Introduction to Resurgence and Non-perturbative Physics

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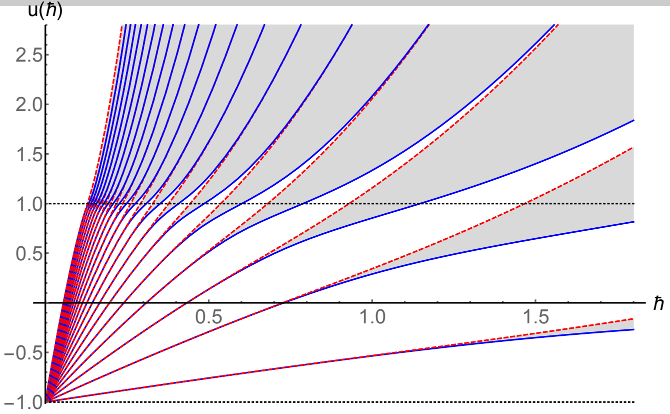
GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414, 1603.04924

recent KITP Program: *Resurgent Asymptotics in Physics and Mathematics*, Fall 2017

future Isaac Newton Institute Programme: *Universal Resurgence*, 2020/2021

1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   - asymptotic series in physics; Borel summation
   - trans-series completions & resurgence
   - examples: linear and nonlinear ODEs
2. Lecture 2: Applications to Quantum Mechanics and QFT
   - instanton gas, saddle solutions and resurgence
   - infrared renormalon problem in QFT
   - Picard-Lefschetz thimbles
3. Lecture 3: Resurgence and Large N
   - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N}=2$ SUSY QFT
4. Lecture 4: Resurgence and Phase Transitions
   - Gross-Witten-Wadia Matrix Model

recall: Mathieu Equation: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$



$$u_\pm(\hbar, N) = u_{\text{pert}}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}}\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right] \mathcal{P}_{\text{inst}}(\hbar, N) + \ldots$$

$$\mathcal{P}_{\text{inst}}(\hbar, N) = \frac{\partial u_{\text{pert}}(\hbar, N)}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3}\left(\frac{\partial u_{\text{pert}}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right.$$

GD & Ünsal (2013); Başar & GD (2015): applies to bands & gaps

recall: Resurgence of $\mathcal{N}=2$ SUSY SU(2)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}(a, \hbar, \Lambda)$:
- Mathieu equation: (Mironov/Morozov)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2 \cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- Mathieu P/NP relation $\equiv$ (quantum) Matone relation:

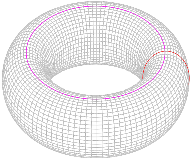
$$u(a, \hbar) = \frac{i\pi}{2}\Lambda\frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

- $\mathcal{N} = 2^*$ $\leftrightarrow$ Lamé equation

Classical Genus 1 Structure

- energy-momentum relation defines a Riemann surface

$$u = \frac{p^2}{2} + V(x)$$
$$p^2 = 2u - 2V(x)$$



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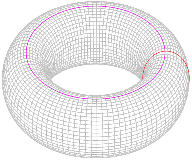

- quartic V (or less) $\Rightarrow$ genus 1: torus
- two independent cycles: $\alpha =$ "well" , $\beta =$ "barrier"

$$a_0(u) = \sqrt{2} \oint_\alpha dx \sqrt{u - V(x)} \quad , \quad \omega_0(u) = \frac{1}{\sqrt{2}} \oint_\alpha \frac{dx}{\sqrt{u - V(x)}}$$

$$a_0^D(u) = \sqrt{2} \oint_\beta dx \sqrt{u - V(x)} \quad , \quad \omega_0^D(u) = \frac{1}{\sqrt{2}} \oint_\beta \frac{dx}{\sqrt{u - V(x)}}$$

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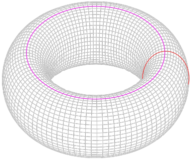

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- periods and actions are elliptic functions: $\mathbb{K}$, $\mathbb{E}$, $\mathbb{\Pi}$
- periods satisfy 2nd order ODE with respect to u
- actions satisfy 2nd/3rd order ODE (Picard-Fuchs) w.r.t. u

Quantization: "All-orders WKB", "Exact WKB"

- formal expansion in $\hbar^2$

$$a(u,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n(u) \qquad , \qquad a^D(u,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n^D(u)$$

- explicit expansion (Dunham, 1932)

$$\begin{aligned} a(u,\hbar) &= \sqrt{2}\left(\oint_\alpha \sqrt{u-V}\,dx - \frac{\hbar^2}{2^6}\oint_\alpha \frac{(V')^2}{(u-V)^{5/2}}dx \right.\\ &\left. -\frac{\hbar^4}{2^{13}}\oint_\alpha \left(\frac{49(V')^4}{(u-V)^{11/2}} - \frac{16V'V'''}{(u-V)^{7/2}}\right)dx - \dots\right) \end{aligned}$$

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- identical integrands !

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$$a_n(u) = \mathcal{D}_u^{(n)}\, a_0(u) \quad , \quad a_n^D(u) = \mathcal{D}_u^{(n)}\, a_0^D(u)$$

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$\Rightarrow$ "perturbation theory encodes all non-perturbative physics"

Exercise 7:

(i) For the Mathieu system, evaluate the classical actions and periods in terms of hypergeometric functions, and identify the 2nd order Picard-Fuchs equation that they satisfy, as a function of energy u.

(ii) Use Dunham's expressions for the all-orders WKB expansion to show that the first quantum correction actions $a_1(u)$ and $a_1^D(u)$ can be expressed as simple differential operators (w.r.t. u) acting on $a_0(u)$ and $a_0^D(u)$, respectively. Note that the differential operator is the same in the two cases.

Quantization of Mathieu System

- P/NP relations in terms of (all-orders) quantum actions
- Mathieu has all-orders quantum Matone relation:

$$\frac{\partial u(a,\hbar)}{\partial a} = \frac{i\pi}{2}\left(a^D(a,\hbar) - a\frac{\partial a^D(a,\hbar)}{\partial a} - \hbar\frac{\partial a^D(a,\hbar)}{\partial \hbar}\right)$$

Flume et al (2004)

- Mathieu has all-orders quantum Wronskian relation:

$$\left[a(u,\hbar) - \hbar\frac{\partial a(u,\hbar)}{\partial \hbar}\right]\frac{\partial a^D(u,\hbar)}{\partial u} - \left[a^D(u,\hbar) - \hbar\frac{\partial a^D(u,\hbar)}{\partial \hbar}\right]\frac{\partial a(u,\hbar)}{\partial u} = \frac{2i}{\pi}$$

Başar & GD (2015)

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial \tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int\left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial \tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial \tau}}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{S+\bar{S}}{2}\right) = -\int\left|\frac{\delta S}{\delta A}\right|^2$$

- Chern-Simons theory (Witten 2010)
- comparison with complex Langevin (Aarts 2013, ...)
- lattice (Aurora, 2013; Tokyo/RIKEN): Bose-gas ✓
- generalized thimble method: (Alexandru, Başar, Bedaque et al, 2016)

Exercise 8:

use complexified gradient flow to find the steepest descent contours for the Airy function integral, as a function of the phase of x, the argument of $Ai(x)$. Compare with the plots in Lecture 2.

Generalized Thimble Method

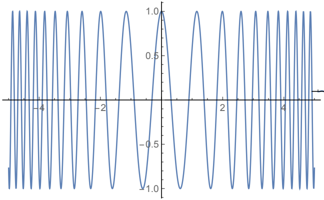
Alexandru, Başar, Bedaque et al, 2016 -

- idea: compromise by "just getting close to" the thimbles

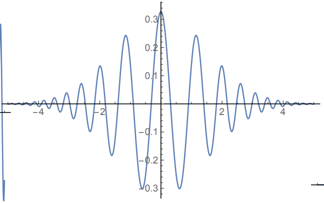
$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\,t^3 + x\,t\right)}\,dt$$

- exact value: $\mathrm{Ai}(5) = 0.000108344428136074$
- real parts of integrand at $x = 5$:

$t \in \mathbb{R}$

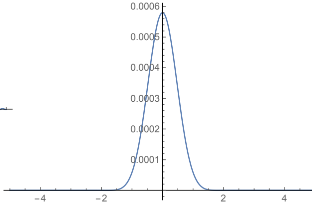


$t = \frac{i}{10}\sqrt{5} + s$



$\mathrm{Ai}(5) = 0.\underline{000108344438}640742$

$t = i\sqrt{5} + s$

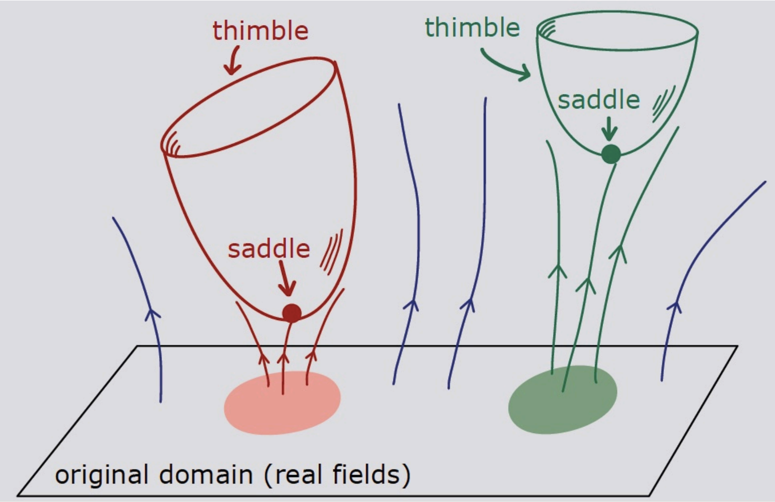


$\mathrm{Ai}(5) = 0.\underline{0001083444281360}76$

- reduces the sign problem to a manageable level

Thimbles from Gradient Flow

- generalized thimble method: (Alexandru, Başar, Bedaque et al, 2016)



idea: phase transition $\longleftrightarrow$ Stokes jump

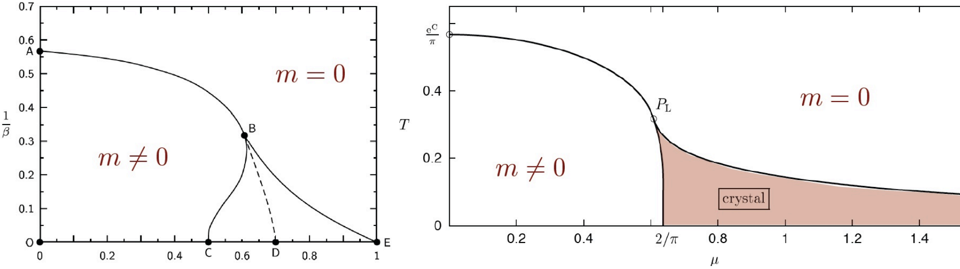
Resurgence and Phase Transitions: Examples

- particle-on-circle (Schulman PhD thesis 1968):
sum over spectrum versus sum over winding (saddles)
- Bose gas (Cristoforetti et al, Alexandru et al)
- Thirring model (Alexandru et al)
- Hubbard model (Tanizaki et al; ...)
- Ising model (GD, 1901.02076; Coger, GD, to appear)
- Hydrodynamics: short time/late time (Heller et al; Basar, GD)
- Large N matrix, localization (Mariño, Schiappa, Couso, Russo, ...)
- Gross-Witten-Wadia model (Mariño, 2008; Ahmed, GD, 2017)
- Painlevé systems (Costin, GD, to appear)
- ...

Phase Transition in 1+1 dim. Gross-Neveu Model

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi}\psi\right)^2$$

- large N_f chiral symmetry breaking phase transition



- saddles: exact solution of inhomogeneous gap equation

$$\sigma(x; T, \mu) = \frac{\delta}{\delta\sigma(x; T, \mu)} \mathrm{Tr}_{T,\mu} \ln\left(i\,\partial\!\!\!/ - \sigma(x; T, \mu)\right)$$

(Thies 2003; Basar, GD, Thies, 2011; Ahmed, 2018)

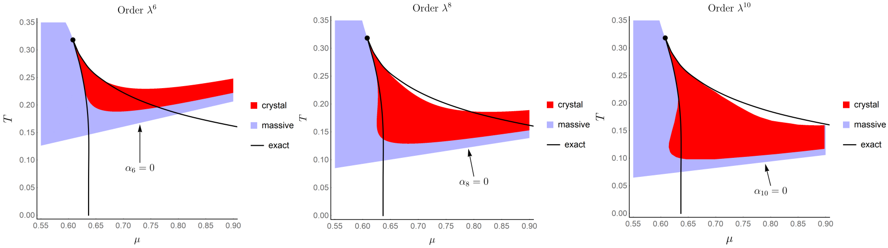
Phase Transition in 1+1 dim. Gross-Neveu Model

- tricritical point: divergent Ginzburg-Landau expansion

$$\Psi(T,\mu) = \sum_n \alpha_n(T,\mu) f_n[\sigma(x;T,\mu)]$$

- successive orders of GL expansion reveal the full crystal phase

(Basar, GD, Thies, 2011; Ahmed, 2018)



- most difficult point: $\mu_c = \frac{2}{\pi}$, $T = 0$

Phase Transition in 1+1 dim. Gross-Neveu Model

- $T = 0$: exact (implicit transcendental) expressions: expansions change character
- large μ expansion $\Rightarrow$ location of critical point $\mu_c = \frac{2}{\pi}$
- non-perturb. $e^{-\frac{1}{\rho}}$ effects at phase transition at $\mu_c = \frac{2}{\pi}$
- high density (convergent !)

$$\mathcal{E}(\rho) \sim \frac{\pi}{2}\,\rho^2\left(1 - \frac{1}{32(\pi\rho)^4} + \frac{3}{8192(\pi\rho)^8} - \dots\right)$$

- low density (non-perturbative !)

$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{e^{-k/\rho}}{\rho^{k-2}} \mathcal{F}_{k-1}(\rho)$$

- analogous for μ expansion

2d Yang-Mills: Douglas-Kazakov Large N Phase Transition

- 2d Yang-Mills on a sphere
- "spectral sum" for partition function

$$Z(a, N) = \sum_R (\dim R)^2 \, e^{-\frac{a}{2N} C_2(R)}$$

- large N phase transition at $a_c = \pi^2$ (Douglas-Kazakov)
- "instanton condensation" (Gross-Matytsin)

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- large N phase transition at $a_c = \pi^2$ (Douglas-Kazakov)
- "instanton condensation" (Gross-Matytsin)
- other limit: saddles = monopole solutions: $A_\mu = \vec{n}\,\mathcal{A}_\mu$

$$Z(a,N) = \sum_{\vec{n}} \mathcal{F}(\vec{n}) \, e^{-\frac{2\pi^2 N}{a} \, \vec{n}^2}$$

- dual descriptions: generalized Poisson duality
- phase transition = change of saddles

Resurgence in Matrix Models: Mariño: 0805.3033, Ahmed & GD: 1710.01812

Gross-Witten-Wadia Unitary Matrix Model

$$Z(g^2, N) = \int_{U(N)} DU \, \exp\left[\frac{1}{g^2} \text{tr}\left(U + U^\dagger\right)\right]$$

- one-plaquette matrix model for 2d lattice Yang-Mills
- two variables: g^2 and N ('t Hooft coupling: $t \equiv g^2 N/2$)
- 3rd order phase transition at $N = \infty$, $t = 1$ (universal!)
- double-scaling limit: Painlevé II
- physics of phase transition = condensation of instantons
- similar to 2d Yang-Mills on sphere and disc

3rd order transition: kink in the specific heat

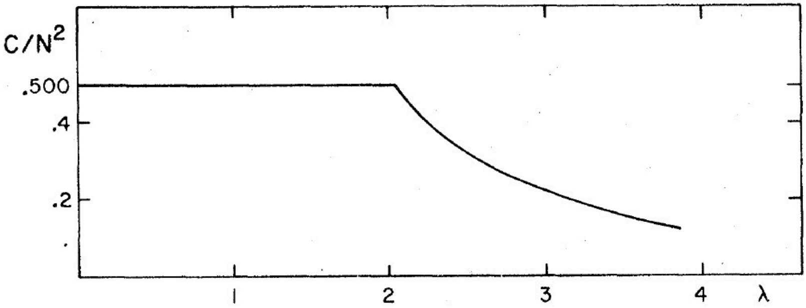


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

D. Gross, E. Witten, 1980

Gross-Witten: beta function at infinite N

- infinite N : $\quad \beta(\lambda) = \begin{cases} -2\lambda \log \lambda & \lambda \geq 2 \\ 2(\lambda - 4) \log \frac{4}{4-\lambda} & \lambda \leq 2 \end{cases}$

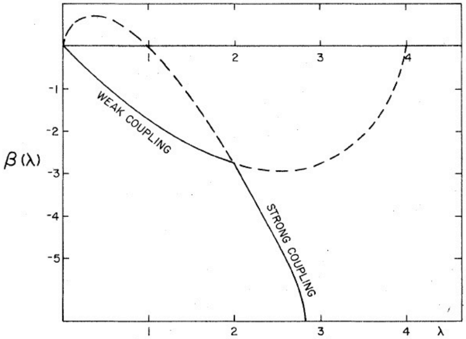


FIG. 1. The β function as a function of λ. The dashed lines are the (invalid) extrapolation of the weak- and strong-coupling results beyond the phase transition at $\lambda = 2$.

Wilson loop:

$$W(\lambda, N) = \langle \mathrm{tr}\, U \rangle := e^{-a^2 \Sigma}$$

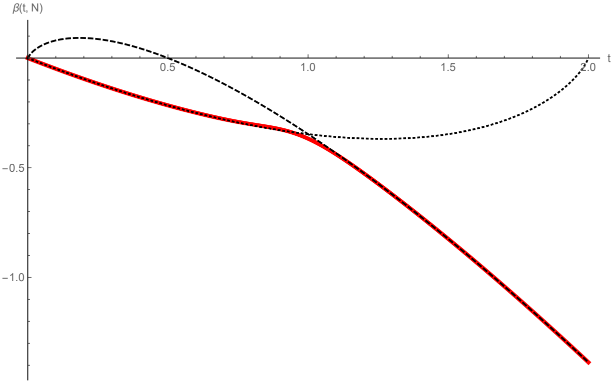
$$\beta(\lambda, N) := -\frac{\partial \lambda(a, N)}{\partial \ln \lambda}$$

- naive large N incorrectly predicts new fixed points at $\lambda = 1$ and $\lambda = 4$

D. Gross, E. Witten, 1980

Gross-Witten: beta function at large N

- for any N: $\beta(\lambda) = \frac{2}{\frac{d}{d\lambda}\log\log W_N(\lambda)}$ $\quad\rightarrow$ trans-series for β



- resurgent large N smoothly passes from weak-coupling to strong-coupling curve, developing a kink at $N = \infty$

Resurgence in Gross-Witten-Wadia Model

- random matrix theory/orthogonal polynomials result:
partition function = $N \times N$ Toeplitz determinant

$$Z(g^2, N) = \det \left(I_{j-k}(x) \right)_{j,k=1,\dots N} \qquad , \quad x \equiv \frac{2}{g^2}$$

- explicit, but not particularly efficient for $N \to \infty$

Gross-Witten-Wadia Phase Transition and Lee-Yang zeros

Lee-Yang: complex zeros of Z pinch the real axis at the phase transition point in the thermodynamic limit

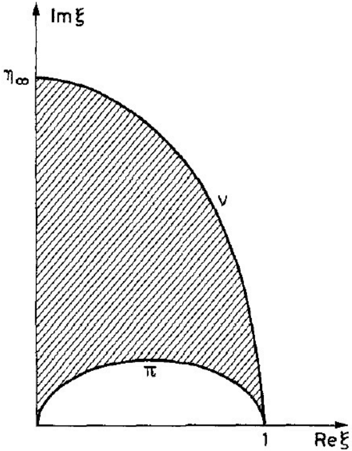


Fig. 1. The first quadrant of the conjectured domain U_∞ that is filled densely with zeros in the limit $N \to \infty$

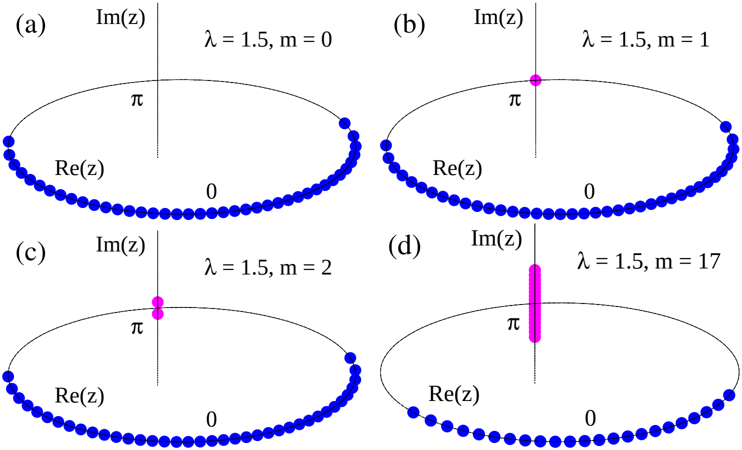
P. Buividovich, GD, S. Valgushev, 1512.09021

- ‘brute force’ numerical search for saddles
- in terms of eigenvalues e^{iz_j}:

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} dz_i \exp\left[-\frac{2N}{\lambda} \sum_i \cos(z_i) + \ln \prod_{i<j} \sin^2\left(\frac{z_i - z_j}{2} \right) \right]$$

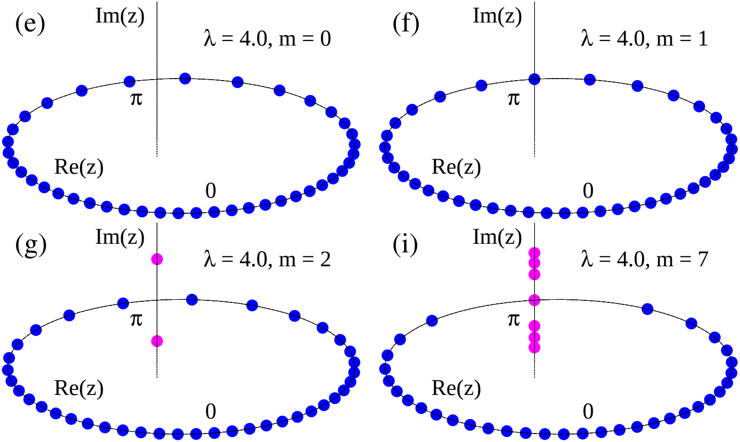
- saddle point approach: $\partial S / \partial z_i = 0$
- which saddles (real/complex?) govern large N behavior?
- how to see the “phase transition” at finite N?

Gross-Witten-Wadia Model: weak coupling: $\lambda < 2$



- "eigenvalue tunneling" of saddles into the complex plane
- number of complex eigenvalues: m = instanton number
- dominant non-perturbative saddle has $m = 1$

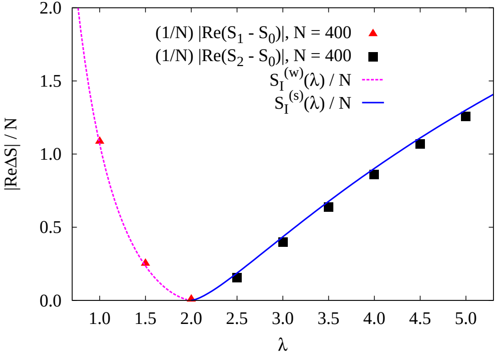
Gross-Witten-Wadia Model: strong coupling: $\lambda > 2$



- "eigenvalue tunneling" of saddles into the complex plane
- number of complex eigenvalues: m = instanton number
- dominant non-perturbative saddle has $m = 2$

Gross-Witten-Wadia Model: non-vacuum saddles

- weak coupling ($\lambda < 2$): $m = 1$ dominant
- strong coupling ($\lambda > 2$): $m = 2$ dominant



$$\lambda < 2: \quad S_I^{(weak)} = 4/\lambda\sqrt{1-\lambda/2} - \text{arccosh}\left((4-\lambda)/\lambda\right)$$

$$\lambda > 2: \quad S_I^{(strong)} = 2\,\text{arccosh}\left(\lambda/2\right) - 2\sqrt{1-4/\lambda^2}$$

- microscopic view of strong-coupling "instanton/saddle"

- random matrix theory/orthogonal polynomials result: partition function = $N \times N$ Toeplitz determinant

$$Z(g^2, N) = \det\left(I_{j-k}(x)\right)_{j,k=1,\ldots N} \qquad , \quad x \equiv \frac{2}{g^2}$$

- weak coupling: resurgent trans-series for $I_{j-k}(x)$
- strong coupling: convergent series for $I_{j-k}(x)$
- interesting transition between the two, esp. at large N

Exercise 3: the modified Bessel function has the large x asymptotic expansion:

$$I_j(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n(j)}{x^n} \pm i e^{ij\pi} \frac{e^{-x}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{\alpha_n(j)}{x^n}, \qquad \left|\arg(x) - \frac{\pi}{2}\right| < \pi$$

where the coefficients are

$$\alpha_j(n) = \frac{\cos(\pi j)}{\pi} \left(-\frac{1}{2}\right)^n \frac{\Gamma\left(n+\frac{1}{2}-j\right)\Gamma\left(n+\frac{1}{2}+j\right)}{\Gamma(n+1)}$$

(i) show that the large-order growth ($n \to \infty$) is

$$\alpha_n(j) \sim \frac{\cos(j\pi)}{\pi} \frac{(-1)^n (n-1)!}{2^n} \left(\alpha_0(j) - \frac{2\,\alpha_1(j)}{(n-1)} + \frac{2^2\,\alpha_2(j)}{(n-1)(n-2)} - \dots \right)$$

(ii) what is the significance of the $\cos(j\pi)$ prefactor?

Resurgence in Gross-Witten-Wadia Model

- partition function = $N \times N$ Toeplitz determinant

$$Z(g^2, N) = \det\left(I_{j-k}(x)\right)_{j,k=1,\ldots N} \quad , \quad x \equiv \frac{2}{g^2}$$

- weak-coupling resurgent trans-series: $N+1$ instantons

$$Z(x,N) \sim Z_0(x,N)\left[\sum_{n=0}^{\infty}\frac{a_n^{(0)}(N)}{x^n} + i\frac{(4x)^{N-1}}{\Gamma(N)}e^{-2x}\sum_{n=0}^{\infty}\frac{a_n^{(1)}(N)}{x^n} + \ldots + \frac{G(N+1)}{\prod_{i=0}^{N-1}\Gamma(N-i)}e^{-2Nx}\sum_{n=0}^{\infty}\frac{a_n^{(N)}(N)}{x^n}\right]$$

- but strong-coupling expansion is **convergent!**

$$Z(x,N) \sim e^{x^2/4}\left[1 - \left(\frac{(x/2)^{N+1}}{(N+1)!}\right)^2\left(1 - \frac{1}{2}\frac{(N+1)\,x^2}{(N+2)^2} + \ldots\right) + \ldots\right]$$

Resurgence in Gross-Witten-Wadia Model

Ahmed & GD: 1710.01812

- idea: map it to a Painlevé function (Painlevé III)

$$\Delta(x,N) \equiv \langle \det U \rangle = \frac{\det\left[I_{j-k+1}(x)\right]_{j,k=1,\dots,N}}{\det\left[I_{j-k}(x)\right]_{j,k=1,\dots,N}}$$

- for any N, $\Delta(x,N)$ satisfies a PIII-type equation:

$$\Delta'' + \frac{1}{x}\Delta' + \Delta\left(1-\Delta^2\right) + \frac{\Delta}{(1-\Delta^2)}\left[\left(\Delta'\right)^2 - \frac{N^2}{x^2}\right] = 0$$

$\Rightarrow$ generate trans-series solutions: weak- & strong-coupling

- N is a parameter ! $\Rightarrow$ large N limit by rescaling

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$\Rightarrow$ generate trans-series solutions: weak- & strong-coupling

- N is a parameter ! $\Rightarrow$ large N limit by rescaling

- direct relation to the partition function:

$$\Delta^2(x,N) = 1 - \frac{Z(x,N-1)\,Z(x,N+1)}{Z^2(x,N)}$$

$$Z(x,N) = \exp\left[\frac{1}{2}\int_0^x x\,dx\left(1-\Delta^2(x,N)\right)\left(1+\Delta(x,N-1)\Delta(x,N+1)\right)\right]$$

Resurgence in Gross-Witten-Wadia Model

- weak-coupling expansion is a divergent series:
  $\rightarrow$ trans-series non-perturbative completion
- strong-coupling expansion is a convergent series:
  but it still has a non-perturbative completion !

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$$\Rightarrow \Delta(x,N)\big]_{\rm strong} \approx \sigma\, J_N(x)$$

- strong-coupling expansion ($x \equiv \frac{2}{g^2}$) is clearly convergent, but only agrees with expansion of $J_N(x)$ to order x^{3N}. Why?

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- strong-coupling expansion ($x \equiv \frac{2}{g^2}$) is clearly convergent, but only agrees with expansion of $J_N(x)$ to order x^{3N}. Why?

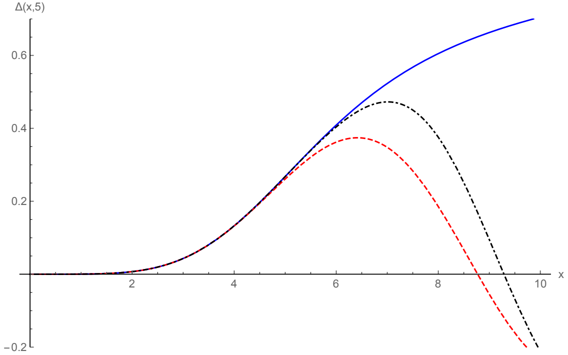
- full solution is a non-perturbative trans-series:

$$\Delta(x,N) = \sum_{k=1,3,5,\dots}^{\infty} \left(\sigma_{\text{strong}}\right)^k \Delta_{(k)}(x,N)$$

Resurgence in Gross-Witten-Wadia Model

- strong-coupling trans-series (convergent !!!):

$$\Delta(x, N) = \sum_{k=1,3,5,\ldots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(x, N)$$



blue: exact , red: $\Delta_{(1)} = J_5(x)$, black: includes $\Delta_{(3)}$

Resurgence in GWW: 't Hooft limit and phase transition

- rescaled PIII equation: $t \equiv N g^2/2 \equiv \frac{N}{x}$

$$t^2 \Delta'' + t\Delta' + \frac{N^2 \Delta}{t^2}\left(1 - \Delta^2\right) = \frac{\Delta}{1 - \Delta^2}\left(N^2 - t^2 \left(\Delta'\right)^2\right)$$

- GWW $N = \infty$ phase transition:

$$\Delta(t, N) \xrightarrow{N \to \infty} \begin{cases} 0 & , \quad t \geq 1 \quad \text{(strong coupling)} \\ \sqrt{1-t} & , \quad t \leq 1 \quad \text{(weak coupling)} \end{cases}$$

- large N:

$$\frac{\Delta}{t^2}\left(1 - \Delta^2\right) = \frac{\Delta}{1 - \Delta^2}$$

$$\Rightarrow \qquad \Delta = 0 \qquad \text{or} \qquad \Delta = \sqrt{1-t}$$

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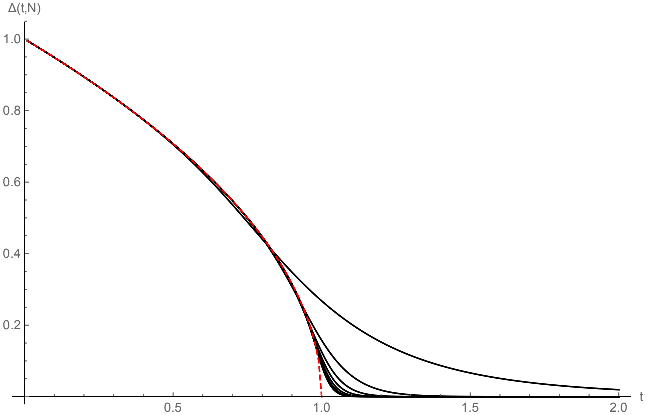

$t \equiv \frac{N}{x} \equiv \frac{N g^2}{2}$

black lines: increasing N

red dashed line: $\Delta = \sqrt{1-t}$

Resurgence in GWW: 't Hooft limit and phase transition

- full large N trans-series at weak-coupling:

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i}{2\sqrt{2\pi N}}\,\sigma_{\rm weak}\,\frac{t\,e^{-NS_{\rm weak}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

- large N weak-coupling action

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\text{arctanh}\left(\sqrt{1-t}\right)$$

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- large N weak-coupling action

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\mathrm{arctanh}\left(\sqrt{1-t}\right)$$

- large-order growth of perturbative coefficients ($\forall t < 1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\,\frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right.$$

- confirm (parametric!) resurgence relations, for all t:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{1}{N} + \dots$$

Resurgence in GWW: 't Hooft limit and phase transition

- large N transseries at strong-coupling: $\Delta(t,N) \approx \sigma J_N\left(\frac{N}{t}\right)$

$$\Delta(t,N) = \sum_{k=1,3,5,\dots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(t,N)$$

  - "Debye expansion" for Bessel function: $J_N(N/t)$

$$\Delta(t,N) \sim \frac{\sqrt{t}\, e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \frac{1}{4(t^2-1)} \left(\frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}}\right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \dots$$

- large N strong-coupling action: $S_{\text{st}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

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$$+\frac{1}{4(t^2-1)}\left(\frac{\sqrt{t} e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}}\right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \ldots$$

- large N strong-coupling action: $S_{\text{st}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

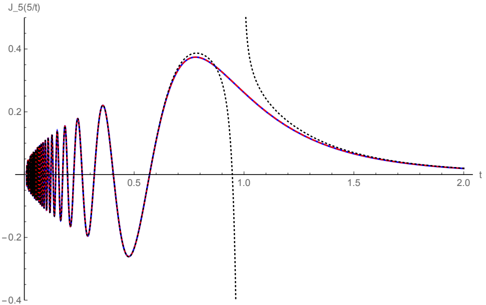
  - large-order/low-order (parametric) resurgence relations:

$$U_n(t) \sim \frac{(-1)^n (n-1)!}{2\pi (2S_{\text{strong}}(t))^n}\left(1 + U_1(t)\frac{(2S_{\text{strong}}(t))}{(n-1)} + U_2(t)\frac{(2S_{\text{strong}}(t))^2}{(n-1)(n-2)} + \right.$$

Resurgence in GWW: 't Hooft limit and phase transition

- Debye expansion has unphysical divergence at $t = 1$
- uniform asymptotic expansion:

$$J_N\left(\frac{N}{t}\right) \sim \left(\frac{4\left(\frac{3}{2}S_{\rm strong}(t)\right)^{2/3}}{1-1/t^2}\right)^{\frac{1}{4}} \frac{\text{Ai}\left(N^{\frac{2}{3}}\left(\frac{3}{2}S_{\rm strong}(t)\right)^{2/3}\right)}{N^{\frac{1}{3}}}$$



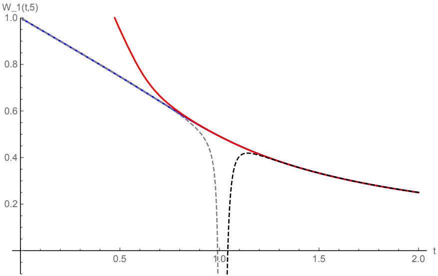
- nonlinear analogue of uniform WKB (coalescing saddles)

- Wilson loop: $\mathcal{W} \equiv \frac{1}{N} \frac{\partial \ln Z}{\partial x}$

$$\mathcal{W}(t, N) = \frac{1}{2t}\left(1 - \Delta^2(t, N)\right)\left(1 + \Delta(t, N-1)\Delta(t, N+1)\right)$$

- uniform large N approximation at strong-coupling:

$$\mathcal{W}(t, N)\Big|^{\text{strong}} \approx \frac{1}{2t}\left(1 - J_N^2(N/t)\right)\left(1 + J_{N-1}(N/t) J_{N+1}(N/t)\right)$$



blue: exact
red: uniform large N
dashed: usual large N

uniform resummation of instantons & fluctuations

Resurgence in GWW: 't Hooft limit and phase transition

- uniform asymptotic expansion:

$$\Delta_{\text{strong}}(N,t) \sim \left(\frac{4\left(\frac{3}{2}S_{\text{strong}}(t)\right)^{2/3}}{1-1/t^2} \right)^{\frac{1}{4}} \frac{\text{Ai}\left(N^{\frac{2}{3}} \left(\frac{3}{2}S_{\text{strong}}(t)\right)^{2/3} \right)}{N^{\frac{1}{3}}}$$

- physical meaning of "uniform large-N instantons" ?
- nonlinear analogue of "uniform WKB"
- technically: coalescence of two saddles $\longrightarrow$ "bion"
- expect similar phenomena in QFT

Resurgence in GWW: double-scaling limit = Painlevé II

- reduction cascade of Painlevé equations
- "zoom in" on vicinity of phase transition:

$$\kappa \equiv N^{2/3}(t-1) \quad ; \quad \Delta(t,N) = \frac{t^{1/3}}{N^{1/3}} y(\kappa)$$

- $N \to \infty$ with κ fixed:

$$\Delta \quad \text{PIII equation} \quad \longrightarrow \quad \frac{d^2 y}{d\kappa^2} = 2\, y^3(\kappa) + 2\kappa\, y(\kappa) \quad \text{(PII)}$$

- e.g. on strong-coupling side:

$$\lim_{N\to\infty} J_N(N - N^{1/3}\kappa) = \left(\frac{2}{N}\right)^{1/3} \text{Ai}\left(2^{1/3}\kappa\right)$$

- integral equation form of PII:

$$y(\chi) = \sigma\, \text{Ai}(\chi) + 2\pi \int_\chi^\infty \left[\text{Ai}(\chi)\text{Bi}(\chi') - \text{Ai}(\chi')\text{Bi}(\chi)\right] y^3(\chi')\, d\chi'$$

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- integral equation form of PII:

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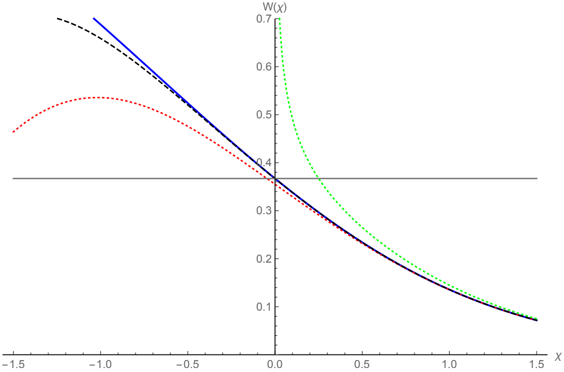

iterate $\longrightarrow$ resummed trans-series instanton expansion

blue: exact

red: leading uniform large N

dashed: sub-leading uniform large N

green dashed: usual large N

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series
- trans-series 'encode' analytic continuation information
- expansions about different saddles are intimately related
- there is extra un-tapped 'magic' in perturbation theory

- QM, matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- $\mathcal{N}=2$ and $\mathcal{N}=2^*$ SUSY gauge theory
- applications to sign problem and non-equil. path integrals

- promising progress & many fascinating open problems

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