Introduction to Resurgence and Non-perturbative Physics

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GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414, 1603.04924

recent KITP Program: Resurgent Asymptotics in Physics and Mathematics, Fall 2017
future Isaac Newton Institute Programme: Universal Resurgence, 2020/2021
1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   ▶ asymptotic series in physics; Borel summation
   ▶ trans-series completions & resurgence
   ▶ examples: linear and nonlinear ODEs

2. Lecture 2: Applications to Quantum Mechanics and QFT
   ▶ instanton gas, saddle solutions and resurgence
   ▶ infrared renormalon problem in QFT
   ▶ Picard-Lefschetz thimbles

3. Lecture 3: Resurgence and Large $N$
   ▶ Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT

4. Lecture 4: Resurgence and Phase Transitions
   ▶ Gross-Witten-Wadia Matrix Model
Physical Motivation

• non-perturbative definition of non-trivial QFT, in the continuum
• analytic continuation of path integrals
• "sign problem" in finite density QFT
• dynamical & non-equilibrium physics from path integrals (strong coupling)
• uncover hidden ‘magic’ in perturbation theory
• new understanding of weak-strong coupling dualities
• infrared renormalon puzzle in asymptotically free QFT
• exponentially improved asymptotics & resummation
Physical Motivation

- sign problem: "complex probability" at finite baryon density?

$$\int \mathcal{D}A \, e^{-S_{YM}[A]} + \ln \det(\mathcal{D} + m + i\mu\gamma^0)$$

- phase transitions and Lee-Yang & Fisher zeroes
Physical Motivation

- equilibrium thermodynamics $\leftrightarrow$ Euclidean path integral
- Kubo-Martin-Schwinger: antiperiodic b.c.’s for fermions

\begin{itemize}
  \item non-equilibrium physics $\leftrightarrow$ Minkowski path integral
  \item Schwinger-Keldysh time contours
  \item quantum transport in strongly-coupled systems
\end{itemize}
Physical Motivation

what does a Minkowski path integral mean, computationally?

\[ \int \mathcal{D}A \exp \left( \frac{i}{\hbar} S[A] \right) \quad \text{versus} \quad \int \mathcal{D}A \exp \left( -\frac{1}{\hbar} S[A] \right) \]

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \left( \frac{1}{3} t^3 + x t \right)} \, dt \sim \begin{cases} 
\frac{e^{-\frac{2}{3} x^{3/2}}}{2\sqrt{\pi} x^{1/4}}, & x \to +\infty \\
\frac{\sin \left( \frac{2}{3} \left(-x\right)^{3/2} + \frac{\pi}{4} \right)}{\sqrt{\pi} \left(-x\right)^{1/4}}, & x \to -\infty
\end{cases} \]

- massive cancellations \Rightarrow \quad \text{Ai}(+5) \approx 10^{-4}
Physical Motivation

• what does a Minkowski path integral mean?

\[ \int \mathcal{D}A \exp \left( \frac{i}{\hbar} S[A] \right) \quad \text{versus} \quad \int \mathcal{D}A \exp \left( -\frac{1}{\hbar} S[A] \right) \]

• since we need complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to expect to require similar tools also for path integrals

• an obvious idea, but how to make it work ... ?
Mathematical Motivation

Resurgence: ‘new’ idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

• perturbation theory generally ⇒ divergent series
• series expansion → trans-series expansion
• trans-series ‘well-defined under analytic continuation’
• perturbative and non-perturbative physics entwined
• applications: ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, String Theory, ...

• philosophical shift:
go beyond the Gaussian approximation and view semiclassical expansions as potentially exact
Perturbation Theory and Asymptotics

- in physical applications, expansions in a small parameter are often, but not always, divergent asymptotic series
- such an expansion is often the best we can do, and sometimes it is the only thing we can do
- it is worth understanding how to extract as much physical information as possible from an asymptotic expansion
- resurgence has the potential to lead to significant improvements, both analytically and numerically
- resurgence relates perturbative expansions to non-perturbative physics in surprisingly explicit ways
It’s a bit of black magic, to figure things out about differential equations even though you can’t solve them

Michael Atiyah
• an interesting observation by Hardy:

No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, *Divergent Series*, 1949

• deep result: “this is all we need” (J. Écalle, 1980)

• also as a closed logic system: Dahn and Göring (1980)
Resurgent Trans-Series

- Écalle: \textit{resurgent functions} closed under all operations:
  
  \[(\text{Borel transform}) + (\text{analytic continuation}) + (\text{Laplace transform})\]

- basic trans-series expansion in QM & QFT applications:
  
  \[
f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} c_{k,l,p} g^{2p} \left( \exp \left[-\frac{c}{g^2}\right] \right)^k \left( \ln \left[\pm \frac{1}{g^2}\right] \right)^l
  \]

  - perturbative fluctuations
  - \(k\)-instantons
  - quasi-zero-modes

- new: analytic continuation encoded in trans-series
- new: trans-series coefficients \(c_{k,l,p}\) highly correlated
- new: exponentially improved asymptotics
Resurgent Trans-Series

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(Borel transform) + (analytic continuation) + (Laplace transform)

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• *trans-monomial elements*: \( g^2, e^{-\frac{1}{g^2}}, \ln(g^2) \), are familiar

• “multi-instanton calculus” in QFT
Resurgent Trans-Series

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  \]

  - *trans-monomial elements*: \( g^2, e^{-\frac{1}{g^2}}, \ln(g^2) \), are familiar
  - “multi-instanton calculus” in QFT
  - **new**: analytic continuation encoded in trans-series
  - **new**: trans-series coefficients \( c_{k,l,p} \) highly correlated
  - **new**: exponentially improved asymptotics
resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980

resurgence = global complex analysis with divergent series
Perturbation theory is generically divergent

- hard problem = easy problem + “small” correction
- perturbation theory generally $\rightarrow$ divergent series

E.g. QM ground state energy: $E = \sum_{n=0}^\infty c_n \text{(coupling)}^n$

- Zeeman: $c_n \sim (-1)^n (2n)!$
- Stark: $c_n \sim (2n)!$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon (Mathieu) potential: $c_n \sim n!$
- symmetric double-well: $c_n \sim n!$

note generic factorial growth of perturbative coefficients
Perturbation theory works

QED perturbation theory:

\[
\frac{g - 2}{2} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right) - (0.32848\ldots) \left( \frac{\alpha}{\pi} \right)^2 + (1.18124\ldots) \left( \frac{\alpha}{\pi} \right)^3 - 1.9097(20) \left( \frac{\alpha}{\pi} \right)^4 + 9.16(58) \left( \frac{\alpha}{\pi} \right)^5 + \ldots
\]

\[
\left[ \frac{1}{2} (g - 2) \right]_{\text{exper}} = 0.00115965218073(28)
\]

\[
\left[ \frac{1}{2} (g - 2) \right]_{\text{theory}} = 0.00115965218178(77)
\]

QCD: asymptotic freedom

\[
\beta(g_s) = -\frac{g_s^3}{16\pi^2} \left( \frac{11}{3} N_C - \frac{4}{3} N_F \right)
\]
Asymptotic Series vs Convergent Series

\[ f(x) = \sum_{n=0}^{N-1} c_n (x - x_0)^n + R_N(x) \]

convergent series:
\[ |R_N(x)| \rightarrow 0 \quad , \quad N \rightarrow \infty \quad , \quad x \quad \text{fixed} \]

asymptotic series:
\[ |R_N(x)| \ll |x - x_0|^N \quad , \quad x \rightarrow x_0 \quad , \quad N \quad \text{fixed} \]

\[ \rightarrow \quad \text{“optimal truncation”:} \]
\[ \text{truncate just before least term (} x \text{ dependent!)} \]
Asymptotic Series vs Convergent Series

**alternating asymptotic series:**

\[
\sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^n} \sim \frac{1}{x} e^{\frac{1}{x}} E_1 \left( \frac{1}{x} \right)
\]

\( (x = 0.1) \)  \hspace{2cm}  \( (x = 0.2) \)

optimal truncation order depends on \( x \):

\[ N_{\text{opt}} \approx \frac{1}{x} \]
Asymptotic Series vs Convergent Series

Exercise 1:

(i) Show that the magnitude of the summand, \( n! \, x^n \), is minimized for \( n \approx \frac{1}{x} \)

(ii) Compute the optimal order of truncation for the series with summand: \((-1)^n (2n)! \, x^n\)
Asymptotic Series vs Convergent Series

non-alternating asymptotic series: \[ \sum_{n=0}^{\infty} n! x^n \sim -\frac{1}{x} e^{-\frac{1}{x}} E_1 \left( -\frac{1}{x} \right) \]

optimal truncation order depends on \( x\): \[ N_{\text{opt}} \approx \frac{1}{x} \]
Asymptotic Series vs Convergent Series

- contrast with behavior of a convergent series, for which more terms always improves the answer, independent of $x$

$$\text{convergent series : } \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} x^n = \text{PolyLog}(2, -x)$$

\[ \begin{array}{c}
\text{Graph for } x = 0.75 \\
\text{Graph for } x = 1
\end{array} \]
asymptotic series are sometimes ‘better’

gold: exact
red: 1 term divergent about $x = -\infty$
green: 10 terms convergent about $x = 0$
black: 50 terms convergent about $x = 0$
Asymptotic Series: exponential precision

\[ \sum_{n=0}^{\infty} (-1)^n \frac{n!}{n!} x^n \sim \frac{1}{x} e^{\frac{1}{x}} E_1 \left( \frac{1}{x} \right) \]

optimal truncation: error term is exponentially small

\[ |R_N(x)|_{N \approx 1/x} \approx N! \frac{x^N}{N} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}} \]

- e.g. alternating exponential integral:

\[ (x = 0.1) \quad (x = 0.2) \]
Borel summation: basic idea

alternating factorially divergent series:

\[ \sum_{n=0}^{\infty} (-1)^n n! x^n = ? \]

write \( n! = \int_0^{\infty} dt \, e^{-t} t^n \)

\[ \sum_{n=0}^{\infty} (-1)^n n! x^n = \int_0^{\infty} dt \, e^{-t} \frac{1}{1 + x \, t} \]  (?)

integral is convergent for all \( x > 0 \): “Borel sum” of the series
Borel Summation: basic idea

\[ \sum_{n=0}^{\infty} (-1)^n n! x^n = \int_0^\infty dt \, e^{-t} \frac{1}{1 + x \, t} = \frac{1}{x} e^{\frac{1}{x}} E_1 \left( \frac{1}{x} \right) \]
Borel summation: basic idea

write $n! = \int_0^\infty dt \ e^{-t} \ t^n$

non-alternating factorially divergent series:

$$\sum_{n=0}^{\infty} n! \ x^n = \int_0^\infty dt \ e^{-t} \ \frac{1}{1 - x \ t} \quad (??)$$

pole on the (real, positive) Borel axis!
Borel summation: basic idea

write $n! = \int_0^\infty dt \, e^{-t} \, t^n$

non-alternating factorially divergent series:

$$
\sum_{n=0}^{\infty} n! \, x^n = \int_0^\infty dt \, e^{-t} \frac{1}{1 - x \, t} \quad (??)
$$

pole on the (real, positive) Borel axis!

$\Rightarrow$ non-perturbative imaginary part $= \pm \frac{i \, \pi}{x} \, e^{-\frac{1}{x}}$

but every term in the series is real !?!
Borel Summation: basic idea

Borel \Rightarrow \Re \left[ \sum_{n=0}^{\infty} n! x^n \right] = \mathcal{P} \int_0^\infty dt \frac{e^{-t}}{1 - x t} = \Re \left[ -\frac{1}{x} e^{-\frac{1}{x}} E_1 \left( -\frac{1}{x} \right) \right]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{borel_summation_diagram.png}
\end{figure}

- note: $E_1 \left( -\frac{1}{x} \right)$ also has an imaginary part $\pm i\pi$.
Borel Summation: basic idea

\[
\text{Borel} \Rightarrow \Re \left[ \sum_{n=0}^{\infty} n! \, x^n \right] = \mathcal{P} \int_0^\infty dt \, e^{-t} \frac{1}{1 - x \, t} = \Re \left[ -\frac{1}{x} \, e^{-\frac{1}{x}} \, E_1 \left( -\frac{1}{x} \right) \right]
\]

- note: \( E_1 \left( -\frac{1}{x} \right) \) also has an imaginary part = \( \pm i\pi \)

\[
-\frac{1}{x} \, e^{-\frac{1}{x}} \, E_1 \left( e^{\pm i \, \pi} \frac{1}{x} \right) = -\frac{1}{x} \, e^{-\frac{1}{x}} \left[ Ein \left( -\frac{1}{x} \right) - \ln x - \gamma \pm i \, \pi \right]
\]

- Borel encodes this non-perturbative "connection formula"
Borel summation

Borel transform of series, where $c_n \sim n!$, $n \to \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n g^n \rightarrow \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has a finite radius of convergence
Borel summation

**Borel transform** of series, where \( c_n \sim n! \), \( n \to \infty \)

\[
f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \quad \rightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n
\]

new series typically has a **finite** radius of convergence

**Borel resummation** of original asymptotic series:

\[
Sf(g) = \frac{1}{g} \int_{0}^{\infty} \mathcal{B}[f](t) e^{-t/g} dt
\]

**note:** \( \mathcal{B}[f](t) \) may have singularities in \((\text{Borel})\ t\) plane
Borel singularities

avoid singularities on $\mathbb{R}^+$: directional Borel sums:

$$S_\theta f (g) = \frac{1}{g} \int_0^{e^{i\theta} \infty} B[f](t)e^{-t/g} dt$$

go above/below the singularity: $\theta = 0^{\pm}$

$\longrightarrow$ non-perturbative ambiguity: $\pm \text{Im}[S_0 f (g)]$

physics challenge: use **physical input** to resolve ambiguity
Borel summation in practice

\[ f(g) \sim \sum_{n=0}^{\infty} c_n g^n, \quad c_n \sim \beta^n \Gamma(\gamma n + \delta) \]

- alternating series: real Borel sum

\[ f(g) \sim \frac{1}{\gamma} \int_{0}^{\infty} \frac{dt}{t} \left( \frac{1}{1 + t} \right) \left( \frac{t}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{t}{\beta g} \right)^{1/\gamma} \right] \]
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- nonalternating series: ambiguous imaginary part

\[ \text{Re } f(-g) \sim \frac{1}{\gamma} \mathcal{P} \int_0^\infty \frac{dt}{t} \left( \frac{1}{1-t} \right) \left( \frac{t}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{t}{\beta g} \right)^{1/\gamma} \right] \]

\[ \text{Im } f(-g) \sim \pm \frac{\pi}{\gamma} \left( \frac{1}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{1}{\beta g} \right)^{1/\gamma} \right] \]
Borel summation in practice

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\[ \text{Im } f(-g) \sim \pm \frac{\pi}{\gamma} \left( \frac{1}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{1}{\beta g} \right)^{1/\gamma} \right] \]

- \( \gamma \) determines power of coupling in the exponent
- \( \beta \) and \( \gamma \) determine coefficient in the exponent
- \( \beta, \gamma \) and \( \delta \) determine the prefactor
Exercise 2:

Use the integral representation of $\Gamma(\gamma n + \delta)$ to derive the expressions on the previous slide.

Hence deduce the meaning of the large-order behavior parameters $\beta$, $\gamma$ and $\delta$.

Compare with the second part of Exercise 1.
another view of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

resurgence = global complex analysis for divergent series

(analytic continuation, transforms, monodromy, ...)

⇒ “the trans-series really IS the function”

question: to what extent is this true/useful in physics?
Stirling expansion for $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ is divergent

$$\psi(1 + z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \cdots$$

- functional relation: $\psi(1 + z) = \psi(z) + \frac{1}{z}$

- reflection formula:
  $$\psi(1 + z) - \psi(1 - z) = \frac{1}{z} - \pi \cot(\pi z)$$

- formal series only has the two "perturbative" terms
- "raw" asymptotics is inconsistent
- resurgence: add infinite series of non-perturbative terms
- "non-perturbative completion"
Stirling expansion for $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ is divergent

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- functional relation: $\psi(1 + z) = \psi(z) + \frac{1}{z}$  \checkmark
- reflection formula: $\psi(1 + z) - \psi(1 - z) = \frac{1}{z} - \pi \cot(\pi z)$

$$\Rightarrow \quad \text{Im} \psi(1 + iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$
Stirling expansion for $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ is divergent

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- formal series only has the two "perturbative" terms
- "raw" asymptotics is inconsistent with analytic continuation
- resurgence: add infinite series of non-perturbative terms

"non-perturbative completion"
Resurgence: Preserving Analytic Continuation

\[ \text{Im } \psi(1 + iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y} \]

- function satisfies infinite order linear ODE
  \[ \Rightarrow \text{infinitely many exponential terms in trans-series} \]

Borel representation:

\[ \psi(1 + z) - \ln z = \int_0^\infty \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-zt} \, dt \]

- Borel transform: poles at \( t = \pm 2n\pi i, n = 1, 2, 3, \ldots \)
- meromorphic (poles, no cuts) \( \Rightarrow \text{no "fluctuation factors"} \)
- this simple example arises often in QFT: Euler-Heisenberg, finite temperature QFT, de Sitter, exact S-matrices, Chern-Simons partition functions, matrix models, \ldots \]
Euler-Heisenberg and Matrix Models, Large N, Strings, ...

• scalar QED EH in self-dual background ($F = \pm \tilde{F}$):

$$S = \frac{F^2}{16\pi^2} \int_0^{\infty} \frac{dt}{t} e^{-t/F} \left( \frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3} \right)$$

• Gaussian matrix model: $\lambda = g N$

$$\mathcal{F} = -\frac{1}{4} \int_0^{\infty} \frac{dt}{t} e^{-2\lambda t/g} \left( \frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3} \right)$$

• $c = 1$ String: $\lambda = g N$

$$\mathcal{F} = \frac{1}{4} \int_0^{\infty} \frac{dt}{t} e^{-2\lambda t/g} \left( \frac{1}{\sin^2(t)} - \frac{1}{t^2} - \frac{1}{3} \right)$$

• Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4} \sum_{m \in \mathbb{Z}} \int_0^{\infty} \frac{dt}{t} e^{-2(\lambda+2\pi i m) t/g} \left( \frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3} \right)$$
Borel summation in practice: Borel cuts

Comments:

• an isolated pole in the Borel transform corresponds to a single "instanton" exponential term (e.g. previous exponential integral function example)

• physically, we may expect fluctuations about instantons. These correspond to branch point singularities, and their associated branch cuts, in the Borel plane
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\[
\Gamma \left( s, \frac{1}{x} \right) = \int_{1/x}^{\infty} t^{s-1} e^{-t} \, dt = x^{-s} e^{-1/x} \int_{0}^{\infty} e^{-t/x} (1 + t)^{s-1} \, dt \\
\sim x^{1-s} e^{-1/x} \sum_{n=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s-n)} x^{n}
\]
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\]

\[
\sim x^{1-s} e^{-1/x} \sum_{n=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s-n)} x^n
\]

• truncation when \( s = \) integer ("SUSY" & "localization")

• resurgence is more interesting with several parameters
• trans-series from 2\textsuperscript{nd} order linear ODE has 2 non-perturbative exponential terms (WKB)

• trans-series from n\textsuperscript{th} order linear ODE has n non-perturbative exponential terms

• the fluctuations about these different ("instanton") exponentials are related by generic large order/low order resurgence relations
All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

  \[ I^{(n)}(g^2) = \int_{C_n} dz \, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}} e^{-\frac{1}{g^2} f_n} T^{(n)}(g^2) \]

- \( T^{(n)}(g^2) \): beyond the usual Gaussian approximation

- asymptotic expansion of fluctuations about the saddle \( n \):

  \[ T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)} g^{2r} \]
All-Orders Steepest Descents: Darboux Theorem

- Berry/Howls: exact resurgent relation between fluctuations about $n^{th}$ saddle and about neighboring saddles $m$

$$T(n)(g^2) = \frac{1}{2\pi i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T(m) \left( \frac{F_{nm}}{v} \right)$$

- proof is based on contour deformation
All-Orders Steepest Descents: Darboux Theorem

• Berry/Howls: exact resurgent relation between fluctuations about \( n^{th} \) saddle and about neighboring saddles \( m \)

\[
T^{(n)}(g^2) = \frac{1}{2\pi i} \sum_m (-1)^\gamma_{nm} \int_0^\infty dv \frac{e^{-v}}{v} \frac{1}{1 - g^2v/(F_{nm})} T^{(m)} \left( \frac{F_{nm}}{v} \right)
\]

• proof is based on contour deformation

• universal factorial divergence of fluctuations

\[
T^{(n)}_r = (r - 1)! \sum_m \frac{(-1)^\gamma_{nm}}{(F_{nm})^r} \left[ T^{(m)}_0 + \frac{F_{nm}}{(r - 1)} T^{(m)}_1 + \frac{(F_{nm})^2}{(r - 1)(r - 2)} T^{(m)}_2 + \ldots \right]
\]

• alternative proof from Darboux’s theorem in the Borel plane

fluctuations about different saddles are explicitly related!
All-Orders Steepest Descents: Darboux Theorem

- example

\( d = 0 \) partition function for periodic potential \( V(z) = \sin^2(z) \)

\[
I(g^2) = \int_0^\pi dz \ e^{-\frac{1}{g^2} \sin^2(z)}
\]

- this is a Bessel function

- two saddle points: \( z_0 = 0 \) and \( z_1 = \frac{\pi}{2} \).
All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle $z_0$:

$$T_r^{(0)} = \frac{\Gamma\left(r + \frac{1}{2}\right)^2}{\sqrt{\pi} \Gamma(r + 1)} \sim \frac{(r - 1)!}{\sqrt{\pi}} \left(1 - \frac{1}{4(r - 1)} + \frac{9}{32} + \frac{75}{128} + \ldots\right) + .$$
All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle \( z_0 \):

\[
T^{(0)}_r = \frac{\Gamma(r + \frac{1}{2})^2}{\sqrt{\pi} \Gamma(r + 1)}
\sim \frac{(r - 1)!}{\sqrt{\pi}} \left( 1 - \frac{1}{4} \frac{9}{32} - \frac{75}{128} \right)
\]

- low order coefficients about saddle \( z_1 \):

\[
T^{(1)}(g^2) \sim i \sqrt{\pi} \left( 1 - \frac{1}{4} g^2 + \frac{9}{32} g^4 - \frac{75}{128} g^6 + \ldots \right)
\]

- fluctuations about the two saddles are explicitly related

- simple example of a generic resurgent large-order/low-order perturbative/non-perturbative relation
Exercise 3: the modified Bessel function has the large $x$ asymptotic expansion:

$$I_j(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n(j)}{x^n} \pm ie^{ij\pi} \frac{e^{-x}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{\alpha_n(j)}{x^n}, \quad \left| \arg(x) - \frac{\pi}{2} \right| < \pi$$

where the coefficients are

$$\alpha_j(n) = \frac{\cos(\pi j)}{\pi} \left(-\frac{1}{2}\right)^n \frac{\Gamma(n + \frac{1}{2} - j) \Gamma(n + \frac{1}{2} + j)}{\Gamma(n + 1)}$$

(i) show that the large-order growth ($n \to \infty$) is

$$\alpha_n(j) \sim \frac{\cos(j\pi)}{\pi} \left(-1\right)^n (n-1)! \left(\alpha_0(j) - \frac{2 \alpha_1(j)}{(n-1)} + \frac{2^2 \alpha_2(j)}{(n-1)(n-2)} - \ldots\right)$$

(ii) what is the significance of the $\cos(j\pi)$ prefactor?
Darboux’s theorem

\[ f(z) \sim \phi(z) \left(1 - \frac{z}{z_0}\right)^{-g} + \psi(z), \quad z \to z_0 \]

large-order growth of Taylor coefficients

\[ b_n \sim \frac{\binom{n + g - 1}{n}}{z_0^n} \left[ \phi(z_0) - \frac{(g - 1) z_0 \phi'(z_0)}{(n + g - 1)} + \frac{(g - 1)(g - 2) z_0^2 \phi''(z_0)}{2!(n + g - 1)(n + g - 2)} - \cdots \right] \]
All-Orders Steepest Descents: Darboux Theorem

- Darboux’s theorem

\[ f(z) \sim \phi(z) \left(1 - \frac{z}{z_0}\right)^{-g} + \psi(z), \quad z \rightarrow z_0 \]

- large-order growth of Taylor coefficients

\[ b_n \sim \left(\frac{n + g - 1}{n}\right) \frac{\phi(z_0)}{z^n_0} - \frac{(g - 1) z_0 \phi'(z_0)}{(n + g - 1)} + \frac{(g - 1)(g - 2) z_0^2 \phi''(z_0)}{2!(n + g - 1)(n + g - 2)} - \ldots \]

- log branch cut \( \Rightarrow \)

\[ b_n \sim \frac{1}{z^n_0} \cdot \frac{1}{n} \left[ \phi(z_0) - \frac{z_0 \phi'(z_0)}{(n - 1)} + \frac{z_0^2 \phi''(z_0)}{(n - 1)(n - 2)} - \ldots \right] \]

- apply this in the Borel plane \( \Rightarrow \) large-order/low-order resurgence relations
Exercise 4:

(i) investigate Darboux’s theorem numerically for the hypergeometric function, which has a branch point at $z = 1$

$$2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n + a)\Gamma(n + b)}{\Gamma(n + c)n!} z^n$$

(ii) what happens if $a + b - c = \text{integer}$?
Resurgence: canonical example = Airy function

- formal large $x$ solution to ODE: "perturbation theory"

$$y'' = x y \Rightarrow \left\{ \begin{array}{l} 2 \text{Ai}(x) \\ \text{Bi}(x) \end{array} \right\} \sim \frac{e^{\mp \frac{2}{3} x^{3/2}}}{2\pi^{3/2} x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma(n + \frac{1}{6}) \Gamma(n + \frac{5}{6})}{n! \left(\frac{4}{3} x^{3/2}\right)^n}$$
Resurgence: canonical example = Airy function

- formal large $x$ solution to ODE: "perturbation theory"

$$y'' = xy \Rightarrow \left\{ \frac{2 \text{Ai}(x)}{\text{Bi}(x)} \right\} \sim \frac{e^{\pm \frac{2}{3} x^{3/2}}}{2\pi^{3/2} x^{1/4}} \sum_{n=0}^{\infty} (\pm 1)^n \frac{\Gamma \left( n + \frac{1}{6} \right) \Gamma \left( n + \frac{5}{6} \right)}{n! \left( \frac{4}{3} x^{3/2} \right)^n}$$

- non-perturbative connection formula:

$$\text{Ai} \left( e^{\pm \frac{2\pi i}{3}} x \right) = \pm \frac{i}{2} e^{\pm \frac{\pi i}{3}} \text{Bi} \left( x \right) + \frac{1}{2} e^{\pm \frac{\pi i}{3}} \text{Ai} \left( x \right)$$

- how do we recover this from the series?
Resurgence: canonical example = Airy function

\[ \text{Ai} \left( e^{\pm \frac{2\pi i}{3}} x \right) = \pm \frac{i}{2} e^{\mp \frac{\pi i}{3}} \text{Bi} (x) + \frac{1}{2} e^{\mp \frac{\pi i}{3}} \text{Ai} (x) \]
Resurgence: canonical example = Airy function

\[\text{Ai}\left(e^{±\frac{2\pi i}{3}} x\right) = ±\frac{i}{2} e^{±\frac{\pi i}{3}} \text{Bi}(x) + \frac{1}{2} e^{±\frac{\pi i}{3}} \text{Ai}(x)\]

Plot:

\[
\text{Plot}\left\{\{\text{Re}[\text{AiryAi}[\text{Exp}[2\pi I / 3] x]] - 1/2 \text{Re}[-I \text{Exp}[I \pi / 3] \text{AiryBi}[x]]\},
\{x, 0, 5\}, \text{PlotStyle} \rightarrow \text{Thick}, \text{AxesStyle} \rightarrow \text{Medium}\right\}
\]
Resurgence: canonical example = Airy function

\[ \text{Ai} \left( e^{\pm \frac{2\pi i}{3} x} \right) = \pm \frac{i}{2} e^{\mp \frac{\pi i}{3}} \text{Bi}(x) + \frac{1}{2} e^{\mp \frac{\pi i}{3}} \text{Ai}(x) \]
• formal large $x$ solution to ODE: "perturbation theory"

$$ y'' = x y \Rightarrow \left\{ \frac{2 \text{Ai}(x)}{\text{Bi}(x)} \right\} \sim \frac{e^{\mp \frac{2}{3} x^{3/2}}}{2 \pi^{3/2} x^{1/4}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma \left( n + \frac{1}{6} \right) \Gamma \left( n + \frac{5}{6} \right)}{n! \left( \frac{4}{3} x^{3/2} \right)^n} $$

• non-perturbative connection formula:

$$ \text{Ai} \left( e^{\mp \frac{2\pi i}{3}} x \right) = \pm \frac{i}{2} e^{\mp \frac{\pi i}{3}} \text{Bi} (x) + \frac{1}{2} e^{\mp \frac{\pi i}{3}} \text{Ai} (x) $$

• how do we recover this from the series?
Resurgence: canonical example = Airy function

• Borel sum of the Ai(x) series factor:

\[
\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + \frac{1}{6}) \Gamma(n + \frac{5}{6})}{n!} \frac{t^n}{n!} = \ _2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
\]

• inverse recovers the Ai(x) formal series:

\[
Z(x) = \frac{4}{3} x^{3/2} \int_0^{\infty} dt \, e^{-\frac{4}{3} x^{3/2} t} \ _2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
\]
Resurgence: canonical example = Airy function

- Borel sum of the Ai(x) series factor:
  \[
  \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + \frac{1}{6}) \Gamma(n + \frac{5}{6})}{n!} \frac{t^n}{n!} = \, _2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
  \]

- inverse recovers the Ai(x) formal series:
  \[
  Z(x) = \frac{4}{3} x^{3/2} \int_0^\infty dt \, e^{-\frac{4}{3} x^{3/2} t} \, _2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
  \]

- cut for \( t \in (-\infty, -1] \): rotate \( t \) contour as \( x \) rotates
Resurgence: canonical example = Airy function

- Borel sum of the Ai(x) series factor:

\[
\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + \frac{1}{6}) \Gamma(n + \frac{5}{6}) t^n}{n! n!} = 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
\]

- inverse recovers the Ai(x) formal series:

\[
Z(x) = \frac{4}{3} x^{3/2} \int_0^\infty dt \ e^{-\frac{4}{3} x^{3/2} t} \ 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
\]

- cut for \( t \in (-\infty, -1] \): rotate \( t \) contour as \( x \) rotates

\[
2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; t + i \epsilon \right) - 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; t - i \epsilon \right) = i \ 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; 1 - t \right)
\]
Resurgence: canonical example = Airy function

• Borel sum of the Ai(x) series factor:

\[
\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + \frac{1}{6}) \Gamma(n + \frac{5}{6})}{n!} \frac{t^n}{n!} = 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
\]

• inverse recovers the Ai(x) formal series:

\[
Z(x) = \frac{4}{3} x^{3/2} \int_0^\infty dt \, e^{-\frac{4}{3} x^{3/2} t} \, 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; -t \right)
\]

• cut for \( t \in (-\infty, -1] \): rotate \( t \) contour as \( x \) rotates

\[
2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; t + i \epsilon \right) - 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; t - i \epsilon \right) = i \, 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; 1 - t \right)
\]

• discontinuity across cut \( \Rightarrow \) non-pert. connection formula

\[
Z \left( e^{\frac{2\pi i}{3}} x \right) - Z \left( e^{-\frac{2\pi i}{3}} x \right) = i \, e^{-\frac{4}{3} x^{3/2}} Z \left( x \right)
\]
Resurgence: canonical example = Airy function

- formal large $x$ solution to ODE: "perturbation theory"

$$y'' = xy \Rightarrow \left\{ \frac{2 \text{Ai}(x)}{\text{Bi}(x)} \right\} \sim \frac{e^{\mp \frac{2}{3} x^{3/2}}}{2\pi^{3/2} x^{1/4}} \sum_{n=0}^{\infty} (\pm 1)^n \frac{\Gamma(n + \frac{1}{6}) \Gamma(n + \frac{5}{6})}{n! \left(\frac{4}{3} x^{3/2}\right)^n}$$

- non-perturbative connection formula:

$$\text{Ai}\left(e^{\mp \frac{2\pi i}{3}} x\right) = \pm \frac{i}{2} e^{\mp \frac{\pi i}{3}} \text{Bi}(x) + \frac{1}{2} e^{\mp \frac{\pi i}{3}} \text{Ai}(x)$$

- Borel summation encodes this non-perturbative effect
Exercise 5:

Use the property of the hypergeometric function for the jump across the cut

\[ 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; t + i \epsilon \right) - 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; t - i \epsilon \right) = i \ 2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1; 1 - t \right) \]

to derive the non-perturbative connection formula for the Airy function on the previous page.