^{(n)} a_0(u), \quad a^D_n(u) = \mathcal{D}_u^{(n)} a_0^D(u)$$

where $\mathcal{D}_u^{(n)}$ and $\mathcal{D}_u^{(n)}$ are the same! (Legendre, Weierstrass, ...
Origin of Perturbative/Non-Perturbative Relation for Genus 1

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4. knowing \( a(u, \hbar) \) to some order \( \Leftrightarrow \) knowledge of \( \mathcal{D}_u^{(n)} \)
    \( \Rightarrow \) we therefore know \( a^D(u, \hbar) \) to the same order
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3. all higher order terms, $a_n(u)$ and $a_n^D(u)$, are generated by action of differential operators on $a_0(u)$ and $a_0^D(u)$

$$a_n(u) = D_u^{(n)} a_0(u) , \quad a_n^D(u) = D_u^{(n)} a_0^D(u)$$

where $D_u^{(n)}$ and $D_u^{(n)}$ are the same! (Legendre, Weierstrass, ...)

4. knowing $a(u, \hbar)$ to some order $\Leftrightarrow$ knowledge of $D_u^{(n)}$

$\Rightarrow$ we therefore know $a^D(u, \hbar)$ to the same order

$\Rightarrow$ “perturbation theory encodes all non-perturbative physics”
Exercise 7:

(i) For the Mathieu system, evaluate the classical actions and periods in terms of hypergeometric functions, and identify the 2nd order Picard-Fuchs equation that they satisfy, as a function of energy $u$.

(ii) Use Dunham’s expressions for the all-orders WKB expansion to show that the first quantum correction actions $a_1(u)$ and $a_1^D(u)$ can be expressed as simple differential operators (w.r.t. $u$) acting on $a_0(u)$ and $a_0^D(u)$, respectively. Note that the differential operator is the same in the two cases.
• P/NP relations in terms of (all-orders) quantum actions

• Mathieu has all-orders quantum Matone relation:

$$\frac{\partial u(a, \hbar)}{\partial a} = \frac{i\pi}{2} \left( a D(a, \hbar) - a \frac{\partial a D(a, \hbar)}{\partial a} - \hbar \frac{\partial a D(a, \hbar)}{\partial \hbar} \right)$$


• Mathieu has all-orders quantum Wronskian relation:

$$\left[ a(u, \hbar) - \hbar \frac{\partial a(u, \hbar)}{\partial \hbar} \right] \frac{\partial a D(u, \hbar)}{\partial u} - \left[ a D(u, \hbar) - \hbar \frac{\partial a D(u, \hbar)}{\partial \hbar} \right] \frac{\partial a(u, \hbar)}{\partial u} = \frac{2i}{\pi}$$

Başar & GD (2015)
Analytic Continuation of Path Integrals: Lefschetz Thimbles

\[ \int \mathcal{D}A e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles } k} \mathcal{N}_k e^{-\frac{i}{g^2} S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A e^{-\frac{1}{g^2} S_{\text{real}}[A]} \]

Lefschetz thimble = “functional steepest descents contour”

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers \( \mathcal{N}_k \) can change with phase of parameters
Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

\[
\frac{\partial}{\partial \tau} A(x; \tau) = - \frac{\delta S}{\delta A(x; \tau)}
\]

- keeps $\text{Im}[S]$ constant, and $\text{Re}[S]$ is monotonic

\[
\frac{\partial}{\partial \tau} \left( \frac{S - \bar{S}}{2i} \right) = - \frac{1}{2i} \int \left( \frac{\delta S \partial A}{\delta A \partial \tau} - \frac{\bar{S} \partial A}{\delta A \partial \tau} \right) = 0
\]

\[
\frac{\partial}{\partial \tau} \left( \frac{S + \bar{S}}{2} \right) = - \int \left| \frac{\delta S}{\delta A} \right|^2
\]

- Chern-Simons theory \(^{(\text{Witten 2010})}\)
- comparison with complex Langevin \(^{(\text{Aarts 2013, ...})}\)
- lattice (Aurora, 2013; Tokyo/RIKEN): Bose-gas ✓
- generalized thimble method: \(^{(\text{Alexandru, Başar, Bedaque et al. 2016})}\)
Exercise 8:

use complexified gradient flow to find the steepest descent contours for the Airy function integral, as a function of the phase of $x$, the argument of $Ai(x)$. Compare with the plots in Lecture 2.
Generalized Thimble Method

- idea: compromise by "just getting close to" the thimbles

\[ \text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(\frac{1}{3} t^3 + x t\right)} dt \]

- exact value: \( \text{Ai}(5) = 0.000108344428136074 \)

- real parts of integrand at \( x = 5 \):

\[ t \in \mathbb{R} \]

\[ t = \frac{i}{10} \sqrt{5} + s \]

\[ t = i \sqrt{5} + s \]

- reduces the sign problem to a manageable level

\[ \text{Ai}(5) = 0.000108344438640742 \]

\[ \text{Ai}(5) = 0.000108344428136076 \]
Thimbles from Gradient Flow

- generalized thimble method:  (Alexandru, Başar, Bedaque et al, 2016)
Resurgence and Phase Transitions

idea: phase transition $\leftrightarrow$ Stokes jump
Resurgence and Phase Transitions: Examples

- particle-on-circle (Schulman PhD thesis 1968): sum over spectrum versus sum over winding (saddles)
- Bose gas (Cristoforetti et al, Alexandru et al)
- Thirring model (Alexandru et al)
- Hubbard model (Tanizaki et al; ...)
- Ising model (GD, 1901.02076; Coger, GD, to appear)
- Hydrodynamics: short time/late time (Heller et al; Basar, GD)
- Large N matrix, localization (Mariño, Schiappa, Couso, Russo, ...)
- Gross-Witten-Wadia model (Mariño, 2008; Ahmed, GD, 2017)
- Painlevé systems (Costin, GD, to appear)
- ...
Phase Transition in 1+1 dim. Gross-Neveu Model

\[ \mathcal{L} = \bar{\psi} i \phi \psi + \frac{g^2}{2} (\bar{\psi} \psi)^2 \]

- large $N_f$ chiral symmetry breaking phase transition

\[ \sigma(x; T, \mu) = \frac{\delta}{\delta \sigma(x; T, \mu)} \text{Tr}_{T, \mu} \ln (i \phi - \sigma(x; T, \mu)) \]

(Thies 2003; Basar, GD, Thies, 2011; Ahmed, 2018)
Phase Transition in 1+1 dim. Gross-Neveu Model

- tricritical point: divergent Ginzburg-Landau expansion

\[ \Psi(T, \mu) = \sum_n \alpha_n(T, \mu) f_n[\sigma(x; T, \mu)] \]

- successive orders of GL expansion reveal the full crystal phase

(Basar, GD, Thies, 2011; Ahmed, 2018)

- most difficult point: \( \mu_c = \frac{2}{\pi}, \ T = 0 \)
Phase Transition in $1+1$ dim. Gross-Neveu Model

- $T = 0$: exact (implicit transcendental) expressions: expansions change character
- large $\mu$ expansion $\Rightarrow$ location of critical point $\mu_c = \frac{2}{\pi}$
- non-perturb. $e^{-\frac{1}{\rho}}$ effects at phase transition at $\mu_c = \frac{2}{\pi}$
- high density (convergent !)

$$E(\rho) \sim \frac{\pi}{2} \rho^2 \left( 1 - \frac{1}{32(\pi \rho)^4} + \frac{3}{8192(\pi \rho)^8} - \cdots \right)$$

- low density (non-perturbative !)

$$E(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{e^{-k/\rho}}{\rho^{k-2}} F_{k-1}(\rho)$$

- analogous for $\mu$ expansion
2d Yang-Mills: Douglas-Kazakov Large $N$ Phase Transition

- 2d Yang-Mills on a sphere
- "spectral sum" for partition function
  \[ Z(a, N) = \sum_{R} (\text{dim } R)^2 e^{-\frac{a}{2N} C_2(R)} \]
- large $N$ phase transition at $a_c = \pi^2$ \textbf{(Douglas-Kazakov)}
- "instanton condensation" \textbf{(Gross-Matytsin)}
2d Yang-Mills: Douglas-Kazakov Large $N$ Phase Transition

- 2d Yang-Mills on a sphere
- "spectral sum" for partition function

$$Z(a, N) = \sum_R (\dim R)^2 e^{-\frac{a}{2N} C_2(R)}$$

- large $N$ phase transition at $a_c = \pi^2$ (Douglas-Kazakov)
- "instanton condensation" (Gross-Matytsin)
- other limit: saddles = monopole solutions: $A_\mu = \vec{n}A_\mu$

$$Z(a, N) = \sum_{\vec{n}} \mathcal{F}(\vec{n}) e^{-\frac{2\pi^2 N}{a} \vec{n}^2}$$

- dual descriptions: generalized Poisson duality
- phase transition = change of saddles
Resurgence in Matrix Models: Mariño: 0805.3033, Ahmed & GD: 1710.01812

Gross-Witten-Wadia Unitary Matrix Model

\[ Z(g^2, N) = \int_{U(N)} DU \exp \left[ \frac{1}{g^2} \text{tr} \left( U + U^\dagger \right) \right] \]

- one-plaquette matrix model for 2d lattice Yang-Mills
- two variables: \( g^2 \) and \( N \) ('t Hooft coupling: \( t \equiv g^2 N/2 \))
- 3rd order phase transition at \( N = \infty, t = 1 \) (universal!)
- double-scaling limit: Painlevé II
- physics of phase transition = condensation of instantons
- similar to 2d Yang-Mills on sphere and disc
3rd order transition: kink in the specific heat

FIG. 2. The specific heat per degree of freedom, \( C/N^2 \), as a function of \( \lambda \) (temperature).

D. Gross, E. Witten, 1980
Gross-Witten: beta function at infinite $N$

- infinite $N$:
  \[
  \beta(\lambda) = \begin{cases} 
  -2\lambda \log \lambda & \lambda \geq 2 \\
  2(\lambda - 4) \log \frac{4}{4-\lambda} & \lambda \leq 2 
  \end{cases}
  \]

Wilson loop:
\[
W(\lambda, N) = \langle \text{tr } U \rangle := e^{-a^2 \Sigma}
\]
\[
\beta(\lambda, N) := -\frac{\partial \lambda(a, N)}{\partial \ln \lambda}
\]

- naive large $N$
  incorrectly predicts new fixed points at $\lambda = 1$ and $\lambda = 4$

**FIG. 1.** The $\beta$ function as a function of $\lambda$. The dashed lines are the (invalid) extrapolation of the weak- and strong-coupling results beyond the phase transition at $\lambda = 2$.

D. Gross, E. Witten, 1980
• for any $N$: $\beta(\lambda) = \frac{2}{d \lambda} \log \log W_N(\lambda) \rightarrow \text{trans-series for } \beta$

• resurgent large $N$ smoothly passes from weak-coupling to strong-coupling curve, developing a kink at $N = \infty$
Random matrix theory/orthogonal polynomials result: partition function $Z(g^2, N) = \det (I_j - k(x))_{j,k=1,...,N}$, $x \equiv \frac{2}{g^2}$

- explicit, but not particularly efficient for $N \to \infty$
Lee-Yang: complex zeros of $Z$ pinch the real axis at the phase transition point in the thermodynamic limit

Fig. 1. The first quadrant of the conjectured domain $U_\infty$ that is filled densely with zeros in the limit $N \to \infty$
• ‘brute force’ numerical search for saddles
• in terms of eigenvalues $e^{iz_j}$:

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} dz_i \exp \left[ -\frac{2N}{\lambda} \sum_{i} \cos(z_i) + \ln \prod_{i<j} \sin^2 \left( \frac{z_i - z_j}{2} \right) \right]$$

• saddle point approach: $\partial S/\partial z_i = 0$
• which saddles (real/complex?) govern large $N$ behavior?
• how to see the “phase transition” at finite $N$?
Gross-Witten-Wadia Model: weak coupling: $\lambda < 2$

- "eigenvalue tunneling" of saddles into the complex plane
- number of complex eigenvalues: $m = \text{instanton number}$
- dominant non-perturbative saddle has $m = 1$
Gross-Witten-Wadia Model: strong coupling: $\lambda > 2$

- "eigenvalue tunneling" of saddles into the complex plane
- Number of complex eigenvalues: $m = \text{instanton number}$
- Dominant non-perturbative saddle has $m = 2$
Gross-Witten-Wadia Model: non-vacuum saddles

- weak coupling ($\lambda < 2$): $m = 1$ dominant
- strong coupling ($\lambda > 2$): $m = 2$ dominant

\begin{align*}
\lambda < 2 : & \quad S_I^{(weak)} = \frac{4}{\lambda} \sqrt{1 - \frac{\lambda}{2}} - \text{arccosh} \left(\frac{4 - \lambda}{\lambda}\right) \\
\lambda > 2 : & \quad S_I^{(strong)} = 2 \arccosh \left(\frac{\lambda}{2}\right) - 2 \sqrt{1 - \frac{4}{\lambda^2}}
\end{align*}

- microscopic view of strong-coupling "instanton/saddle"
Resurgence in Gross-Witten-Wadia Model

- random matrix theory/orthogonal polynomials result: partition function $= N \times N$ Toeplitz determinant

$$Z(g^2, N) = \det (I_{j-k}(x))_{j,k=1,...,N} , \quad x \equiv \frac{2}{g^2}$$

- weak coupling: resurgent trans-series for $I_{j-k}(x)$
- strong coupling: convergent series for $I_{j-k}(x)$
- interesting transition between the two, esp. at large $N$
Exercise 3: the modified Bessel function has the large $x$ asymptotic expansion:

$$I_j(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n(j)}{x^n} \pm i e^{ij\pi} \frac{e^{-x}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{\alpha_n(j)}{x^n}, \quad |\arg(x) - \frac{\pi}{2}| < \pi$$

where the coefficients are

$$\alpha_j(n) = \frac{\cos(\pi j)}{\pi} \left(-\frac{1}{2}\right)^n \frac{\Gamma(n + \frac{1}{2} - j) \Gamma(n + \frac{1}{2} + j)}{\Gamma(n + 1)}$$

(i) show that the large-order growth ($n \to \infty$) is

$$\alpha_n(j) \sim \frac{\cos(j\pi)}{\pi} \left(-1\right)^n \frac{(n-1)!}{2^n} \left(\alpha_0(j) - \frac{2 \alpha_1(j)}{(n-1)} + \frac{2^2 \alpha_2(j)}{(n-1)(n-2)} - \cdots\right)$$

(ii) what is the significance of the $\cos(j\pi)$ prefactor?
Resurgence in Gross-Witten-Wadia Model

- partition function = $N \times N$ Toeplitz determinant

$$Z(g^2, N) = \det (I_{j-k}(x))_{j,k=1,...N}, \quad x \equiv \frac{2}{g^2}$$

- weak-coupling resurgent trans-series: $N + 1$ instantons

$$Z(x, N) \sim Z_0(x, N) \left[ \sum_{n=0}^{\infty} \frac{a_n^{(0)}(N)}{x^n} + i \frac{(4x)^{N-1}}{\Gamma(N)} e^{-2x} \sum_{n=0}^{\infty} \frac{a_n^{(1)}(N)}{x^n} + \ldots + \frac{G(N + 1)}{\prod_{i=0}^{N-1} \Gamma(N - i)} e^{-2Nx} \sum_{n=0}^{\infty} \frac{a_n^{(N)}(N)}{x^n} \right]$$

- but strong-coupling expansion is convergent!

$$Z(x, N) \sim e^{x^2/4} \left[ 1 - \left( \frac{(x/2)^{N+1}}{(N + 1)!} \right)^2 \left( 1 - \frac{1}{2} \frac{(N + 1) x^2}{(N + 2)^2} + \ldots \right) + \ldots \right]$$
Resurgence in Gross-Witten-Wadia Model

- Idea: map it to a Painlevé function (Painlevé III)

\[ \Delta(x, N) \equiv \langle \det U \rangle = \frac{\det [I_{j-k+1}(x)]_{j,k=1,\ldots,N}}{\det [I_{j-k}(x)]_{j,k=1,\ldots,N}} \]

- For any \( N \), \( \Delta(x, N) \) satisfies a PIII-type equation:

\[ \Delta'' + \frac{1}{x} \Delta' + \Delta (1 - \Delta^2) + \frac{\Delta}{(1 - \Delta^2)} \left[ (\Delta')^2 - \frac{N^2}{x^2} \right] = 0 \]

⇒ generate trans-series solutions: weak- & strong-coupling

- \( N \) is a parameter! ⇒ large \( N \) limit by rescaling
• idea: map it to a Painlevé function (Painlevé III)

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\Delta(x, N) \equiv \langle \det U \rangle = \frac{\det [I_{j-k+1}(x)]_{j,k=1,\ldots,N}}{\det [I_{j-k}(x)]_{j,k=1,\ldots,N}}
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• for any \( N \), \( \Delta(x, N) \) satisfies a PIII-type equation:

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\]

⇒ generate trans-series solutions: weak- & strong-coupling

• \( N \) is a parameter! ⇒ large \( N \) limit by rescaling

• direct relation to the partition function:

\[
\Delta^2(x, N) = 1 - \frac{Z(x, N - 1) Z(x, N + 1)}{Z^2(x, N)}
\]

\[
Z(x, N) = \exp \left[ \frac{1}{2} \int_0^x x \, dx \left( 1 - \Delta^2(x, N) \right) \left( 1 + \Delta(x, N - 1) \Delta(x, N + 1) \right) \right]
\]
Resurgence in Gross-Witten-Wadia Model

- weak-coupling expansion is a **divergent** series:
  → trans-series non-perturbative completion

- strong-coupling expansion is a **convergent** series:
  but it still has a non-perturbative completion!

\[\Delta'' + 1/x \Delta' + \Delta(1 - \Delta^2) + \Delta(1 - \Delta^2)((\Delta')^2 - N^2x^2) = 0\]
Resurgence in Gross-Witten-Wadia Model

- weak-coupling expansion is a **divergent** series:
  \[ \rightarrow \text{trans-series non-perturbative completion} \]

- strong-coupling expansion is a **convergent** series:
  \[ \text{but it still has a non-perturbative completion!} \]

- \( \Delta \) small \( \Rightarrow \) linearize \( \rightarrow \) Bessel equation

  \[
  \Delta'' + \frac{1}{x} \Delta' + \Delta \left(1 - \Delta^2\right) + \frac{\Delta}{(1 - \Delta^2)} \left[(\Delta')^2 - \frac{N^2}{x^2}\right] = 0
  \]

  \[ \Rightarrow \Delta(x, N) \big|_{\text{strong}} \approx \sigma J_N(x) \]

- strong-coupling expansion (\( x \equiv \frac{2}{g^2} \)) is clearly convergent, but only agrees with expansion of \( J_N(x) \) to order \( x^{3N} \). Why?
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- strong-coupling expansion \((x \equiv \frac{2}{g^2})\) is clearly convergent, but only agrees with expansion of \(J_N(x)\) to order \(x^{3N}\). Why?

- full solution is a non-perturbative trans-series:

\[
\Delta(x, N) = \sum_{k=1,3,5,...}^{\infty} (\sigma_{\text{strong}})^k \Delta_k(x, N)
\]
Resurgence in Gross-Witten-Wadia Model

• strong-coupling trans-series (convergent !!!):

\[ \Delta(x, N) = \sum_{k=1,3,5,...}^{\infty} (\sigma_{\text{strong}})^k \Delta(k)(x, N) \]

blue: exact , red: \( \Delta(1) = J_5(x) \) , black: includes \( \Delta(3) \)
Resurgence in GWW: ’t Hooft limit and phase transition

- rescaled PIII equation: \( t \equiv N g^2 / 2 \equiv \frac{N}{x} \)

\[
t^2 \Delta'' + t \Delta' + \frac{N^2 \Delta}{t^2} (1 - \Delta^2) = \frac{\Delta}{1 - \Delta^2} \left( N^2 - t^2 (\Delta')^2 \right)
\]

- GWW \( N = \infty \) phase transition:

\[
\Delta(t, N) \xrightarrow{N \to \infty} \begin{cases} 
0 & , \quad t \geq 1 \quad (\text{strong coupling}) \\
\sqrt{1 - t} & , \quad t \leq 1 \quad (\text{weak coupling}) 
\end{cases}
\]

- large \( N \):

\[
\frac{\Delta}{t^2} (1 - \Delta^2) = \frac{\Delta}{1 - \Delta^2}
\]

\[\Rightarrow \quad \Delta = 0 \quad \text{or} \quad \Delta = \sqrt{1 - t}\]
Resurgence in GWW: ’t Hooft limit and phase transition

- Gross-Witten-Wadia $N = \infty$ phase transition:

$$\Delta(t, N) \xrightarrow{N \to \infty} \begin{cases} 0 , & t \geq 1 \text{ (strong coupling)} \\ \sqrt{1-t} , & t \leq 1 \text{ (weak coupling)} \end{cases}$$

$$t \equiv \frac{N}{x} \equiv \frac{Ng^2}{2}$$

- black lines: increasing $N$
- red dashed line: $\Delta = \sqrt{1-t}$
Resurgence in GWW: ’t Hooft limit and phase transition

- full large $N$ trans-series at weak-coupling:

$$\Delta(t, N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^0(t)}{N^{2n}} - \frac{i}{2\sqrt{2\pi N}} \sigma_{\text{weak}} \frac{t e^{-NS_{\text{weak}}(t)}}{(1-t)^{1/4}} \sum_{n=0}^{\infty} \frac{d_n^1(t)}{N^n} + \ldots$$

- large $N$ weak-coupling action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2 \text{arctanh} \left(\sqrt{1-t}\right)$$
Resurgence in GWW: ’t Hooft limit and phase transition

- full large $N$ trans-series at weak-coupling:

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\Delta(t, N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i}{2 \sqrt{2\pi N}} \sigma_{\text{weak}} \frac{t e^{-NS_{\text{weak}}(t)}}{(1-t)^{1/4}} \sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} + \ldots
$$

- large $N$ weak-coupling action

$$
S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2 \arctanh \left( \sqrt{1-t} \right)
$$

- large-order growth of perturbative coefficients ($\forall t < 1$):

$$
d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}} \frac{\Gamma(2n-\frac{5}{2})}{(S_{\text{weak}}(t))^{2n-\frac{5}{2}}} \left[ 1 + \frac{(3t^2 - 12t - 8)}{96(1-t)^{3/2}} \frac{S_{\text{weak}}(t)}{(2n-\frac{7}{2})} \right] + \ldots
$$

- confirm (parametric!) resurgence relations, for all $t$:

$$
\sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2 - 12t - 8)}{96(1-t)^{3/2}} \frac{1}{N} + \ldots
$$
Resurgence in GWW: ’t Hooft limit and phase transition

- large $N$ transseries at strong-coupling: $\Delta(t, N) \approx \sigma J_N \left( \frac{N}{t} \right)$

\[
\Delta(t, N) = \sum_{k=1,3,5,...}^{\infty} (\sigma_{\text{strong}})^k \Delta_k(t, N)
\]

- "Debye expansion" for Bessel function: $J_N(N/t)$

\[
\Delta(t, N) \sim \frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N} (t^2 - 1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n}
\]  
\[
+ \frac{1}{4(t^2 - 1)} \left( \frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N} (t^2 - 1)^{1/4}} \right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \ldots
\]

- large $N$ strong-coupling action: $S_{st}(t) = \text{arccosh}(t) - \sqrt{1 - \frac{1}{t^2}}$
Resurgence in GWW: 't Hooft limit and phase transition

- large $N$ transseries at strong-coupling: $\Delta(t, N) \approx \sigma J_N \left(\frac{N}{t}\right)$

$$\Delta(t, N) = \sum_{k=1,3,5,...}^{\infty} (\sigma_{\text{strong}})^k \Delta(k)(t, N)$$

- "Debye expansion" for Bessel function: $J_N(N/t)$

$$\Delta(t, N) \sim \frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N} \left(t^2 - 1\right)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n}$$

$$+ \frac{1}{4(t^2 - 1)} \left(\frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N} \left(t^2 - 1\right)^{1/4}}\right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \ldots$$

- large $N$ strong-coupling action: $S_{\text{st}}(t) = \text{arccosh}(t) - \sqrt{1 - \frac{1}{t^2}}$

- large-order/low-order (parametric) resurgence relations:

$$U_n(t) \sim \frac{(-1)^n (n - 1)!}{2\pi (2S_{\text{strong}}(t))^n} \left(1 + U_1(t) \frac{2S_{\text{strong}}(t)}{(n - 1)} + U_2(t) \frac{(2S_{\text{strong}}(t))^2}{(n - 1)(n - 2)} + \ldots\right)$$
Resurgence in GWW: ’t Hooft limit and phase transition

- Debye expansion has unphysical divergence at $t = 1$
- Uniform asymptotic expansion:

$$J_N \left( \frac{N}{t} \right) \sim \left( \frac{4 \left( \frac{3}{2} S_{\text{strong}}(t) \right)^{2/3}}{1 - 1/t^2} \right)^{\frac{1}{4}} \frac{\text{Ai} \left( N^{\frac{2}{3}} \left( \frac{3}{2} S_{\text{strong}}(t) \right)^{2/3} \right)}{N^{\frac{1}{3}}}$$

- Nonlinear analogue of uniform WKB (coalescing saddles)
Resurgence in GWW: 't Hooft limit and phase transition

- Wilson loop: $\mathcal{W} \equiv \frac{1}{N} \frac{\partial \ln Z}{\partial x}$

$$\mathcal{W}(t, N) = \frac{1}{2t} \,(1 - \Delta^2(t, N)) \,(1 + \Delta(t, N - 1)\Delta(t, N + 1))$$

- uniform large $N$ approximation at strong-coupling:

$$\mathcal{W}(t, N) \bigg|_{\text{strong}} \approx \frac{1}{2t} \,(1 - J^2_N(N/t)) \,(1 + J_{N-1}(N/t)J_{N+1}(N/t))$$

**Graph:**
- blue: exact
- red: uniform large $N$
- dashed: usual large $N$

uniform resummation of instantons & fluctuations
Resurgence in GWW: ’t Hooft limit and phase transition

• uniform asymptotic expansion:

\[ \Delta_{\text{strong}}(N, t) \sim \left( \frac{4 \left( \frac{3}{2} S_{\text{strong}}(t) \right)^{2/3}}{1 - 1/t^2} \right)^{\frac{1}{4}} \frac{\text{Ai} \left( N^{\frac{2}{3}} \left( \frac{3}{2} S_{\text{strong}}(t) \right)^{2/3} \right)}{N^{\frac{1}{3}}} \]

• **physical** meaning of "uniform large-N instantons"?

• nonlinear analogue of "uniform WKB"

• technically: coalescence of two saddles \( \rightarrow \) "bion"

• expect similar phenomena in QFT
Resurgence in GWW: double-scaling limit = Painlevé II

- reduction cascade of Painlevé equations
- "zoom in" on vicinity of phase transition:
  \[ \kappa \equiv N^{2/3}(t-1) \quad ; \quad \Delta(t, N) = \frac{t^{1/3}}{N^{1/3}} y(\kappa) \]
- \( N \to \infty \) with \( \kappa \) fixed:
  \[ \Delta \quad \text{PIII equation} \quad \to \quad \frac{d^2 y}{d\kappa^2} = 2 y^3(\kappa) + 2\kappa y(\kappa) \quad \text{(PII)} \]
- e.g. on strong-coupling side:
  \[ \lim_{N \to \infty} J_N(N - N^{1/3} \kappa) = \left( \frac{2}{N} \right)^{1/3} \text{Ai} \left( 2^{1/3} \kappa \right) \]
- integral equation form of PII:
  \[ y(\chi) = \sigma \text{Ai}(\chi) + 2\pi \int_{\chi}^{\infty} \left[ \text{Ai}(\chi)\text{Bi}(\chi') - \text{Ai}(\chi')\text{Bi}(\chi) \right] y^3(\chi') d\chi' \]
Resurgence in GWW: double-scaling limit = Painlevé II

- "zoom in" on vicinity of phase transition:
- integral equation form of PII:

\[ y(\chi) = \sigma \text{Ai}(\chi) + 2\pi \int_{\chi}^{\infty} \left[ \text{Ai}(\chi)\text{Bi}(\chi') - \text{Ai}(\chi')\text{Bi}(\chi) \right] y^3(\chi') d\chi' \]

iterate \(\rightarrow\) resummed trans-series instanton expansion
blue: exact
red: leading uniform large \(N\)
dashed: sub-leading uniform large \(N\)
green dashed: usual large \(N\)
Conclusions

- **Resurgence** systematically unifies perturbative and non-perturbative analysis, via trans-series
- trans-series ‘encode’ analytic continuation information
- expansions about different saddles are intimately related
- there is extra un-tapped ‘magic’ in perturbation theory

- QM, matrix models, large $N$, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory
- applications to sign problem and non-equil. path integrals

- promising progress & many fascinating open problems
A Few Selected References: books

- J.C. Le Guillou and J. Zinn-Justin (Eds.), *Large-Order Behaviour of Perturbation Theory*
- C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*
- R. B. Dingle, *Asymptotic expansions: their derivation and interpretation*
- O. Costin, *Asymptotics and Borel Summability*
- R. B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*
A Few Selected References: papers


▶ E. Witten, “A New Look At The Path Integral Of Quantum Mechanics,” arXiv:1009.6032
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