Topics on Integrability and N=4 SYM

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Topics on Integrability & $\mathcal{N}=4$ SYM

I. Intro to Integrability ($O(N)$ model)

II. From $\mathcal{N}=4$ SYM to Spin Chain

III. ODE/IM correspondence

Yesterday

$\mathcal{N}=4$ SYM $\longrightarrow$ $H \propto \sum_i \left(\| - \times \right)$

$Z \to |\uparrow\rangle$

$X \to |\downarrow\rangle$

$$= \sum_i \left(\frac{1}{4} - \vec{S}_i \cdot \vec{S}_{i+1} \right)$$

① Ground State

$|\uparrow \cdots \uparrow\rangle \quad (= \mathrm{Tr}\, Z^L) \qquad E = 0$

② 1 magnon state

magnon

$$|p\rangle = \sum_n e^{ipn} |\uparrow \cdots \underset{\sim}{\overset{n}{\downarrow}} \cdots \uparrow\rangle$$

- $E(p) = 4g^2 \times (1 - \cos p)$
- $e^{ipL} = 1$

Reference (I & II) : S.K. arXiv:1710.03853

S.K, Integrability Lecture III

Bootstrap School 2018

@ Caltech

(iii) 2 magnon state.

$$|n, m\rangle = |\uparrow \cdots \underset{n}{\downarrow} \cdots \underset{m}{\downarrow} \cdots \uparrow\rangle \qquad (n<m)$$

If $n \ll m$, $\mapsto$ $\hat{H}$ acts separately on each $\downarrow$'s

$$|p_1, p_2\rangle = \sum_{n<m} \underbrace{\left(e^{ip_1 n + ip_2 m} + (\ast)\, e^{ip_2 n + ip_1 m} \right)}_{\psi(n,m)} |n, m\rangle$$

$$\rightsquigarrow E(p_1, p_2) = E(p_1) + E(p_2)$$

$$E(p_1) \to 4g^2(1-\cos p_1)$$

$$\hat{H} |p_1, p_2\rangle = E(p_1, p_2) \ |p_1, p_2\rangle$$

$$E(p_1, p_2) = E(p_1) + E(p_2)$$

Look @ Coeff. of $|\uparrow \cdots \underset{n}{\downarrow} \underset{n+1}{\downarrow} \cdots \uparrow\rangle$

(r.h.s) $\rightarrow$ $E(p_1, p_2) \times \psi(n, n+1)$

$\|$

(l.h.s) $\rightarrow$ $3 \times \psi(n, n+1)$

$$X \rightsquigarrow \begin{cases} -\psi(n, n+1) - \psi(n-1, n+1) \\ -\psi(n, n+2) \end{cases}$$

$$\circledast = - \frac{1 + e^{i(p_1+p_2)} - 2e^{ip_2}}{1 + e^{i(p_1-p_2)} - 2e^{ip_1}} = S(p_1, p_2)$$

$$\psi(n, m) = e^{ip_1 n + ip_2 m} + \circledast\, e^{ip_2 n + ip_1 m}$$

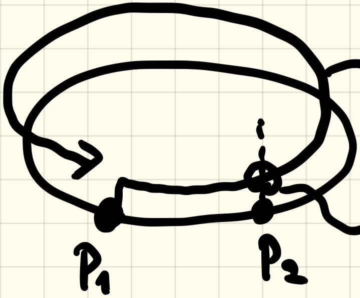
$\hookrightarrow n<m$

Periodicity

$0<n<m<L$ $\quad$ $m<n+L$

$$\psi(n, m) = \psi(m, n+L)$$

$$\rightsquigarrow \begin{cases} e^{ip_1 L}\, S(p_1, p_2) = 1 \\ e^{ip_2 L}\, S(p_2, p_1) = 1 \end{cases}$$



$e^{ip_1 L}$

$S(p_1, p_2)$ ← S-matrix

$$e^{ip_1 L} S(p_1, p_2) = 1$$

← Bethe eq.

Multi-magnon

$H \rightsquigarrow E(p_1 \cdots p_n) = \sum_i E(p_i)$

$\rightsquigarrow e^{ip_i L} \prod_{j \neq i} S(p_i, p_j) = 1$

Integrability

II. 2. "Bootstrapping" $\mathcal{N}=4$ SYM spin chain

- Higher loops $\to$ Γ $\to$ H $\to$ diagonalize

  $\leftarrow$ Hard.

- Idea : Use Symmetry

  (Superconformal + "Large gauge")

  to constrain the dynamics.

1. Symmetry.

① Classification of particles using Poincaré

$$k^\mu \xrightarrow{\text{Step 1.}} \quad k^\mu = (M, 0, 0, 0) \xrightarrow{\text{Step 2.}}$$

Step 1. Go to the rest frame.

Step 2. Understand the "Little Group" (L.G.)

Step 3. Classify internal motion (spin) using rep. of the L.G.

Step 1. Rest frame.

$$\langle \mathrm{Tr}[Z^L](0) \qquad \mathrm{Tr}[\bar{Z}^L](\infty) \rangle$$

$Z = \phi_1 + i\phi_2$ $\quad$ $\bar{Z} = \phi_1 - i\phi_2$

R-charge

$\frac{1}{2}$ BPS op. $\quad \Delta = L = J \quad \begin{cases} J \cdot Z \to +Z \\ J \cdot \bar{Z} \to -\bar{Z} \end{cases}$

Step 2. Little Group.

Space Time

$$SO(4) = SU(2) \times SU(2)$$

R-symmetry

$$\Phi_1 \; \Phi_2 \; \underbrace{\Phi_3 \; \Phi_4 \; \Phi_5 \; \Phi_6}$$

$\to SO(4) \simeq SU(2) \times SU(2)$

Fermionic

8 susy Q's + 8 superconformal S's

$$\rightsquigarrow \boxed{\begin{array}{c} PSU(2|2)_L \times PSU(2|2)_R \\ + \\ D-J \longleftarrow \text{central charge} \end{array}}$$

$= \Delta$

Step 3. Internal Motion

$$\mathrm{Tr}[Z \cdots Z] \longrightarrow \mathrm{Tr}[Z \cdots X Z Z \psi \cdots Z D Z]$$

$\phi_3 + i\phi_4$

without spin.

insertion

Covariant derivative.

Insertion : $M^{A\dot{A}}$ ← fund. $psu(2|2)_R$.

fund. $psu(2|2)_L$

$A = (1\,2|\hat{1}\,\hat{2})$

$\dot{A}$ 〃

$$= \chi^A \cdot \chi^{\dot{A}}$$: Bifundamental.

$$X^A = (\varphi^1, \varphi^2, \psi^1, \psi^2) \Longleftrightarrow A = \{1, 2\}\{\hat{1}, \hat{2}\}$$

$$\begin{cases} X = \varphi^1 \varphi^{\dot{1}} & \bar{X} = \varphi^2 \varphi^{\dot{2}} \\ D_\mu \rightsquigarrow (D)^{\alpha\dot{\beta}} = \psi^\alpha \psi^{\dot{\beta}} \\ \Psi \sim \varphi^a \psi^{\dot{\beta}}, \text{ or } \psi^\alpha \varphi^{\dot{b}} \end{cases}$$

→ Try to constrain dynamics of Spin Chain using $PSU(2|2)^2$.

But, it's not very powerful.

$\longrightarrow$ No coupling constant.

Resolution : "Large gauge transf. / symmetry"

2. Large Gauge Symmetry :-

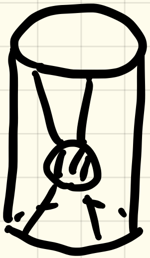
Key Idea. Send $L \to \infty$

L : length of op.

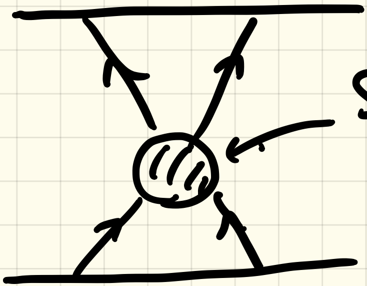
$$\mathrm{Tr}\,[\underbrace{Z \cdots Z}_{L}] \to \cdots Z \cdots\cdots Z \cdots\cdots$$

Advantages :

I. S-matrix description.



$\uparrow H$: messy $\longrightarrow$



S-matrix

2. Large Gauge Symmetry.

- Gaug Transf. which doesn't die off @ infinity

  $\rightsquigarrow$ Behaves like global symmetry.

① SUSY tranf.

fermion

$$\delta \Phi = i \bar{\epsilon} \underbrace{\Gamma}_{\text{Gamma}} \psi$$

bosons

$$\delta \psi = \cdots + \frac{i}{2} g_{YM} [\Phi_I, \Phi_J] \times \Gamma^{IJ} \epsilon$$

$$\delta^2 \Psi \sim g_{YM} [\underset{\cdots}{\Phi}^{\delta}, \underline{\Psi}] \, \dot{\cdot}$$

field-dependent gauge transf.

For $Q \in PSU(2|2)^2$

$$\underset{\cdots}{Q}^2 \cdot M^{A\dot{A}} \sim g [Z, M] \left(=: \underline{\hat{P}} \cdot M \right)$$

④ Action of $\hat{P}$

- $\hat{P} \, \mathrm{Tr}[\cdots] = \mathrm{Tr}[[Z, \cdots]] = 0$

$$\hat{p}\cdot\left(\sum_n e^{ipn}\,|Z\cdots Z\,\underset{n}{M}\,Z\cdots Z\cdots\rangle\right)$$

$$= g\sum_n e^{ipn}\,|\,[Z,\ Z\cdots Z\,M\,Z\cdots Z\cdots]\rangle\ )$$

$$= g\sum_n e^{ipn}\left(|Z\cdots Z\,Z\,\overset{(n+1)\text{-th}}{M}\,Z\cdots Z\cdots\rangle - |Z\cdots Z\,\underset{n\text{-th}}{M}\,Z\,Z\cdots Z\rangle\right)$$

$$= g\,(\underbrace{e^{-ip}-1}_{\text{Discrete Translation}})\left(\sum_n e^{ipn}\,|Z\cdots Z\,M\,Z\cdots Z\rangle\right)$$

2-loop analysis

$$(S)^2 M \sim g\,[Z^{-1}, M] \quad (=: \hat{K} M)$$

$$\boxed{\Rightarrow \quad PSU(2|2)^2 + D-J + \hat{P} + \hat{K} \cdots}$$

gauge transf.

central charges.

[Beisert, 2006]

- $\{Q^{\alpha}{}_{a}, Q^{\beta}{}_{b}\} = \epsilon^{\alpha\beta}\,\epsilon_{ab}\,\hat{P}$ Transl. of spin chain
- $\{S^{a}{}_{\alpha}, S^{b}{}_{\beta}\} = \epsilon^{ab}\,\epsilon_{\alpha\beta}\,\hat{K}$
- $\{Q^{\alpha}{}_{a}, S^{b}{}_{\beta}\} = \delta^{\alpha}_{\beta}\,R^{a}{}_{b} + \delta^{b}_{a}\,L^{\alpha}{}_{\beta} + \delta^{b}_{a}\,\delta^{\alpha}_{\beta}\,\hat{C}$

R-sym.

Lorentz

Energy of spin chain

$\Delta = D - J$

- Unifies 4d sym & 2d sym.
- Looks like discretized extended 2d SUSY

$\leftarrow$ [D'Adda, Kawamoto, Saito]

$$\begin{cases} Q^{\alpha}{}_{a} |\phi^{b}\rangle = \mathbb{a}\, \delta_{a}^{b} |\psi^{\alpha}\rangle \\ Q^{\alpha}{}_{a} |\psi^{\beta}\rangle = \mathbb{b}\, \epsilon^{\alpha\beta} \epsilon_{ab} |\phi^{b}\rangle \\ S^{a}{}_{\alpha} |\phi^{b}\rangle = \mathbb{c}\, \epsilon^{ab} \epsilon_{\alpha\beta} |\psi^{\beta}\rangle \\ S^{a}{}_{\alpha} |\psi^{\beta}\rangle = \mathbb{d}\, \delta_{\alpha}^{\beta} |\phi^{a}\rangle \end{cases}$$

$$\begin{cases} \hat{C} |\chi\rangle = \frac{1}{2}(\mathbb{a}\mathbb{d} + \mathbb{b}\mathbb{c}) |\chi\rangle \\ \hat{P} |\chi\rangle = \mathbb{a}\mathbb{b} |Z\chi\rangle \\ \hat{K} |\chi\rangle = \mathbb{c}\mathbb{d} |Z^{-1}\chi\rangle \end{cases}$$

- Closure of algebra.

$\rightarrow \quad \mathfrak{a}\mathfrak{d} - \mathfrak{b}\mathfrak{c} = 1$

- Unitarity of rep.

$\rightsquigarrow$ Another condition.

$\rightarrow 4 - 2 = 2$

p magnon

g coupling

Impose.

$$\hat{P} \sim g\left(e^{-ip} - 1\right) = \mathfrak{a}\mathfrak{b}$$

$\rightsquigarrow$ Nice parametrization of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$

$$\hat{C} = \sqrt{1 + 16 g^2 \sin^2 \frac{p}{2}}.$$

Energy of Spin Chain

D-J

$$\Big\downarrow \; g \ll 1.$$

$$1 + 8 g^2 \underbrace{\sin^2 \frac{p}{2}}_{= \frac{1}{2}(1-\cos p)}$$

$E_{\mathrm{Heisenberg}}(p)$ $=$ $\frac{1}{2}(1-\cos p)$

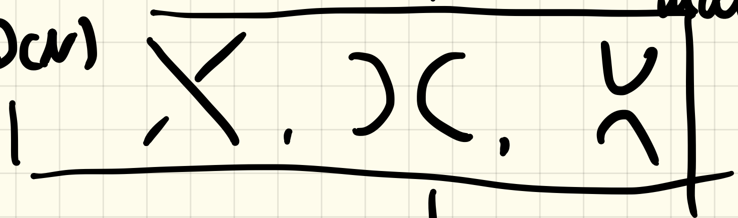
$$\rightsquigarrow 1 + 4 g^2 (1 - \cos p)$$

We can fix $S_{2\to2}$ by imposing

$$PSU(2|2)^2 + \text{D-J} + \hat{P} + \hat{K}$$

$$\underline{S} = \sigma \, \underbrace{\$_L \otimes \$_R}_{\text{matrix}} \quad \leftarrow \text{[Beisert 2006]}$$

O(N)



Crossing sym.
+
Unitary

This S satisfies

Y.B. !

$\leftarrow$ $\mathcal{N}=4$ SYM @ Large N

support

is integrable

- Same strategy works for 3pt
[Basso, S.K, Vieira

Les houches Lecture Note. 2015]

So far

$O(N) \leftarrow$ Yang-Baxter, Unitarity, Crossing

$\mathcal{N}=4$ SYM $\leftarrow$ Superconformal + "Large gauge" sym

$\rightsquigarrow$ "Bootstrap"