Introduction to Resurgence and Non-perturbative Physics

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GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414, 1603.04924

recent KITP Program: *Resurgent Asymptotics in Physics and Mathematics*, Fall 2017

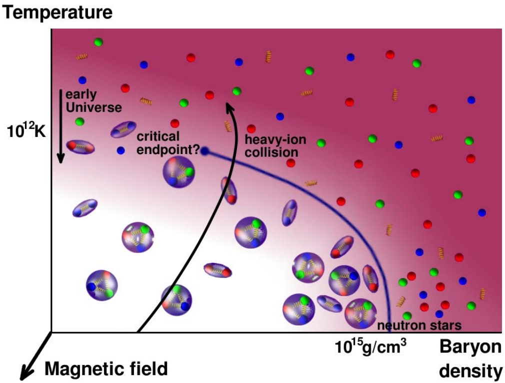
future Isaac Newton Institute Programme: *Universal Resurgence*, 2020/2021

1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   - asymptotic series in physics; Borel summation
   - trans-series completions & resurgence
   - examples: linear and nonlinear ODEs
2. Lecture 2: Applications to Quantum Mechanics and QFT
   - instanton gas, saddle solutions and resurgence
   - infrared renormalon problem in QFT
   - Picard-Lefschetz thimbles
3. Lecture 3: Resurgence and Large N
   - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT
4. Lecture 4: Resurgence and Phase Transitions
   - Gross-Witten-Wadia Matrix Model

Physical Motivation

- non-perturbative definition of non-trivial QFT, in the continuum
- analytic continuation of path integrals
- "sign problem" in finite density QFT
- dynamical & non-equilibrium physics from path integrals (strong coupling)
- uncover hidden ‘magic’ in perturbation theory
- new understanding of weak-strong coupling dualities
- infrared renormalon puzzle in asymptotically free QFT
- exponentially improved asymptotics & resummation

Physical Motivation



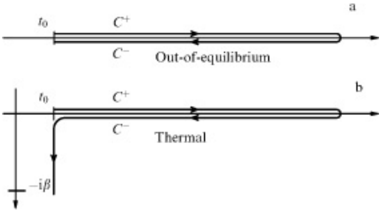
- sign problem: "complex probability" at finite baryon density?

$$\int \mathcal{D}A\, e^{-S_{YM}[A]+\ln\det(\not{D}+m+i\,\mu\gamma^0)}$$

- phase transitions and Lee-Yang & Fisher zeroes

Physical Motivation

- equilibrium thermodynamics $\leftrightarrow$ Euclidean path integral
- Kubo-Martin-Schwinger: antiperiodic b.c.'s for fermions

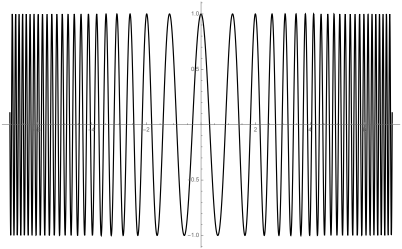


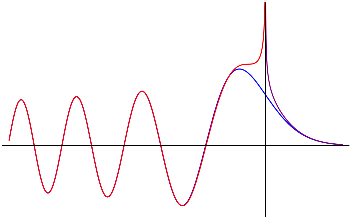
- non-equilibrium physics $\leftrightarrow$ Minkowski path integral
- Schwinger-Keldysh time contours
- quantum transport in strongly-coupled systems

Physical Motivation

what does a Minkowski path integral mean, computationally?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$





$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\,t^3+x\,t\right)}\,dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x \to +\infty \\[2em] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

- massive cancellations $\Rightarrow$ $\quad \mathrm{Ai}(+5) \approx 10^{-4}$

Physical Motivation

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$

- since we need complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to expect to require similar tools also for path integrals

- an obvious idea, but how to make it work ... ?

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

go beyond the Gaussian approximation and view semiclassical expansions as potentially exact

Perturbation Theory and Asymptotics

- in physical applications, expansions in a small parameter are often, but not always, divergent asymptotic series
- such an expansion is often the best we can do, and sometimes it is the only thing we can do
- it is worth understanding how to extract as much physical information as possible from an asymptotic expansion
- resurgence has the potential to lead to significant improvements, both analytically and numerically
- resurgence relates perturbative expansions to non-perturbative physics in surprisingly explicit ways

It's a bit of black magic, to figure things out about differential equations even though you can't solve them

Michael Atiyah

Trans-series

- an interesting observation by Hardy:

No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, *Divergent Series*, 1949

- deep result: “this is all we need” (J. Écalle, 1980)
- also as a closed logic system: Dahn and Göring (1980)

Resurgent Trans-Series

- Écalle: *resurgent functions* closed under all operations:

(Borel transform) + (analytic continuation) + (Laplace transform)

- basic trans-series expansion in QM & QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

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- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT

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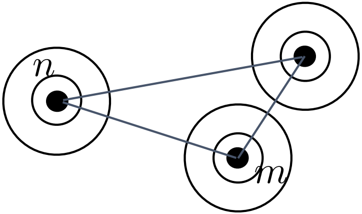
- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponentially improved asymptotics

Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



resurgence = global complex analysis with divergent series

Perturbation theory is generically divergent

- hard problem = easy problem + "small" correction
- perturbation theory generally $\rightarrow$ divergent series

e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$
- Stark: $c_n \sim (2n)!$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon (Mathieu) potential: $c_n \sim n!$
- symmetric double-well: $c_n \sim n!$

note generic factorial growth of perturbative coefficients

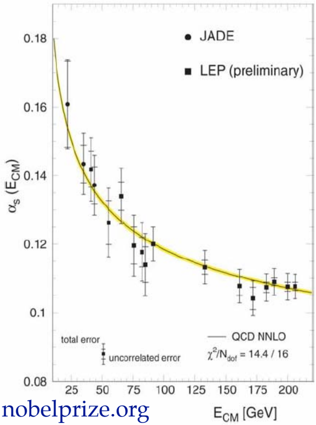
Perturbation theory works

QED perturbation theory:

$$\frac{g-2}{2} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.18124\ldots)\left(\frac{\alpha}{\pi}\right)^3 - 1.9097(20)\left(\frac{\alpha}{\pi}\right)^4 + 9.16(58)\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{theory}} = 0.001\,159\,652\,181\,78(77)$$



nobelprize.org

QCD: asymptotic freedom

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$

Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n \, (x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \to 0 \qquad , \quad N \to \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

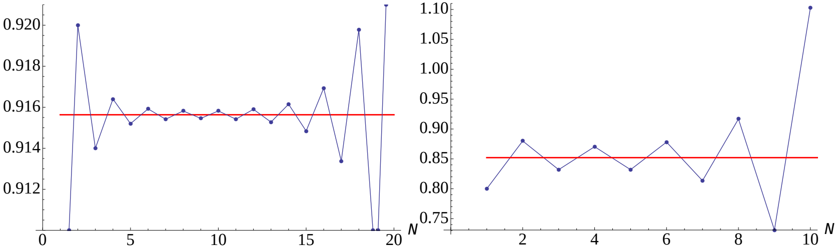
$$|R_N(x)| \ll |x - x_0|^N \qquad , \quad x \to x_0 \quad , \quad N \quad \text{fixed}$$

$\longrightarrow$ "optimal truncation":

truncate just before least term (x dependent!)

Asymptotic Series vs Convergent Series

alternating asymptotic series : $$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$



$(x = 0.1)$ $(x = 0.2)$

optimal truncation order depends on x: $N_{\text{opt}} \approx \frac{1}{x}$

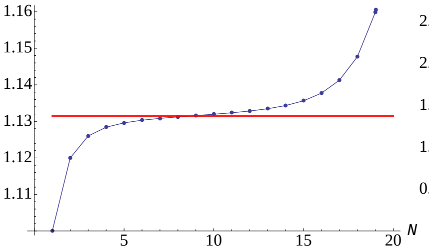
Exercise 1:

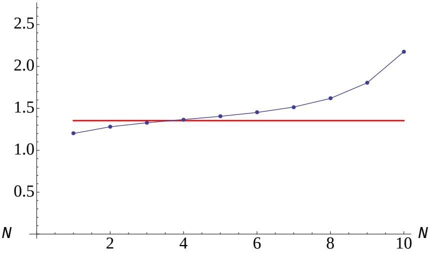
(i) Show that the magnitude of the summand, $n!\,x^n$, is minimized for $n \approx \frac{1}{x}$

(ii) Compute the optimal order of truncation for the series with summand: $(-1)^n(2n)!\,x^n$

Asymptotic Series vs Convergent Series

non-alternating asymptotic series : $$\sum_{n=0}^{\infty} n!\, x^n \sim -\frac{1}{x}\, e^{-\frac{1}{x}}\, E_1\left(-\frac{1}{x}\right)$$





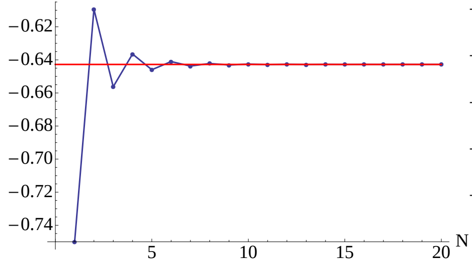
$x = 0.1$ $\qquad$ $x = 0.2$

optimal truncation order depends on x: $\quad N_{\text{opt}} \approx \frac{1}{x}$

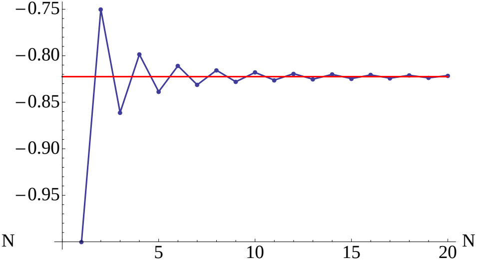
Asymptotic Series vs Convergent Series

- contrast with behavior of a convergent series, for which more terms always improves the answer, independent of x

convergent series : $$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} x^n = \mathrm{PolyLog}(2, -x)$$



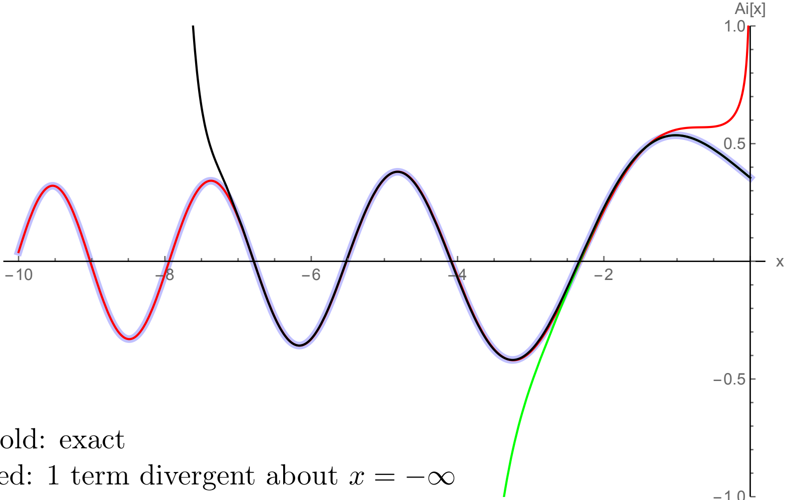
$(x = 0.75)$



$(x = 1)$

Asymptotic Series vs Convergent Series

asymptotic series are sometimes 'better'



gold: exact

red: 1 term divergent about $x = -\infty$

green: 10 terms convergent about $x = 0$

black: 50 terms convergent about $x = 0$

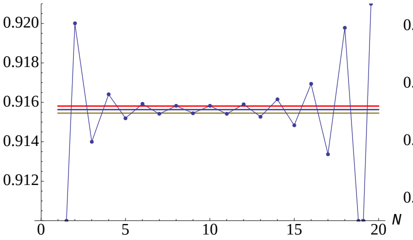
Asymptotic Series: exponential precision

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$

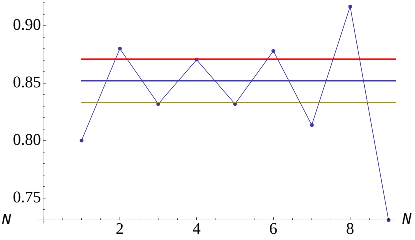
optimal truncation: error term is <u>exponentially small</u>

$$\left|R_N(x)\right|_{N\approx 1/x} \approx N!\, x^N\Big|_{N\approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$

- e.g. alternating exponential integral:



$(x = 0.1)$



$(x = 0.2)$

Borel summation: basic idea

alternating factorially divergent series:

$$\sum_{n=0}^{\infty}(-1)^n \, n! \, x^n = ?$$

write $n! = \int_0^\infty dt \, e^{-t} \, t^n$

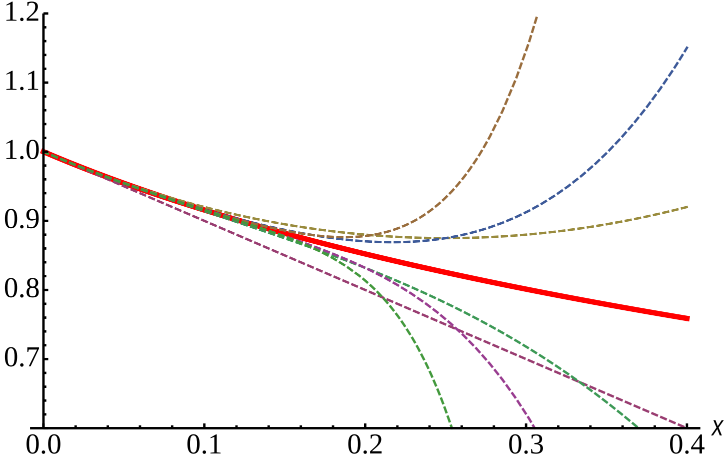
Emile Borel

$$\sum_{n=0}^{\infty}(-1)^n \, n! \, x^n = \int_0^\infty dt \, e^{-t} \, \frac{1}{1 + x\, t} \qquad (?)$$

integral is convergent for all $x > 0$: "Borel sum" of the series

Borel Summation: basic idea

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+x\,t} = \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$



Borel summation: basic idea

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<u>non-alternating</u> factorially divergent series:

$$\sum_{n=0}^{\infty} n!\, x^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1 - x\, t} \qquad (??)$$

pole on the (real, positive) Borel axis!

Emile Borel

Borel summation: basic idea

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pole on the (real, positive) Borel axis!

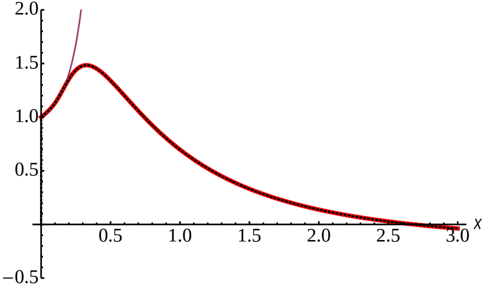
Emile Borel

$$\Rightarrow \text{non-perturbative imaginary part} = \pm \frac{i\,\pi}{x}\, e^{-\frac{1}{x}}$$

but every term in the series is real !?!

Borel Summation: basic idea

$$\text{Borel} \Rightarrow \mathcal{R}e\left[\sum_{n=0}^{\infty} n!\, x^n\right] = \mathcal{P}\int_0^\infty dt\, e^{-t}\, \frac{1}{1-x\,t} = \mathcal{R}e\left[-\frac{1}{x}\, e^{-\frac{1}{x}}\, E_1\left(-\frac{1}{x}\right)\right]$$



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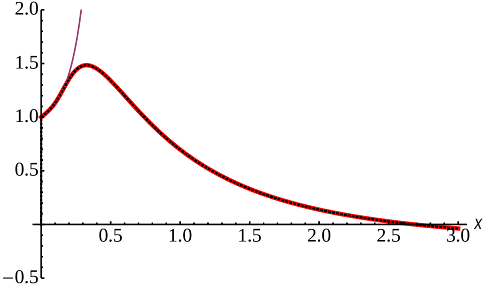

- note: $E_1\left(-\frac{1}{x}\right)$ also has an imaginary part $= \pm i\pi$

$$-\frac{1}{x}e^{-\frac{1}{x}}E_1\left(e^{\pm i\,\pi}\frac{1}{x}\right) = -\frac{1}{x}e^{-\frac{1}{x}}\left[Ein\left(-\frac{1}{x}\right) - \ln x - \gamma \mp i\,\pi\right]$$

- Borel encodes this non-perturbative "connection formula"

Borel summation

Borel transform of series, where $c_n \sim n!\quad,\quad n \to \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \quad \longrightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has a finite radius of convergence

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Borel resummation of original asymptotic series:

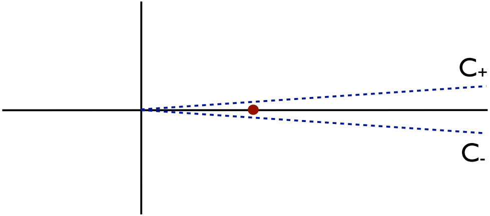
$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

note: $\mathcal{B}[f](t)$ may have singularities in (Borel) t plane

Borel singularities

avoid singularities on $\mathbb{R}^+$: directional Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](t) e^{-t/g} dt$$



go above/below the singularity: $\theta = 0^\pm$

$\longrightarrow$ non-perturbative ambiguity: $\pm \mathrm{Im}[\mathcal{S}_0 f(g)]$

physics challenge: use physical input to resolve ambiguity

Borel summation in practice

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\, \Gamma(\gamma\, n + \delta)$$

- alternating series: real Borel sum

$$f(g) \sim \frac{1}{\gamma} \int_0^{\infty} \frac{dt}{t} \left(\frac{1}{1+t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

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- nonalternating series: ambiguous imaginary part

$$\mathrm{Re}\, f(-g) \;\sim\; \frac{1}{\gamma}\, \mathcal{P} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1-t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

$$\mathrm{Im}\, f(-g) \;\sim\; \pm\frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

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$$\mathrm{Im}\, f(-g) \;\sim\; \pm\frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

- γ determines **power** of coupling in the exponent
- β and γ determine **coefficient** in the exponent
- β, γ and δ determine the **prefactor**

Exercise 2:

Use the integral representation of $\Gamma(\gamma\, n + \delta)$ to derive the expressions on the previous slide.

Hence deduce the meaning of the large-order behavior parameters β, γ and δ.

Compare with the second part of Exercise 1.

another view of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

resurgence = global complex analysis for divergent series

(analytic continuation, transforms, monodromy, ...)

$\Rightarrow$ "the trans-series really IS the function"

question: to what extent is this true/useful in physics?

Resurgence: Preserving Analytic Continuation

Stirling expansion for $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \dots + \frac{174611}{6600z^{20}} - \dots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$ ✓

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- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

$$\Rightarrow \quad \operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

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- formal series only has the two "perturbative" terms

“raw” asymptotics is <u>inconsistent</u> with analytic continuation

- resurgence: add infinite series of non-perturbative terms

"non-perturbative completion"

Resurgence: Preserving Analytic Continuation

$$\mathrm{Im}\,\psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi\, k\, y}$$

• function satisfies infinite order linear ODE

$\Rightarrow$ infinitely many exponential terms in trans-series

Borel representation:

$$\psi(1+z) - \ln z = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-zt}\, dt$$

• Borel transform: poles at $t = \pm 2n\pi i$, $n = 1, 2, 3, \ldots$

• meromorphic (poles, no cuts) $\Rightarrow$ no "fluctuation factors"

• this simple example arises often in QFT: Euler-Heisenberg, finite temperature QFT, de Sitter, exact S-matrices, Chern-Simons partition functions, matrix models, ...

- scalar QED EH in self-dual background ($F = \pm\tilde{F}$):

$$S = \frac{F^2}{16\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t/F}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- Gaussian matrix model: $\lambda = g\,N$

$$\mathcal{F} = -\frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- $c = 1$ String: $\lambda = g\,N$

$$\mathcal{F} = \frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sin^2(t)} - \frac{1}{t^2} - \frac{1}{3}\right)$$

- Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4}\sum_{m\in\mathbb{Z}}\int_0^\infty \frac{dt}{t}\, e^{-2(\lambda+2\pi\, i\, m)\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

Borel summation in practice: Borel cuts

Comments:

• an isolated pole in the Borel transform corresponds to a single "instanton" exponential term (e.g. previous exponential integral function example)

• physically, we may expect fluctuations about instantons. These correspond to branch point singularities, and their associated branch cuts, in the Borel plane

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$$\Gamma\left(s, \frac{1}{x}\right) = \int_{\frac{1}{x}}^{\infty} t^{s-1} e^{-t}\, dt \quad = \quad x^{-s}\, e^{-\frac{1}{x}} \int_0^{\infty} e^{-t/x}\, (1+t)^{s-1}\, dt$$

$$\sim \quad x^{1-s}\, e^{-1/x} \sum_{n=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s-n)}\, x^n$$

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$$\begin{aligned}\Gamma\left(s, \frac{1}{x}\right) = \int_{\frac{1}{x}}^{\infty} t^{s-1} e^{-t}\, dt &= x^{-s}\, e^{-\frac{1}{x}} \int_0^{\infty} e^{-t/x}\, (1+t)^{s-1}\, dt \\ &\sim x^{1-s}\, e^{-1/x} \sum_{n=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s-n)}\, x^n\end{aligned}$$

• truncation when $s =$ integer (“SUSY” & “localization”)

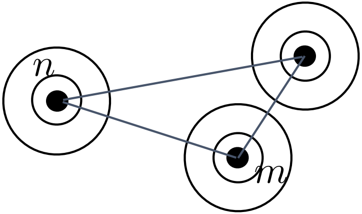
• resurgence is more interesting with several parameters

Resurgence in Differential Equations I

• trans-series from $2^{\rm nd}$ order <u>linear</u> ODE has 2 non-perturbative exponential terms (WKB)

• trans-series from $n^{\rm th}$ order <u>linear</u> ODE has n non-perturbative exponential terms

• the fluctuations about these different ("instanton") exponentials are related by generic large order/low order resurgence relations



All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

<u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- Berry/Howls: exact resurgent relation between fluctuations about $n^{\rm th}$ saddle and about neighboring saddles m

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- proof is based on contour deformation

All-Orders Steepest Descents: Darboux Theorem

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- proof is based on contour deformation

- universal factorial divergence of fluctuations

$$T_r^{(n)}=\frac{(r-1)!}{2\pi\, i}\sum_m\frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r}\left[T_0^{(m)}+\frac{F_{nm}}{(r-1)}\,T_1^{(m)}+\frac{(F_{nm})^2}{(r-1)(r-2)}\,T_2^{(m)}+\dots\right.$$

- alternative proof from Darboux's theorem in the Borel plane

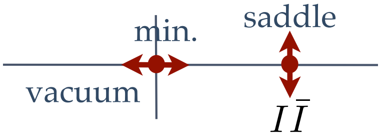
fluctuations about different saddles are explicitly related !

- example

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz\, e^{-\frac{1}{g^2}\sin^2(z)}$$

- this is a Bessel function
- two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$

$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \ldots\right.$$

All-Orders Steepest Descents: Darboux Theorem

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- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}g^2 + \frac{9}{32}g^4 - \frac{75}{128}g^6 + \ldots\right)$$

- fluctuations about the two saddles are explicitly related

- simple example of a generic resurgent large-order/low-order perturbative/non-perturbative relation

Exercise 3: the modified Bessel function has the large x asymptotic expansion:

$$I_j(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n(j)}{x^n} \pm i e^{ij\pi} \frac{e^{-x}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{\alpha_n(j)}{x^n}, \qquad \left|\arg(x) - \frac{\pi}{2}\right| < \pi$$

where the coefficients are

$$\alpha_j(n) = \frac{\cos(\pi j)}{\pi} \left(-\frac{1}{2}\right)^n \frac{\Gamma\left(n + \frac{1}{2} - j\right) \Gamma\left(n + \frac{1}{2} + j\right)}{\Gamma(n+1)}$$

(i) show that the large-order growth ($n \to \infty$) is

$$\alpha_n(j) \sim \frac{\cos(j\pi)}{\pi} \frac{(-1)^n (n-1)!}{2^n} \left(\alpha_0(j) - \frac{2\,\alpha_1(j)}{(n-1)} + \frac{2^2\,\alpha_2(j)}{(n-1)(n-2)} - \dots \right)$$

(ii) what is the significance of the $\cos(j\pi)$ prefactor?

All-Orders Steepest Descents: Darboux Theorem

- Darboux's theorem

$$f(z) \sim \phi(z) \left(1 - \frac{z}{z_0}\right)^{-g} + \psi(z) \qquad , \quad z \to z_0$$

- large-order growth of Taylor coefficients

$$b_n \sim \frac{\binom{n+g-1}{n}}{z_0^n}\left[\phi(z_0) - \frac{(g-1)\, z_0\, \phi'(z_0)}{(n+g-1)} + \frac{(g-1)(g-2)\, z_0^2\, \phi''(z_0)}{2!(n+g-1)(n+g-2)} - \dots\right.$$

All-Orders Steepest Descents: Darboux Theorem

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- log branch cut $\Rightarrow$

$$b_n \sim \frac{1}{z_0^n} \cdot \frac{1}{n} \left[\phi(z_0) - \frac{z_0\, \phi'(z_0)}{(n-1)} + \frac{z_0^2\, \phi''(z_0)}{(n-1)(n-2)} - \dots \right]$$

- apply this in the Borel plane $\Rightarrow$ large-order/low-order resurgence relations

Exercise 4:

(i) investigate Darboux's theorem numerically for the hypergeometric function, which has a branch point at $z = 1$

$$ {}_2F_1\left(a, b, c; z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)\, n!} z^n $$

(ii) what happens if $a + b - c =$ integer?

Resurgence: canonical example = Airy function

- formal large x solution to ODE: "perturbation theory"

$$y'' = x\,y \;\Rightarrow\; \begin{Bmatrix} 2\,\mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} \sim \frac{e^{\mp\frac{2}{3}x^{3/2}}}{2\pi^{3/2}\,x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!\,\left(\frac{4}{3}\,x^{3/2}\right)^n}$$

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- non-perturbative connection formula:

$$\mathrm{Ai}\left(e^{\mp \frac{2\pi i}{3}}\, x\right) = \pm \frac{i}{2} e^{\mp \frac{\pi i}{3}} \mathrm{Bi}\,(x) + \frac{1}{2} e^{\mp \frac{\pi i}{3}} \mathrm{Ai}\,(x)$$

- how do we recover this from the series?

Resurgence: canonical example = Airy function

$$\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}\,x\right)=\pm\frac{i}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Bi}\left(x\right)+\frac{1}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Ai}\left(x\right)$$

```
Plot[{Re[AiryAi[Exp[2 π I / 3] x]], 1 / 2 Re[-I Exp[I π / 3] AiryBi[x]]},
 {x, 0, 5}, PlotStyle → {{Blue, Thickness[0.02], Opacity[.3]}, {Red, Thick}},
 AxesStyle → Medium]
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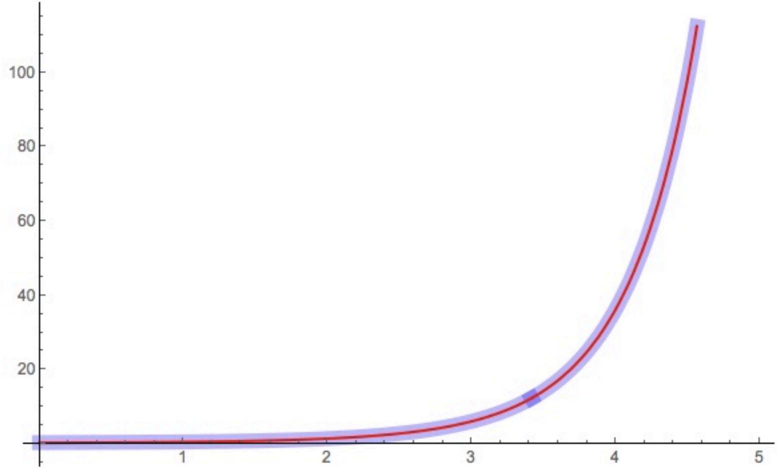

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```
Plot[{Re[AiryAi[Exp[2 π I / 3] x]] - 1 / 2 Re[-I Exp[I π / 3] AiryBi[x]]},
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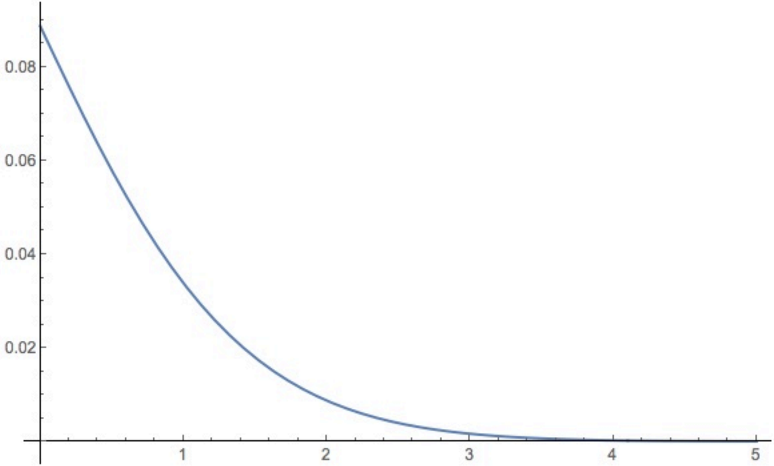
```



Resurgence: canonical example = Airy function

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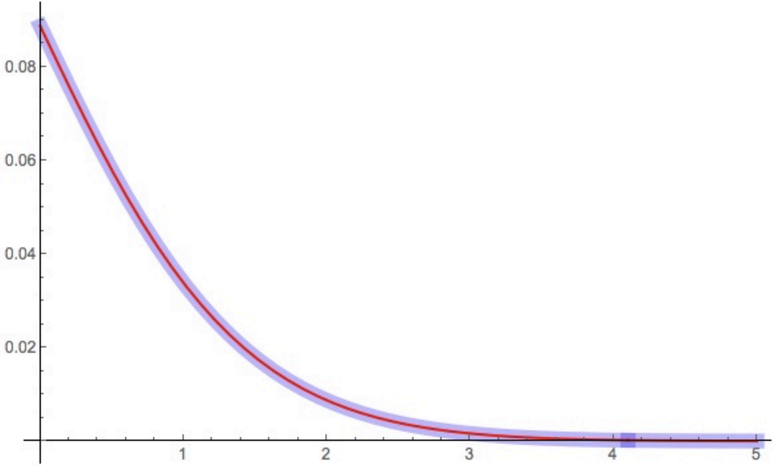

Resurgence: canonical example = Airy function

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- how do we recover this from the series?

Resurgence: canonical example = Airy function

- Borel sum of the Ai(x) series factor:

$$\sum_{n=0}^{\infty} (-1)^n \, \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!}\,\frac{t^n}{n!} = \ {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -t\right)$$

- inverse recovers the Ai(x) formal series:

$$Z(x) = \frac{4}{3}x^{3/2}\int_0^\infty dt\, e^{-\frac{4}{3}x^{3/2}t}\ {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -t\right)$$

Resurgence: canonical example = Airy function

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- cut for $t\in(-\infty,-1]$: rotate t contour as x rotates

Resurgence: canonical example = Airy function

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$${}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t+i\,\epsilon\right)-{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t-i\,\epsilon\right)=i\;{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;1-t\right)$$

Resurgence: canonical example = Airy function

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- discontinuity across cut $\Rightarrow$ non-pert. connection formula

$$Z\left(e^{\frac{2\pi i}{3}}\,x\right)-Z\left(e^{-\frac{2\pi i}{3}}\,x\right)=i\,e^{-\frac{4}{3}x^{3/2}}Z\left(x\right)$$

Resurgence: canonical example = Airy function

- formal large x solution to ODE: "perturbation theory"

$$y'' = x\,y \;\Rightarrow\; \left\{\begin{matrix} 2\,\mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix}\right\} \sim \frac{e^{\mp\frac{2}{3}x^{3/2}}}{2\pi^{3/2}\,x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!\,\left(\frac{4}{3}\,x^{3/2}\right)^n}$$

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- Borel summation encodes this non-perturbative effect

Exercise 5:

Use the property of the hypergeometric function for the jump across the cut

$$ {}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t+i\,\epsilon\right) - {}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t-i\,\epsilon\right) = i\;{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;1-t\right) $$

to derive the non-perturbative connection formula for the Airy function on the previous page.