Topics on Integrability and N=4 SYM

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Topics on Integrability & $\mathcal{N}=4$ SYM

I. Intro to Integrability ($O(N)$ model)

II. From $\mathcal{N}=4$ SYM to Spin Chain

III. ODE/IM correspondence

I. Intro to Integrability.

1. What is integrability?

Classical (Quantum) Mechanis

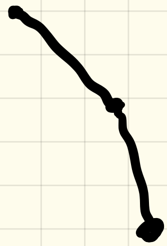
$\sim$ # of conserved charges

= # of d.o.f. (Liouville integrability)

e.g. $H = \sum_k \frac{P_k^2}{2m} + \frac{m}{2} \omega_k^2 x_k^2$

$\rightsquigarrow H_k = \frac{P_k^2}{2m} + \frac{m}{2} \omega_k^2 x_k^2$

e.g. double pendulum



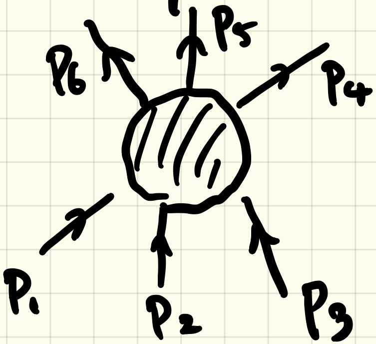
- QFT ?

→ ∞ many d.o.f.

→ ∞ many conserved charges ?

⇢ abstract ... ? physical consequence

◎ Physical consequence



$3 \to 3$ scattering.

in non-relativistic QFT

Momentum : $\sum_{i=1}^{3} p_i = \sum_{j=4}^{6} p_j$

Energy : $\sum_{i=1}^{3} p_i^2 = \sum_{j=4}^{6} p_j^2$

Suppose $\exists\, Q_n$ ($\forall n$: integer)

$$Q_n |p_1, \cdots, p_m\rangle = \sum_{i=1}^{m} p_i^n |p_1, \cdots, p_m\rangle$$

1. $\underline{\{P_1, P_2, P_3\} = \{P_4, P_5, P_6\}}$.

- Consider a wave packet

$$|p_k\rangle \longrightarrow \int_{-\infty}^{\infty} dp \; e^{-c(p-p_k)^2} \; \underline{|p\rangle}$$

$$\Rightarrow \quad \psi(x,t) = \int_{-\infty}^{\infty} dp \; e^{-c(p-p_k)^2} \times e^{\underbrace{-\frac{i\hbar}{2m}p^2 t + ipx}_{\equiv\, f(p)}}$$

peak of the wave packet:

$$\partial_p f(p)\Big|_{p=p_k} = 0$$

$$\Rightarrow \quad x - \underbrace{\frac{\hbar p_k}{m}}_{\to\, v_k} t = 0.$$

$$p^n$$

wave-packet

• Now, act $e^{i\alpha_n Q_n}$: $|p\rangle \to$?

$$f(p) \longrightarrow f(p) + i\alpha_n p^n$$

$\Rightarrow$ peak :

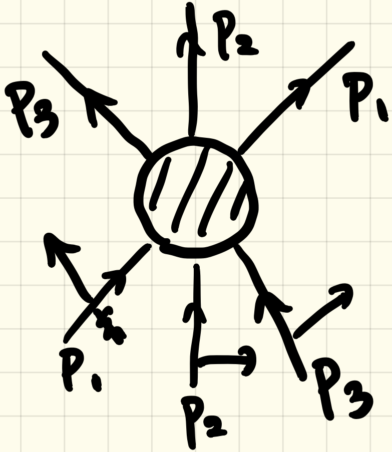
$$0 = x - \frac{\hbar p_k}{m} t + n \alpha_n p_k^{n-1}$$

$n=1$: Momentum $\rightarrow$ Shift in x direction

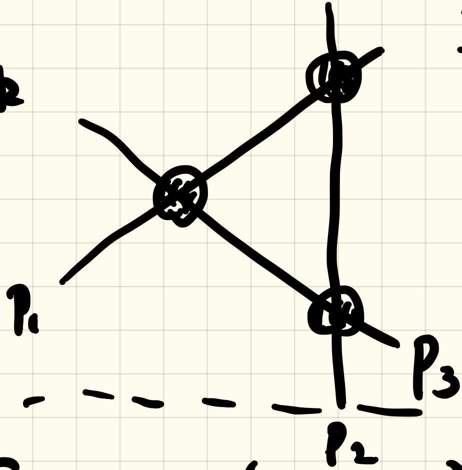
$n=2$: Energy. $\rightarrow$ Shift in t

$n>2$: Shift depends on p_k

1+1d

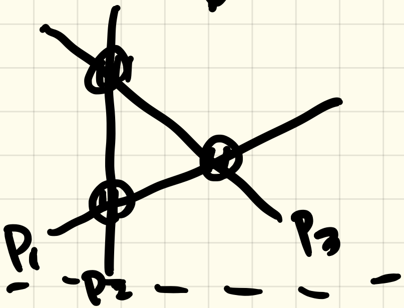


$$\xrightarrow{e^{i\sum_k \alpha_k Q_k}}$$



$$S_{3\to 3} = (S_{2\to 2})^3$$

$$\Big\downarrow\, e^{i\sum_k \alpha'_k Q_k}$$



Yang-Baxter eq.

$S^{kl}_{ij} \sim n^4$, Y.B. $\sim n^6$

2. O(N) non linear σ-model (1+1d)

Sigma model w/. Target space S^{N-1}

$\rightsquigarrow \vec{n}(z, \bar{z})$ $\qquad \vec{n} \cdot \vec{n} = 1$

1+1d coordinates

Goal: Determine $2 \to 2$ S-matrix of O(N) model

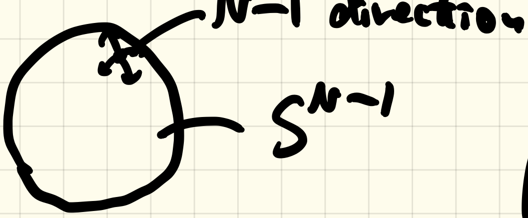
$\leftarrow$ "Bootstrap"

$$Z = \int \mathcal{D}\vec{n}\,\mathcal{D}\sigma \; e^{-\underbrace{\frac{1}{\kappa}}_{\text{coupling}} \int d^2z\, \mathcal{L}}$$

$$\mathcal{L} = \partial\vec{n}\cdot\partial\vec{n} + \sigma(\vec{n}\cdot\vec{n} - 1)$$

Input 0: Classically $N-1$ massles particle

Wrong



$\rightsquigarrow$ Wrong

$$\left(SO(N) \to SO(N-1) \right)$$

Input 1: Large N.

$$Z \xrightarrow[\text{out } \vec{n}]{\text{integrate}} \int \mathcal{D}\sigma \; e^{-S_{\text{eff}}[\sigma]}$$

$$S_{\text{eff}}[\sigma] = -\int d^2z \, \frac{\sigma}{K} + N \underbrace{\log\det}_{\text{Tr}\log}[\partial^2 + \sigma]$$

$$\frac{\sigma}{K} \; " \; \frac{N}{\lambda}$$

__Large N limit__

$N \to \infty$ $\quad$ $K \cdot N$: fixed

$K \to 0$ $\quad$ " λ 't Hooft coupling

Saddle pnt

$$\frac{1}{\lambda} = \mathrm{Tr}\left[\frac{1}{\partial^2 + \sigma_*}\right] = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+\sigma_*}$$

σ_*: s.p. value.

$$= \frac{1}{2\pi} \log\left[\frac{\Lambda}{\sigma_*} + 1\right]$$

Λ: UV cut off

$$\Rightarrow \quad \sigma_* = \Lambda \times \underbrace{e^{-\frac{2\pi}{\lambda}}}_{}$$

$\rightarrow$ nonperturbative

$\sigma_* \neq 0 \rightsquigarrow \vec{n}$: massive

$\uparrow$ $\Rightarrow N$ massive particles

$\sigma_* (\vec{n}\cdot\vec{n} - 1)$

Input 1 : At large N, N massive particles.

(Assume that this is true @ finite N)

Input 2 : $\exists\ \infty$ many conserved charges

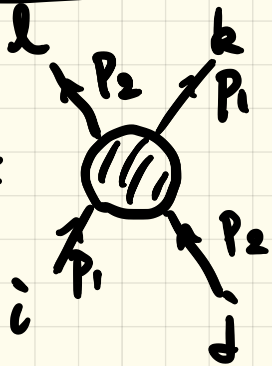
$\to$ tomorrow.

$\to$ Yang - Baxter eq.

3. Integrable Bootstrap for O(N) model

Kinematics

$S_{ij}^{kl}(s) =$

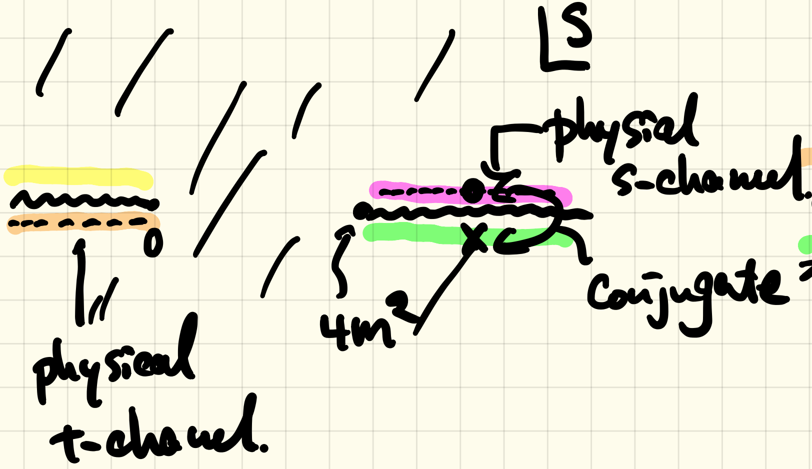


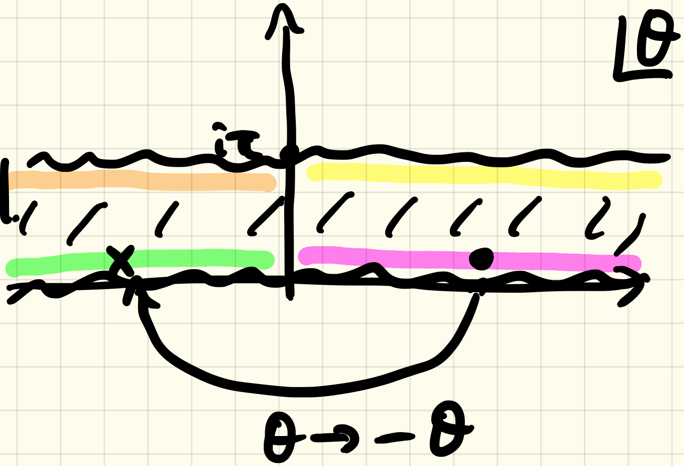
$i, j, k, l = 1, \cdots, N$

$s = (p_1+p_2)^2 \qquad p_i^2 = m^2$

$t = (p_1-p_2)^2$

$s + t = 4m^2$





$$P_{1,2} = m \sinh \theta_{1,2}$$

$$E_{1,2} = m \cosh \theta_{1,2}.$$

$$\Rightarrow \quad s = 4m^2 \cosh^2 \frac{\theta_1 - \theta_2}{2}$$

$$S(\theta_1 - \theta_2)$$

- Axioms

* Unitarity

$$\mathbb{S}(\theta)\cdot\mathbb{S}(-\theta) = \mathbb{1}$$

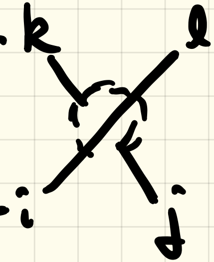
matrices

* Crossing Sym

$$S_{i\hat{j}}^{kl}(\theta) = S_{il}^{k\hat{j}}(i\pi-\theta)$$

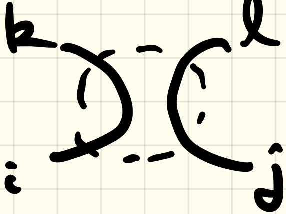
* Yang-Baxter eq.

$$S_{ij}^{kl}(\theta) = \underline{\sigma(\theta)} \cdot \Big[$$



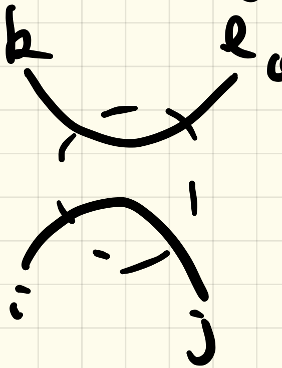
$\sim \delta_i^l \, \delta_j^k$

$$+ \ \underline{a(\theta)}$$



$\delta_i^k \, \delta_j^l$

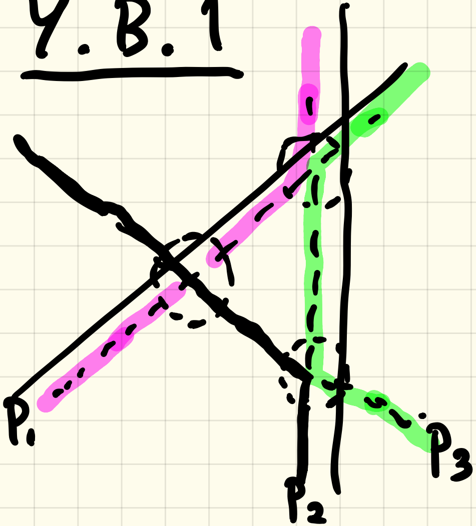
$$+ \ \underline{b(\theta)}$$



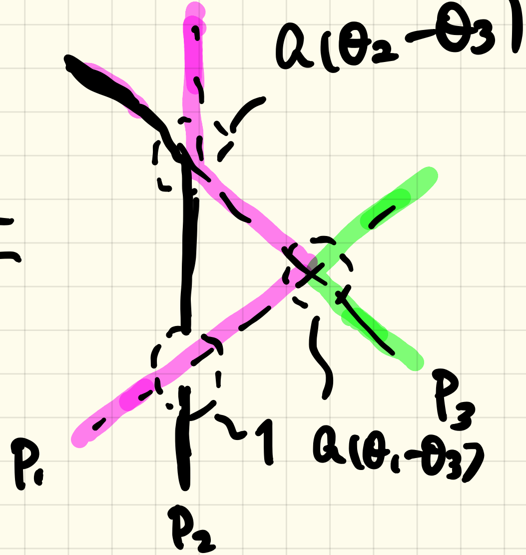
$$\Big]$$

$\sim \delta_{ij} \, \delta^{kl}$

Y. B. 1

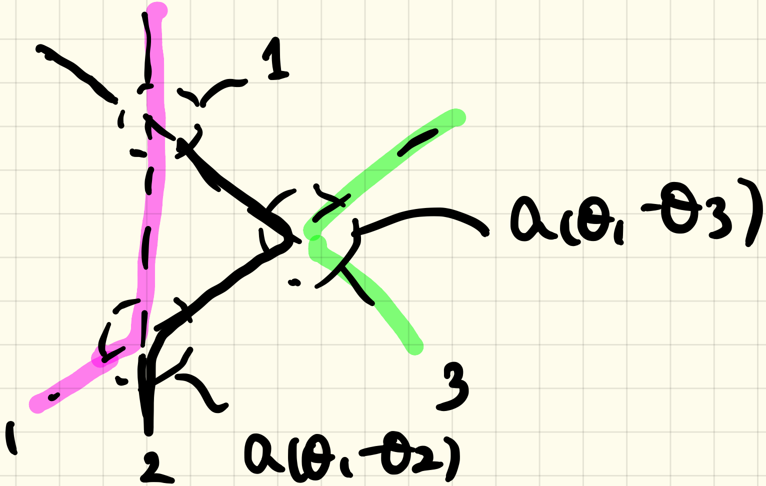


$=$



$$1 \times a(\theta_2-\theta_3) \times a(\theta_1-\theta_2) = a(\theta_1-\theta_3)\, a(\theta_2-\theta_3) + a(\theta_1-\theta_2)\, a(\theta_1-\theta_3)$$

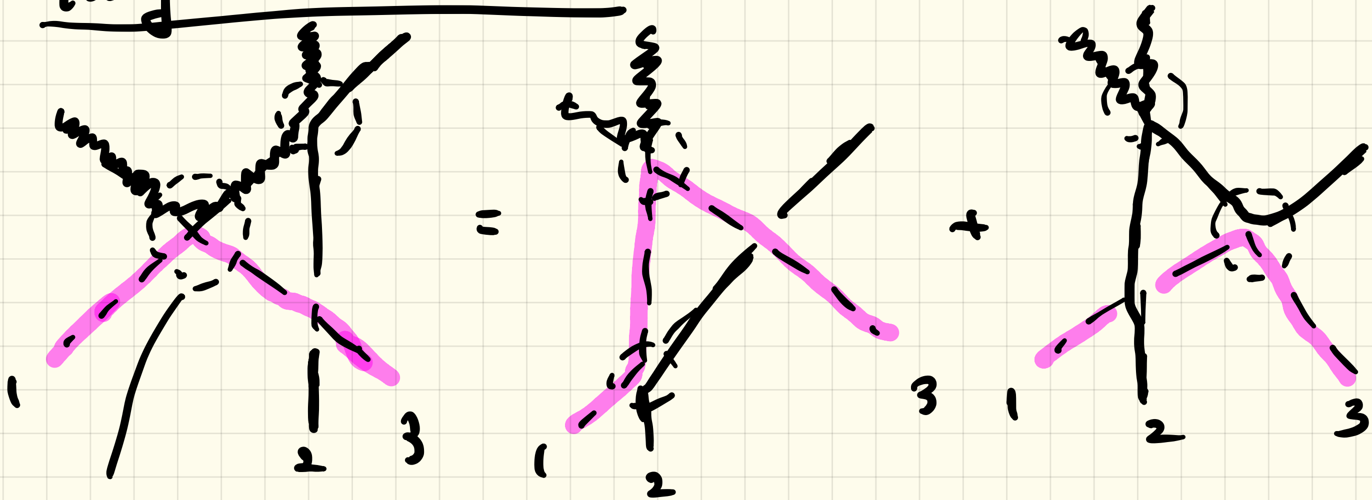
$$\Rightarrow \quad \frac{1}{a(\theta_1-\theta_3)} = \frac{1}{a(\theta_1-\theta_2)} + \frac{1}{a(\theta_2-\theta_3)}$$



$$a(\theta_1-\theta_2)\times a(\theta_1-\theta_3)$$

$$\rightsquigarrow \quad \frac{1}{a(\theta)} = \underbrace{c}_{\text{unfixed coeff.}} \times \theta$$

Yang-Baxter 2.



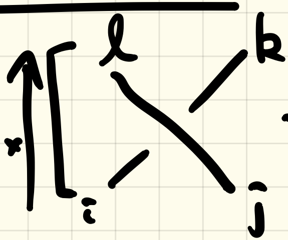
$$b(\theta_1-\theta_3)\times a(\theta_1-\theta_2) = b(\theta_2-\theta_3)\, a(\theta_1-\theta_2) + b(\theta_2-\theta_3)\, b(\theta_1-\theta_3)$$

$c\cdot(\theta_1-\theta_2)$

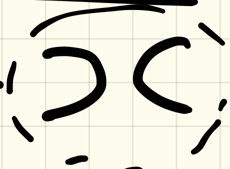
$$\Rightarrow \quad \frac{1}{a(\theta_1-\theta_2)} = \frac{1}{b(\theta_2-\theta_3)} - \frac{1}{b(\theta_1-\theta_3)}$$

$$\frac{1}{b(\theta)} = -c\theta + d$$

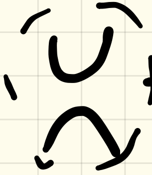
$$S_{ij}^{kl}(\theta) = \sigma(\theta)\Big[$$



$$+\ \frac{1}{c\theta}$$



$$+\ \frac{1}{d-c\theta}$$



$$\Big]$$

- Another Y. B. (Exercise) $\rightsquigarrow$ $d = -\dfrac{N-2}{2}$
- Crossing sym. $S_{ij}^{kl}(\theta) = S_{il}^{kj}(i\pi-\theta)$

$\boxed{\sigma(\theta) = \sigma(i\pi-\theta)}$ $\quad \dfrac{1}{d-c\theta} = \dfrac{1}{c(i\pi-\theta)} \rightsquigarrow \boxed{c = \dfrac{d}{i\pi}}$

- Unitarity

$$S(\theta)\, S(-\theta) = 1$$

$$\boxed{\sigma(\theta)\cdot\sigma(-\theta) = \frac{\theta^2}{\theta^2 + 4\pi^2\Delta^2}} \quad \Delta = \frac{1}{N-2}.$$

$$\sigma(\theta) = \frac{\Gamma \times \Gamma \times -\Gamma}{\Gamma \times \Gamma \times \Gamma}$$

Similar Func. Eq. $\rightsquigarrow$ Lecture IV.