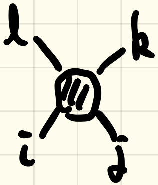
Yesterday

$\exists\ \infty$ many higher spin charges $\xrightarrow{2d}$ Yang-Baxter.

O(N)

- Yang-Baxter
- Crossing



$$s \leftrightarrow t$$

$$\boxed{S_{ij}^{lk}(\theta) = S_{li}^{jk}(i\pi - \theta)}$$

$$s = 4m^2 \cosh^2\frac{\theta}{2}, \quad t = -4m^2 \sinh^2\frac{\theta}{2}$$

- Unitarity $\mathbb{S}(\theta)\,\mathbb{S}(-\theta) = \mathbb{1}$

$\rightsquigarrow$ $2 \to 2$ S-matrix

4. Classical vs. Quantum integrability in O(N)

[Goldschmidt & Witten]

[w/. R. Mahajan & S. Shao]

- Classical

$$S = \frac{1}{K}\int d^2z \; \partial_+\vec{n}\cdot\partial_-\vec{n} + \sigma\,(\vec{n}\cdot\vec{n} - 1)$$

$\downarrow$ rescal.

$$= \int d^2x \; \partial_+\vec{n}\cdot\partial_-\vec{n} + \sigma\left(\vec{n}\cdot\vec{n} - \frac{1}{K}\right)$$

want to eliminate

$$\rightsquigarrow \quad \partial_+\partial_-\vec{n} - \sigma\,\vec{n} = 0$$

$$\vec{n}\cdot\partial_+\partial_-\vec{n} - \sigma\,(\vec{n}\cdot\vec{n}) = 0$$

$\frac{1}{K}$

$$\vec{n}\cdot\partial_+\partial_-\vec{n} = -\partial_+\vec{n}\cdot\partial_-\vec{n}$$

$$\partial_+\partial_-(\vec{n}\cdot\vec{n}) = 0$$

$$\rightsquigarrow \boxed{\underbrace{\vec{n}_{+-}}_{\partial_+\partial_-\vec{n}} + K(\vec{n}_+\cdot\vec{n}_-)\times\vec{n} = 0}$$

$$\partial_-\left((T_{++})^k\right) = 0 \quad \forall k\in\mathbb{Z}_{>0}$$

$$T_{++} = \vec{n}_+\cdot\vec{n}_+ \qquad \vec{n}_+\cdot\vec{n}$$

$$k=1 \Rightarrow \quad \vec{n}_{+-}\cdot\vec{n}_+ = 0 \quad = \tfrac{1}{2}\partial_+(\vec{n}\cdot\vec{n}) = 0$$

$$\underline{\partial_-(T_{++}^{k}) = 0}$$

<u>Quantum</u> $(\vec{n}_+ \cdot \vec{n}_-)^k$

$$\underbrace{\partial_-}_{-1}\big(\underbrace{(T_{++})^k}_{+2k}\big) = \overbrace{\underbrace{A}_{\to 0} + \partial_- B + \partial_+ B'}$$

Task : $\boxed{\text{List all ops with } S = 2k-1,\ \Delta = 2k+1 \text{ which are } \underline{\underline{\text{not}}} \text{ derivatives}}$

→ Conformal Primaries ....

Constraint 1: Op must be $O(N)$ inv.

$$\vec{n}_+ \cdot \vec{n}_{+-} \quad \text{OK}$$

$$\vec{n} \quad \text{bad} \quad \text{not allowed}$$

Constraint 2: $\vec{n} \cdot \vec{n} = 1$

$$0 = \partial_+^{k_1} \partial_-^{k_2} (\vec{n} \cdot \vec{n})$$

$$= \underbrace{\vec{n} \cdot \partial_+^{k_1} \partial_-^{k_2} \vec{n}}_{\text{eliminate}} + \partial_+ \vec{n} \cdot \partial_+^{k_1 - 1} \partial_-^{k_2} \vec{n}_{+-}$$

$\rightsquigarrow$ <u>All $\vec{n}$'s must come w. derivatives</u>

Constraint 3 : E.O.M.

$$\vec{n}_{+-} + \underbrace{K(\vec{n}_+ \cdot \vec{n}_-)\cdot\vec{n}}_{\text{if } K=0} = 0$$

coupling.

we can ignore

"quantum anomaly"

absorb.

- For counting ops, we can set $K=0$. $K\neq 0$ effect can be absorbed into anomaly

$$\boxed{\vec{n}_{+-} = 0}$$

$\partial_- \left((T_{++})^2 \right) = ?$

All possible ops.

$(\vec{n}_- \cdot \vec{n}_{++++})$, $(\vec{n}_- \cdot \vec{n}_+)(\vec{n}_{++} \cdot \vec{n}_+)$

$(\vec{n}_- \cdot \vec{n}_{++})(\vec{n}_+ \cdot \vec{n}_+)$ $\quad \partial_- \partial_+^2 T_{++}$

Derivates

$\partial_- [\vec{n}_+ \cdot \vec{n}_{+++}]$ $\quad -\partial_- [\vec{n}_{++} \cdot \vec{n}_{++}]$

$-\partial_- [(\vec{n}_+ \cdot \vec{n}_+)(\vec{n}_+ \cdot \vec{n}_+)]$ $\quad \partial_+ [\vec{n}_- \cdot \vec{n}_{+++}]$

$\partial_+ [(\vec{n}_- \cdot \vec{n}_+)(\vec{n}_+ \cdot \vec{n}_+)]$

$\rightsquigarrow \partial_-\left((T_{++})^2\right) = 0$ ✓ quantum mechanically.

Spin 4 conserved current

$\rightsquigarrow$ Spin 6 ✓

[Parke 1980] 2 higher conserved currents $\xrightarrow{}$ Yang-Baxter

$\uparrow$

Causality.

More generally,

if $N_{\Delta=2k,\, S=2k} - N_{\Delta=2k+1,\, S=2k-1} > 0$

$\rightarrow \quad \exists$ spin $2k$ current.

$N_{\Delta, S}$: # of primaries w/. Δ, S

II. Spin Chain from $\mathcal{N}=4$ SYM

1. $\mathcal{N}=4$ SYM in 3+1d.

- Maximally SUSY gauge theory in 4d.

  $\rightarrow$ 16 SUSY

- R-symmetry. $SO(6) \simeq SU(4)$
- Conformal. $SO(4,2) \simeq SU(2,2)$

$\mathcal{N}=4$ superconformal.

$PSU(2,2|4)$

Field content

- 6 scalars Φ_I $\quad I = 1, \cdots, 6$
- gauge field A_μ

(+ 8 fermions $\psi, \bar{\psi}$)

$\rightsquigarrow$ adjoint rep. of $SU(N)$

$$(\Phi_I)^a{}_b \qquad \begin{array}{l} a = 1, \cdots, N \\ b = 1, \cdots, N \end{array}$$

Gauge transf. $\quad \Phi_I \to g \Phi_I g^{-1}$

2. Single-trace operators

CFT $\rightsquigarrow$ correlation fns.

$$\underline{\mathcal{O}^b(x) = \mathrm{Tr}\left[\Phi_{I_1} \cdots\cdots \Phi_{I_k}\right](x)}$$

$\hookrightarrow$ gauge inv.

$\hookrightarrow$ same position

@ tree-level

$$\langle \mathcal{O}^b(x) \cdots\cdots \rangle = \text{Wick contractions.}$$

@ 1-loop.

$$\langle \mathcal{O}^b(x) \cdots \rangle = \text{UV-divergent}$$

$\rightsquigarrow$ Renormalize the operator!

coupling

$$\langle \mathcal{O}^b_I(x)\ \mathcal{O}^b_J(y) \rangle = \frac{1}{|x-y|^{2\Delta^0_I}} \left[\delta_{IJ} - g^2 \times (2\Gamma_{IJ}) \times \log(\Lambda |x-y|) \right]$$

diagonalize

eigenvalue of Γ_{IJ}

$$\langle \tilde{\mathcal{O}}^b_I(x)\ \tilde{\mathcal{O}}^b_J(y) \rangle = \frac{\delta_{IJ}}{|x-y|^{2\Delta^0_I}} \times \left(1 - 2g^2 \gamma_I \log(\Lambda |x-y|)\right)$$

$$\mathcal{O}^r_I(x) = e^{\gamma_I \log \Lambda}\, \tilde{\mathcal{O}}^b_I \quad : \text{renormalization}$$

$$\langle \mathcal{O}^r_I(x)\, \mathcal{O}^r_J(y) \rangle = \frac{\delta_{IJ}}{|x-y|^{2\Delta^0_I}}$$

$$\times \left(1 - 2 g^2 \gamma_I \log |x-y| \right)$$

$$= \frac{\delta_{IJ}}{|x-y|^{2(\Delta^0_I + \underset{g^2}{\wedge}\gamma_I)}}$$

corrected dim

Summary To compute 2pt fn.

$$\left[\begin{array}{l} \bullet\ \underline{\text{Compute log div. } \Gamma_{IJ}}. \\ \bullet\ \text{Diagonalize } \Gamma_{IJ} \end{array}\right.$$

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3. Computation @ Large N.

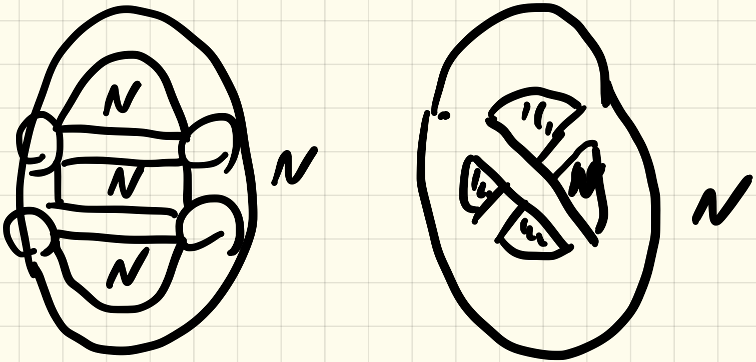
Consider $\mathcal{O}_I = \mathrm{Tr}\left[\Phi_{I_1} \cdots\cdots \Phi_{I_k}\right]$

$I_1 = 1, \dots, 6$

$$\langle \mathcal{O}_I \ \mathcal{O}_J \rangle \underset{\text{large}N}{=} \text{(1 l.)} \ \log \Lambda$$

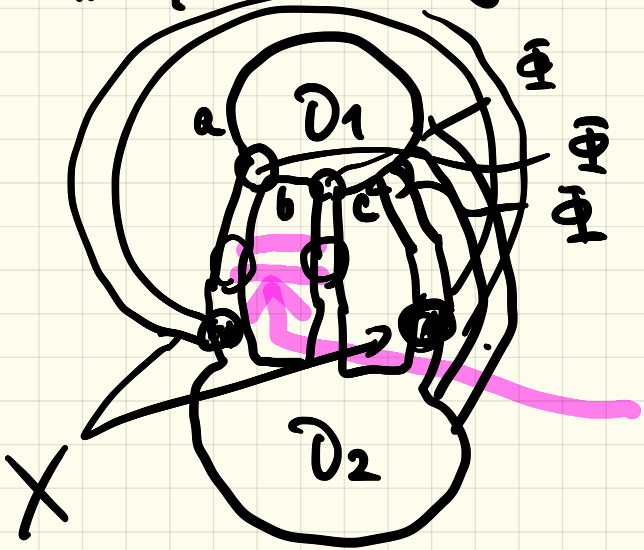
Obs 1. Large N, $\left[\begin{array}{l} N \to \infty \\ \lambda = g_{YM}^2 N \text{ : fixed} \end{array}\right.$

$$\Phi^a{}_b \;=\!=\!=\; \Phi^c{}_d.$$



$g_{YM}^4 N^4$ $\qquad$ $g_{YM}^4 N^2 \leftarrow$ suppressed.

$$\mathcal{O}_1 = \mathrm{Tr}\left[\Phi\,\Phi\,\Phi\right] = \Phi^a{}_b\,\Phi^b{}_c\,\Phi^c{}_a.$$

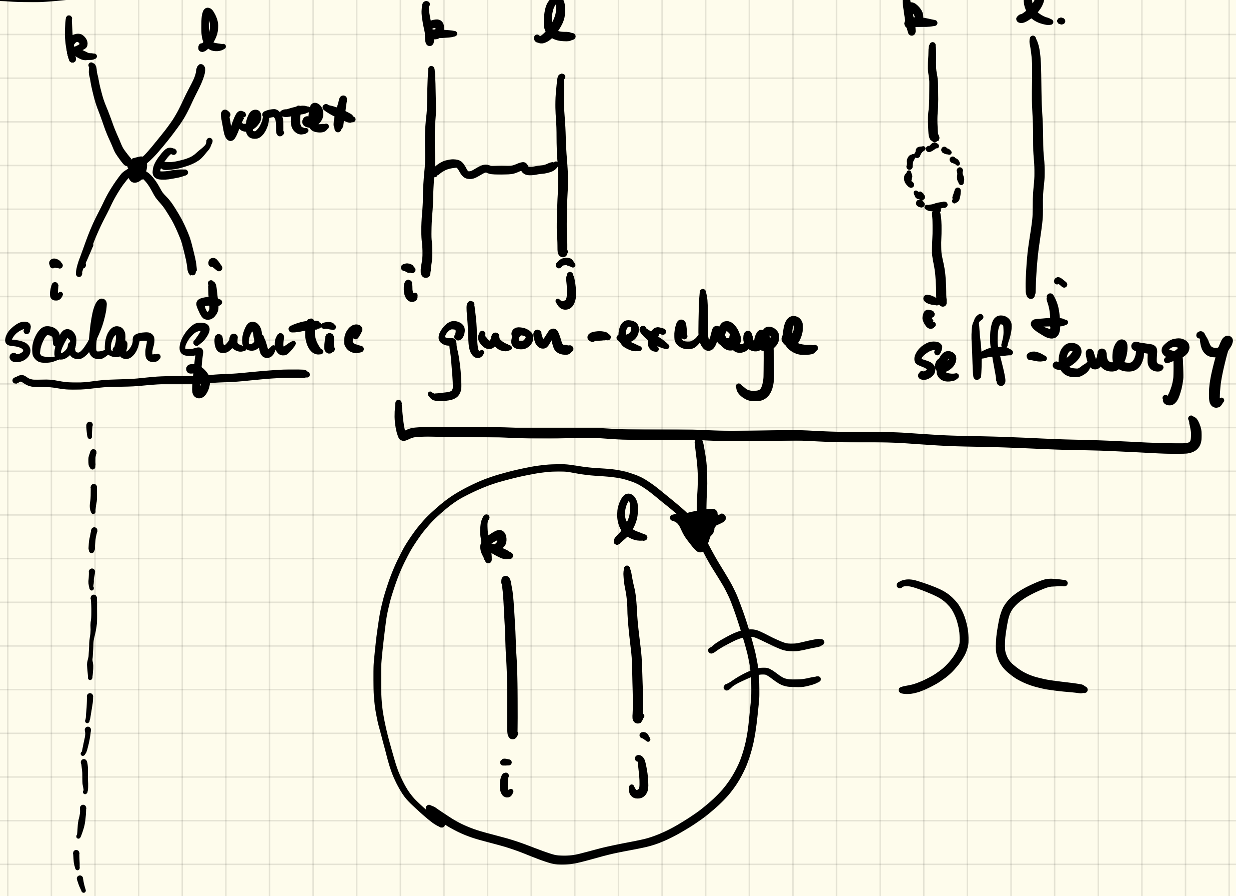


← Tree-level

@ 1-loop

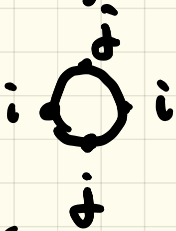
$\rightsquigarrow$ "<u>Nearest-Neighbor</u>"

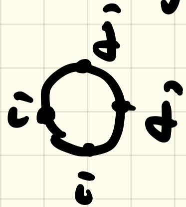
Obs 2. SO(6) index structure



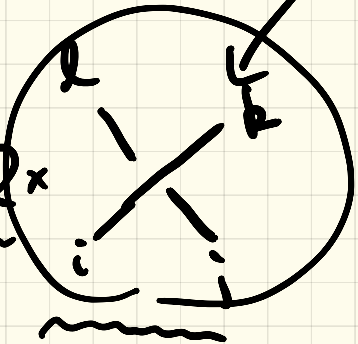
$$\lambda \sum_{i,j} \mathrm{Tr}\left[[\phi_i, \phi_j]^2 \right]$$

$$\Rightarrow \mathrm{Tr}\left[\phi_i \phi_j \phi_i \phi_j\right] - \mathrm{Tr}\left[\phi_i \phi_i \phi_j \phi_j\right]$$





λ 2×



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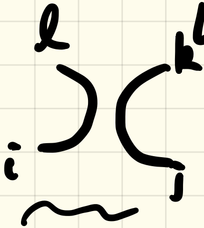

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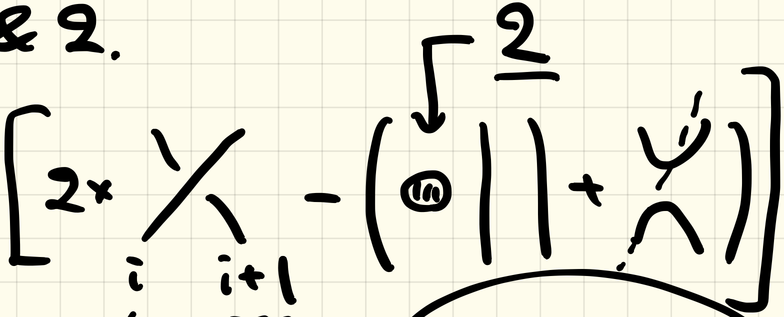
Combining obs. 1 & 2.

$$\Gamma_* = \sum_{i=1}^{L} \blacksquare$$



$$I_{L+1} = I_1$$

$$\mathrm{Tr}\left[\Phi_{I_1} \cdots \Phi_{I_L}\right] \quad -\frac{\lambda}{16\pi^2} \quad \Phi_{I_i} \quad \Phi_{I_{i+1}}$$

<u>Obs. 3</u> SUSY $= -g^2$

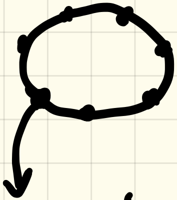
$\mathrm{Tr}\left[Z^L\right]$: protected $\leftarrow$ 0 eigenvector of Γ

$Z \sim \Phi_1 + i\Phi_2$

$\textcircled{n} = 2$

$$\Gamma = -\sum_{i=1}^{L} \frac{\lambda}{16\pi^2}\left[2\times \; \times \; - 2 \;)(\; + \; \asymp \right]$$

$\downarrow$ Diagonalize



1, ..., 6

$\Longleftrightarrow \quad \mathrm{Tr}\left[\Phi_{I_1} \cdots \Phi_{I_{4k}}\right]$

$H_{\text{spin chain}} \quad \Longleftrightarrow \quad \Gamma$

4. Reduction to SU(2) chain

$$\mathrm{Tr}\left[\Phi_{I_1} \cdots \Phi_{I_6}\right]$$

I 1, ..., 6

Consider ops. made out of

$$Z = \phi_1 + i\phi_2 , \quad X = \phi_3 + i\phi_4$$

restricted

$$\Gamma = 2g^2 \sum_{i=1}^{L} \left[\underset{i}{|}\;\underset{i+1}{|} - \times \right]$$

$$Z \Rightarrow |\uparrow\rangle$$

$$X \Rightarrow |\downarrow\rangle$$

$$H_{\text{Heisenberg}} = 2g^2 \sum_{i=1}^{L} \left[\frac{1}{4} - \mathbf{S}_i \cdot \mathbf{S}_{i+1} \right]$$

spin ½

SU(2) op.

$$S_i^z S_{i+1}^z + S_i^x S_{i+1}^x + S_i^y S_{i+1}^y$$

Heisenberg Spin Chain

→ Integrable

↳ . ∃ ∞ conserved charges

+

Systematic way to construct eigenstate & compute eigenvalue of H

Bethe Ansatz

$$H = 2g^2 \sum_{i=1}^{L} \left[\frac{1}{4} - \mathbf{S}_i \cdot \mathbf{S}_{i+1} \right]$$

◎ Groudd state.

$$\underbrace{|\uparrow \cdots \uparrow\rangle}_{\mathrm{Tr}[Z^L]}, \quad \left(\underbrace{|\downarrow \cdots \downarrow\rangle}_{\mathrm{Tr}[X^L]} \right)$$

$$E = 0$$

$$\mathrm{Tr}[Z^L] \qquad \qquad \mathrm{Tr}[X^L]$$

$$\mathrm{Tr}[(\Phi_5 + i\Phi_6)^L]$$

① 1st excited state.

$$\hat{H} \propto \sum_{i=1}^{L} \left[\underbrace{1_i} - X_i \right]$$

$$\underset{\sim\sim}{|n\rangle} = |\uparrow \cdots \uparrow \downarrow \uparrow \cdots\rangle$$

└ n-th.

$$\hat{H}\,|n\rangle \propto 2\,|n\rangle - \boxed{|n-1\rangle - |n+1\rangle}$$

X

$$|\uparrow \cdots \uparrow \downarrow \uparrow \cdots \rangle$$

X

$\downarrow \uparrow$

Discretized Laplacian

$\rightsquigarrow$ eigenstate $|p\rangle = \sum_n e^{ipn} |n\rangle$.

$$\rightsquigarrow \underbrace{E(p) = 4g^2(1-\cos p)}_{\substack{\sim \; p^2. \\ p \ll 1}}$$

Periodicity: $|L+1\rangle = |1\rangle$

$$e^{ip(L+1)} = e^{ip}.$$

$$\boxed{e^{ipL} = 1}$$