Asian Winter School 2019: Integrability Problem Sets

Here are some exercises related to my first and second lectures: The first set of problems is about the S-matrix of the O(N) model while the second is about the CP^{N-1} model.

1 O(N) S-matrix

Yang-Baxter

As explained in the lecture, by imposing two different Yang-Baxter equations, one can fix the form of the S-matrix of the O(N) model to be

$$S_{ij}^{kl}(\theta) = \sigma(\theta)\left[\delta_i^k\delta_j^l + \frac{1}{c\theta}\delta_i^l\delta_j^k + \frac{1}{d-c\theta}\delta_{ij}\delta^{kl}\right]. \tag{1}$$

The goal of this exercise is to determine the unfixed coefficients c, d and $\sigma(\theta)$ by imposing the combination of the Yang-Baxter equation, crossing symmetry and the unitarity condition.

1. By considering an appropriate 3-to-3 scattering process and imposing the Yang-Baxter equation, show that $d = -\frac{N-2}{2}$.
   (*Hint*: In order to obtain a number that depends on N, one has to consider a scattering process which contains a closed loop of the O(N) index.)

Now, after fixing d, one can impose the crossing symmetry and determine c as explained in the lecture. After doing so, the S-matrix is given by

$$S_{ij}^{kl}(\theta) = \sigma(\theta)\left[\delta_i^k\delta_j^l - \frac{2\pi i\Delta}{\theta}\delta_i^l\delta_j^k + \frac{2\pi i\Delta}{\theta - i\pi}\delta_{ij}\delta^{kl}\right], \tag{2}$$

with $\Delta = 1/(N-2)$. The crossing symmetry also gives a constraint on $\sigma(\theta)$:

$$\sigma(\theta) = \sigma(i\pi - \theta). \tag{3}$$

Unitarity

The next step is to impose the unitarity and solve for $\sigma(\theta)$.

4. Impose the unitarity condition[1]

   $$S(-\theta)S(\theta) = \mathbf{1}, \tag{4}$$

   and show that it gives the following condition on σ:

   $$\sigma(-\theta)\sigma(\theta) = \frac{\theta^2}{\theta^2 + 4\pi^2\Delta^2}. \tag{5}$$

   (*Hint*: It is simpler to work with the pictorial notation introduced in the lecture rather than working with indices.)

[1] One should interpret $S(-\theta)S(\theta)$ as the matrix multiplication and $\mathbf{1}$ is the unit matrix.

Solving the equations for $\sigma(\theta)$

Now we are going to solve the equations for $\sigma(\theta)$, (3) and (5). Before doing this, it is useful to introduce the S-matrix projected onto the symmetric-traceless representation.

5. Consider the S-matrix in the symmetric-traceless sector $s(\theta)$, defined by

$$s(\theta) \equiv \sigma(\theta)\frac{\theta - 2\pi i\Delta}{\theta}\,, \tag{6}$$

and show that $s(\theta)$ satisfies

$$s(\theta)s(-\theta) = 1\,, \tag{7}$$

$$s(\theta^+)s(\theta^-) = \frac{(\theta^+ - 2\pi i\Delta)\theta^-}{(\theta^- + 2\pi i\Delta)\theta^+} \tag{8}$$

Here the notation $\theta^\pm$ means $\theta^\pm \equiv \theta \pm i\pi/2$.

6. (Optional) Physically, $s(\theta)$ denotes the S-matrix projected onto the symmetric-traceless sector. Construct the projection operator (which acts on the indices of S_{ij}^{kl}), act on (2) and derive (6).

We now try to solve (7) and (8).

7. Take the logarithm of the equation (8), differentiate with respect to θ and Fourier-transform both sides. Once you do it, you will find that the equation can be easily solved in the Fourier space. Find a solution and transform it back to the θ space.

The solution you will get by doing the exercise 5 is not unique: From any solution to (7) and (8), one can always construct another solution by multiplying to it a factor which satisfies

$$\begin{aligned} f(\theta)f(-\theta) &= 1\,,\\ f(\theta^+)f(\theta^-) &= 1\,. \end{aligned} \tag{9}$$

8. Check that the following product satisfies the equations for $f(\theta)$:

$$f(\theta) = \prod_{k=1}^{M}\frac{\sinh\theta - i\cos\alpha_k}{\sinh\theta + i\cos\alpha_k}\,. \tag{10}$$

The factors on the right hand side of (10) are called the CDD (Castillejo-Dalitz-Dyson) factors. By multiplying them, one can change the analyticity of the S-matrix (namely one can introduce or remove poles in the S-matrix) without affecting the unitarity and the crossing.

9. Analyze the analytic properties of $s(\theta)$ you obtained and try to remove the poles in the physical strip ($\mathrm{Im}\,\theta \in [0,\pi]$) by multiplying the minimal number of the CDD factor.

In the end, you would get

$$s(\theta) = -\frac{\Gamma\left(1+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}-\frac{i\theta}{2\pi}\right)\Gamma\left(\Delta-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\Delta+\frac{i\theta}{2\pi}\right)}{\Gamma\left(1-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\frac{i\theta}{2\pi}\right)\Gamma\left(\Delta+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\Delta-\frac{i\theta}{2\pi}\right)}\,. \tag{11}$$

This is the S-matrix obtained first by (Zamolodchikov)^2 [1].

2 Non-Integrability of CP^{N-1} Model

The CP^{N-1} model is a sigma model whose target space is a N-component complex vector satisfying

$$\sum_{i=1}^{N} z_i z_i^* = R^2\,, \tag{12}$$

where R is the radius of the target space. Furthermore, the vector $(z_1,\ldots,z_N)$ are subject to the following $U(1)$ identification,

$$(z_1,\ldots,z_N) \sim e^{i\alpha}(z_1,\ldots,z_N)\,. \tag{13}$$

In other words, this $U(1)$ symmetry is a redundency of the description, namely the gauge symmetry.

The easiest way to incorporate the $U(1)$ identification is to introduce a gauge field A_μ and uplift the ordinary derivative to the covariant derivative:

$$\begin{aligned} \partial_\mu \vec{z} \quad &\to \quad D_\mu \vec{z} \equiv (\partial_\mu + iA_\mu)\,\vec{z}\,, \\ \partial_\mu \vec{z^*} \quad &\to \quad D_\mu \vec{z^*} \equiv (\partial_\mu - iA_\mu)\,\vec{z^*}\,. \end{aligned} \tag{14}$$

Using these covariant derivatives, the action can be expressed as

$$S = \frac{1}{2}\int d^2x\,\left(D_\mu \vec{z^*}\cdot D^\mu \vec{z}\right) + \sigma(\vec{z^*}\cdot\vec{z} - R^2)\,. \tag{15}$$

Here again σ is the Lagrange multiplier which implements the constraint.

10. Using the equation of motion, show that A_μ can be expressed by $\vec{z}$'s as

$$A_\mu = \frac{i}{2R^2}\left(\vec{z^*}\cdot\partial_\mu\vec{z} - \partial_\mu\vec{z^*}\cdot\vec{z}\right)\,. \tag{16}$$

11. Derive the equation of motion for $\vec{z}$ and show that it simplifies in the large R limit to

$$\partial_+\partial_-\vec{z} = 0\,. \tag{17}$$

As discussed in the lecture, for the purpose of counting the operators, one can take the $R\to\infty$ limit and simplify the analysis. Using this idea, let us now consider whether the conservation law

$$\partial_-(T_{++}^2) = 0 \tag{18}$$

with

$$T_{++} = D_+\vec{z}^* \cdot \vec{z}\,, \tag{19}$$

is violated or not at the quantum level. When doing so, we also need to take into account the discrete charge conjugation symmetry which maps

$$\vec{z} \to \vec{z}^* \tag{20}$$

and vice versa.

12. Write down all possible anomaly terms (consistent with the symmetry of the theory) that can appear on the right hand side of (18)[2]. You will find 8 operators if you do not impose the charge conjugation symmetry (20) while you will find 4 operators if you impose the charge conjugation.

13. List all the derivative operators that can appear on the right hand side of (18). If you will get 3 operators after imposing the charge conjugation and eliminating the operators that actually vanish.

This computation shows that there are more anomaly terms than the derivative terms, and therefore the conservation law can in general be violated. This is in fact what was found in the large N analysis of the CP^{N-1} model, and it is known that the CP^{N-1} model is not quantum integrable [2].

References

[1] A. B. Zamolodchikov and A. B. Zamolodchikov, "Relativistic Factorized S Matrix in Two-Dimensions Having O(N) Isotopic Symmetry," Nucl. Phys. B **133**, 525 (1978) [JETP Lett. **26**, 457 (1977)].

[2] Y. Y. Goldschmidt and E. Witten, "Conservation Laws in Some Two-dimensional Models," Phys. Lett. **91B**, 392 (1980).

[2] As in the O(N) model, the constraint $\vec{z}^* \cdot \vec{z}$ allows us to eliminate $\vec{z}$'s without any derivatives.