Introduction to Resurgence and Non-perturbative Physics

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GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414, 1603.04924

recent KITP Program: *Resurgent Asymptotics in Physics and Mathematics*, Fall 2017

future Isaac Newton Institute Programme: *Universal Resurgence*, 2020/2021

Resurgence, Trans-series and Non-perturbative Physics

1. Lecture 1: Basic Formalism of Trans-series and Resurgence
   - asymptotic series in physics; Borel summation
   - trans-series completions & resurgence
   - examples: linear and nonlinear ODEs
2. Lecture 2: Applications to Quantum Mechanics and QFT
   - instanton gas, saddle solutions and resurgence
   - infrared renormalon problem in QFT
   - Picard-Lefschetz thimbles
3. Lecture 3: Resurgence and Large N
   - Mathieu equation and Nekrasov-Shatashvili limit of $\mathcal{N} = 2$ SUSY QFT
4. Lecture 4: Resurgence and Phase Transitions
   - Gross-Witten-Wadia Matrix Model

Resurgence: canonical example = Airy function

"path integral"

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\left(x\, t+\frac{t^3}{3}\right)} = \frac{\sqrt{r}}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, e^{r^{3/2}\left(e^{i\theta}\, z-\frac{z^3}{3}\right)}$$

- we have written $x \equiv r\, e^{i\theta}$, $t \equiv -i\sqrt{r} z$

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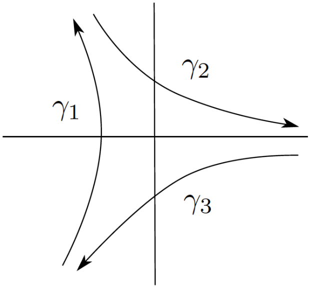
- we have written $x \equiv r\,e^{i\theta}$, $t \equiv -i\sqrt{r} z$
- basis of allowed contours

$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i} \int_{\gamma_k} dz\, e^{r^{3/2}\left(e^{i\theta}\,z - \frac{z^3}{3}\right)}$$



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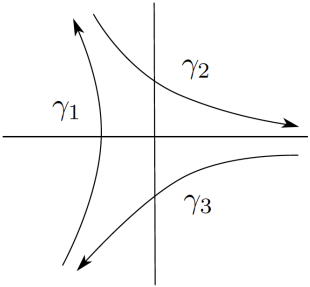

- saddles at $z = \pm e^{i\theta/2}$; these move as θ varies
- saddle exponent ($\equiv$ "action") $= \pm\frac{2}{3} r^{3/2} e^{3i\theta/2}$
- steepest descent contours: $\mathrm{Im}\left[e^{i\theta}\, z - \frac{z^3}{3}\right] = \pm\frac{2}{3}\sin\left(\frac{3\theta}{2}\right)$

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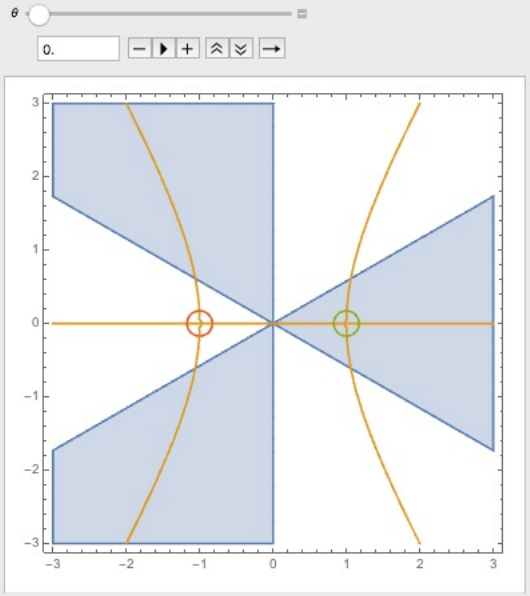

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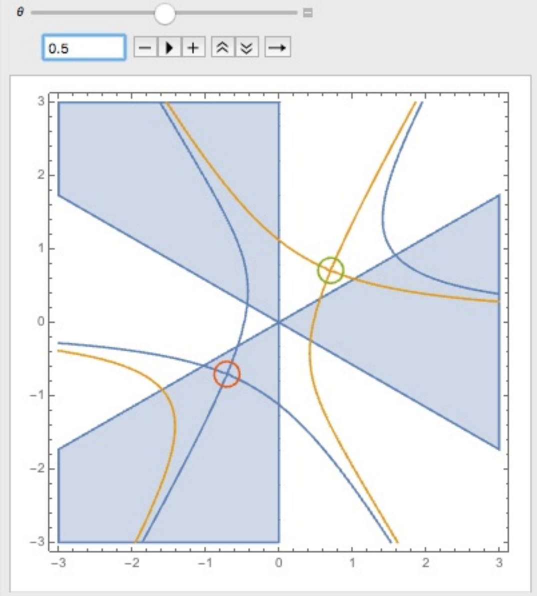

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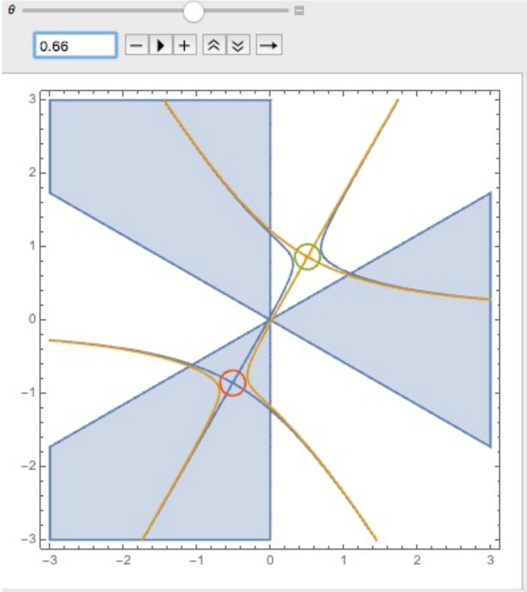

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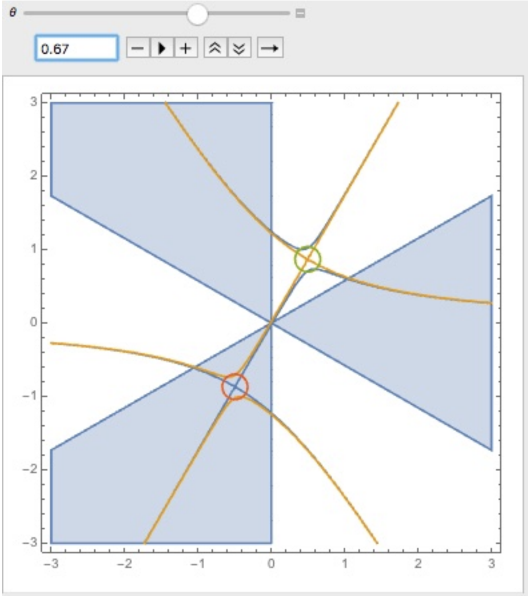

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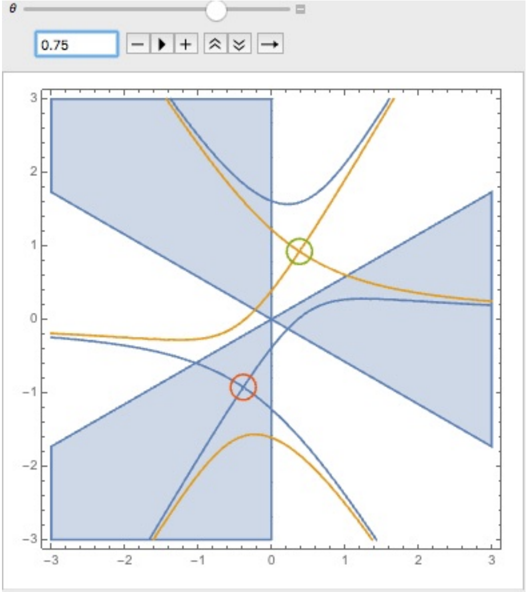

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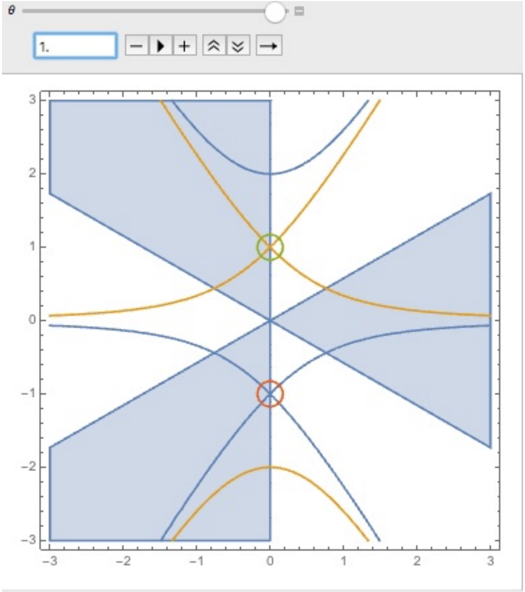

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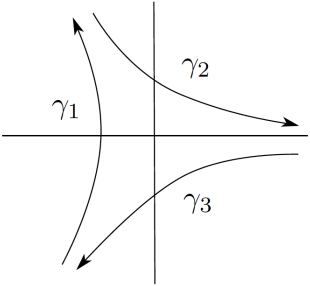


- saddles at $z = \pm e^{i\theta/2}$
- saddle exponent ($\equiv$ "action") $= \pm\frac{2}{3} r^{3/2} e^{3i\theta/2}$

$x > 0 \Rightarrow \theta = 0 \Rightarrow$ contour through only 1 saddle $(z = -1)$

$$\Rightarrow \text{action} = -\frac{2}{3} r^{3/2} = -\frac{2}{3} x^{3/2}$$

$x < 0 \Rightarrow \theta = \pm\pi \Rightarrow$ contour through 2 saddles $(z = \pm i)$

$$\Rightarrow \text{action} = \pm i \frac{2}{3} r^{3/2} = \pm i \frac{2}{3} (-x)^{3/2}$$

Resurgence: canonical example = Airy function

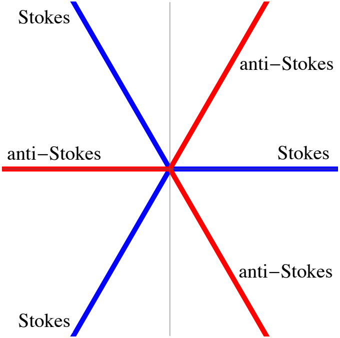
$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i}\int_{\gamma_k} dz\, e^{r^{3/2}\left(e^{i\theta}\, z - \frac{z^3}{3}\right)}$$

- saddles at $z = \pm e^{i\theta/2}$, action $= \pm\frac{2}{3}r^{3/2}e^{3i\theta/2}$
- real action when $\theta = 0, \pm\frac{2\pi}{3}$: "Stokes lines"
- imaginary action when $\theta = \pi, \pm\frac{\pi}{3}$: "anti-Stokes lines"

Stokes lines in complex x-plane

$$x = r\, e^{i\,\theta}$$

moral: keep track of **both** saddle contributions as we analytically continue in complex x plane



Phase Transitions as Stokes Transitions

Stokes jumps: as a parameter in the "path integral" changes (possibly in the complex plane) saddles can "appear" and "disappear".

idea: associate Stokes transitions with physical phase transitions

$$\text{phase transition} = \text{change of dominant saddle(s)}$$

Resurgence: canonical example = Airy function

- expansions about the two saddles are explicitly related

$$a_n = \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^n n!} = \left\{1, \frac{5}{48}, \frac{385}{4608}, \frac{85085}{663552}, \dots\right\}$$

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- large order behavior:

$$a_n \sim \frac{(n-1)!}{(2\pi)\left(\frac{4}{3}\right)^n}\left(1 - \frac{5}{36}\frac{1}{n} + \frac{25}{2592}\frac{1}{n^2} - \dots\right)$$

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- large order/low order relation: generic resurgence

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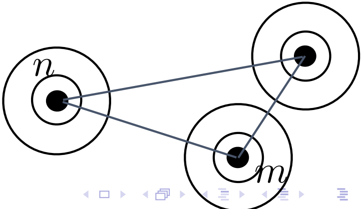
generic large order/low order relation



- trans-series from n^{th} order linear ODE has n non-perturbative exponential terms

- trans-series from nonlinear ODE has infinitely many non-perturbative exponential terms

- e.g.: $y_1(x) \times y_2(x)$ satisfies 3^{rd} order **linear** ODE
  but $y_1(x)/y_2(x)$ satisfies 2^{nd} order **non-linear** ODE

- also generalizes to (some) PDE's, linear and non-linear

- Painlevé = "special functions of nonlinear ODE's"
many physical applications: fluids, statistical physics, gravity, random matrices, matrix models, optics, QFT, strings, ...

- resurgent trans-series are the natural language for their asymptotics

Resurgence in Nonlinear ODEs: e.g. Painlevé II

Painlevé II:

$$y'' = x\, y(x) + 2\, y^3(x)$$

- "non-linear Airy function"
- Tracy-Widom law for statistics of max. eigenvalue for Gaussian random matrices
- correlators in polynuclear growth; directed polymers (KPZ)
- double-scaling limit in unitary matrix models
- double-scaling limit in 2d Yang-Mills
- double-scaling limit in 2d supergravity
- non-intersecting Brownian motions
- longest increasing subsequence in random permutations
- ... universal !

Resurgence in Nonlinear ODEs: e.g. Painlevé II

$$y'' = x\, y(x) + 2\, y^3(x)$$

- $x \to +\infty$ asymptotics: $y'' \approx x\, y(x) + \ldots$

$$y \to 0 \quad \text{as} \quad x \to +\infty \quad \Rightarrow \quad y_+^{(1)}(x) \sim \sigma_+ \, \mathrm{Ai}(x) + \ldots$$

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- trans-series solution generated from ODE:

$$y_+(x) \sim \sum_{k=1}^{\infty} \left(\sigma_+ \, \frac{e^{-\frac{2}{3} x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} \right)^{2k-1} y_+^{(k)}(x)$$

- infinite number of non-perturbative terms
- fluctuations factorially divergent & **alternating**
- σ_+ = **real** trans-series parameter (for real solution)
- higher fluctuations determined by lower fluctuations

Resurgence in Nonlinear ODEs: e.g. Painlevé II

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$$y_-^{(0)}(x) \sim \sqrt{\frac{-x}{2}}\left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \dots\right)$$

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- non-pert. corrections "beyond all orders": $y = y_{\text{pert}} + \delta y$

$$\delta y'' = \left(x + 6 y_{\text{pert}}^2\right) \delta y \sim \left(-2x - \frac{3}{4x^2} + \dots \right) \delta y$$

$$\delta y \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2}\frac{2}{3}(-x)^{3/2}} \left(1 - \frac{\frac{17}{72}}{\sqrt{2}\frac{2}{3}(-x)^{3/2}} + \frac{\frac{1513}{10368}}{(\sqrt{2}\frac{2}{3}(-x)^{3/2})^2} - \dots \right)$$

using resurgence and Exercise 2, we can now make a non-trivial prediction for the large-order growth of the perturbative expansion coefficients:

$$c_n \sim \beta^n \, \Gamma(\gamma \, n + \delta)$$

$$\mathrm{Im}\, f(-g) \sim \pm \frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

$$\delta y \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2}\frac{2}{3}(-x)^{3/2}}$$

(i) we learn that: $\gamma = 2$, $\beta = \frac{9}{8}$, and $\delta = -\frac{5}{2}$

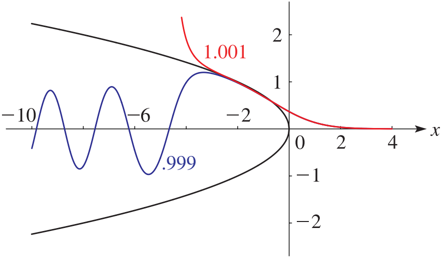
(ii) subleading large-order growth terms also:

$$c_n^{(0)} \sim \# \left(\frac{9}{8}\right)^n \Gamma\left(2n - \frac{5}{2}\right) \left(1 + \frac{\frac{17}{72}}{\left(2n - \frac{7}{2}\right)} + \frac{\frac{1513}{10368}}{\left(2n - \frac{7}{2}\right)\left(2n - \frac{9}{2}\right)} - \dots\right)$$

Resurgence in Nonlinear ODEs: e.g. Painlevé II

$$y'' = x\, y(x) + 2\, y^3(x) \quad , \quad y(x) \sim \sigma_+ \, \mathrm{Ai}(x) \quad , \quad x \to +\infty$$

- trans-series structurally different as $x \to \pm\infty$
- note different exponents!



$$x \to +\infty \Rightarrow \frac{e^{-\frac{2}{3} x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}$$

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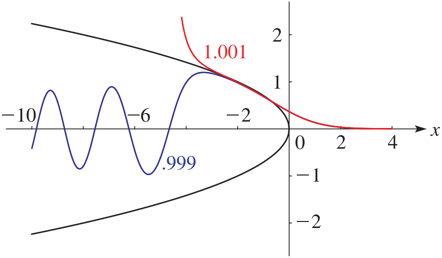

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- Hastings-McLeod: $\sigma_+ = 1$ unique real solution on $\mathbb{R}$
- intricate "condensation of instantons" across transition

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Dyson's argument (QED)

- Dyson (1952): *physical argument* for divergence of QED perturbation theory

$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots$$

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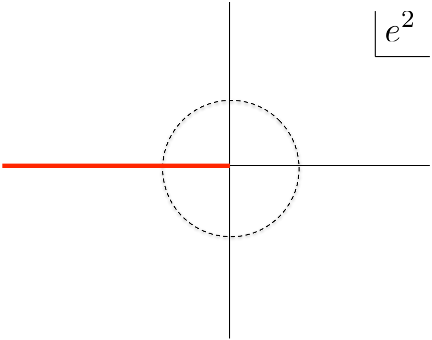


$e^2 < 0$

unstable

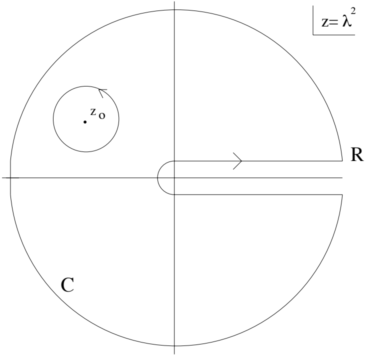
Dyson's argument (QED)

- Dyson (1952): *physical argument* for divergence of QED perturbation theory

$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots$$

> *Thus [for $e^2 < 0$] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.*

- *suggests* perturbative expansion cannot be convergent cubic oscillator: $V = x^2 + \lambda\, x^3$ A. Vainshtein, 1964



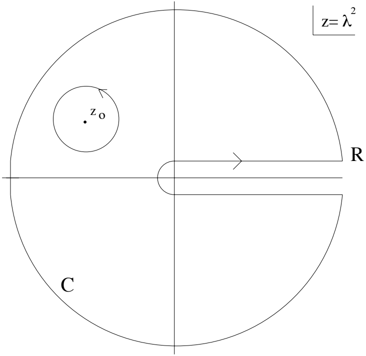
$$E(z_0) = \frac{1}{2\pi i}\oint_C dz\, \frac{E(z)}{z - z_0}$$

$$= \frac{1}{\pi}\int_0^R dz\, \frac{Im\, E(z)}{z - z_0}$$

$$= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi}\int_0^R dz\, \frac{\mathrm{Im}\, E(z)}{z^{n+1}} \right)$$

Borel Summation and Dispersion Relations: QM examples

cubic oscillator: $V = x^2 + \lambda\, x^3$ A. Vainshtein, 1964



$$
\begin{aligned}
E(z_0) &= \frac{1}{2\pi i}\oint_C dz\, \frac{E(z)}{z - z_0} \\
&= \frac{1}{\pi}\int_0^R dz\, \frac{Im\, E(z)}{z - z_0} \\
&= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi}\int_0^R dz\, \frac{\mathrm{Im}\, E(z)}{z^{n+1}} \right)
\end{aligned}
$$

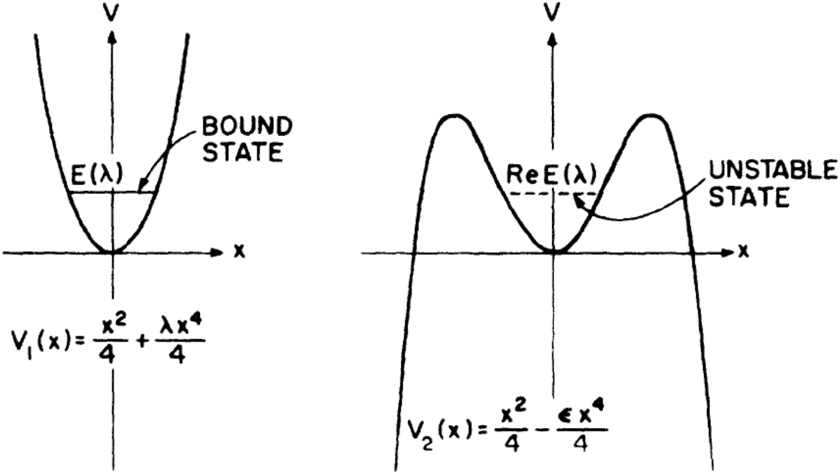
$$
\text{WKB} \Rightarrow \mathrm{Im}\, E(z) \sim \frac{a}{\sqrt{z}}\, e^{-b/z} \quad , \quad z \to 0 \quad \leftrightarrow \quad n \to \infty
$$

$$
\Rightarrow \qquad c_n \sim \frac{a}{\pi}\int_0^\infty dz\, \frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi}\, \frac{\Gamma(n+\frac{1}{2})}{b^{n+1/2}} \qquad \checkmark
$$

Instability and Divergence of Perturbation Theory

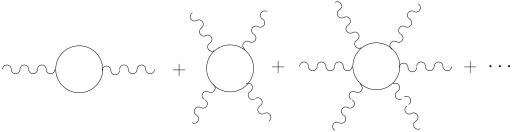
quartic AHO: $V(x) = \frac{x^2}{4} + \lambda\frac{x^4}{4}$

Bender/Wu, 1969



Euler-Heisenberg Effective Action (1935)

review: hep-th/0406216



- 1-loop QED effective action in uniform emag field
- the birth of *effective field theory*

$$L = \frac{\vec{E}^2 - \vec{B}^2}{2} + \frac{\alpha}{90\pi}\frac{1}{E_c^2}\left[\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\left(\vec{E}\cdot\vec{B}\right)^2\right] + \dots$$

- encodes nonlinear properties of QED/QCD vacuum

QFT Application: Euler-Heisenberg

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2 (\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$

$$\left(\begin{array}{l} \mathfrak{E}, \mathfrak{B} \quad \text{Kraft auf das Elektron.} \\ |\mathfrak{E}_k| = \frac{m^2 c^3}{e\hbar} = \frac{1}{\text{„}137\text{“}}\frac{e}{(e^2/mc^2)^2} = \text{„Kritische Feldstärke“.} \end{array}\right)$$

- Borel transform of a (doubly) asymptotic series
- resurgent trans-series: analytic continuation $B \longleftrightarrow E$

Euler-Heisenberg Effective Action: Borel summation

- e.g., constant B field:

$$S = -\frac{B^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)\exp\left[-\frac{m^2 s}{B}\right]$$

- perturbative (weak field) expansion:

$$S \sim -\frac{B^2}{2\pi^2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2B}{m^2}\right)^{2n+2}$$

Euler-Heisenberg Effective Action: Borel summation

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- characteristic factorial divergence

$$c_n = \frac{(-1)^{n+1}}{8}\sum_{k=1}^{\infty}\frac{\Gamma(2n+2)}{(k\,\pi)^{2n+4}}$$

- instructive exercise: reconstruct Borel transform

$$\sum_{k=1}^{\infty}\frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

Euler-Heisenberg Effective Action: Borel summation

- e.g., constant B field: characteristic factorial divergence

$$c_n = \frac{(-1)^{n+1}}{8} \sum_{k=1}^{\infty} \frac{1}{(k\,\pi)^{2n+4}}\,\Gamma(2n+2)$$

- recall Borel summation:

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\,\Gamma(\gamma\, n + \delta)$$

$$\to \quad f(g) \sim \frac{1}{\gamma} \int_0^{\infty} \frac{ds}{s} \left(\frac{1}{1+s}\right) \left(\frac{s}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{s}{\beta g}\right)^{1/\gamma}\right]$$

- for each k, reconstruct Borel transform:

$$\sum_{k=1}^{\infty} \frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

Exercise 6:

(i) fill in these steps for the Borel summation of the Euler-Heisenberg effective action

(ii) deduce the imaginary part of the effective action when the background field changes from magnetic to electric

(iii) repeat for the case of *scalar* QED in a background magnetic field, where the Euler-Heisenberg effective action is instead

$$S = \frac{B^2}{16\pi^2} \int_0^\infty \frac{ds}{s^2} \left(\frac{1}{\sinh s} - \frac{1}{s} + \frac{s}{6} \right) \exp\left[-\frac{m^2 s}{B} \right]$$

Euler-Heisenberg Effective Action and Schwinger Effect

B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \mathrm{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

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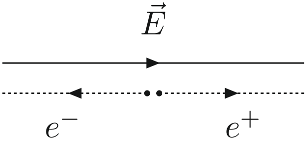
Schwinger effect:



$$2eE \frac{\hbar}{mc} \sim 2mc^2$$

$$\Rightarrow$$

$$E_c \sim \frac{m^2 c^3}{e\hbar} \approx 10^{16} \mathrm{V/cm}$$

WKB tunneling from Dirac sea
$\mathrm{Im}\, S \to$ physical pair production rate

- Euler-Heisenberg series must be divergent

de Sitter/ anti de Sitter effective actions (Das & GD, hep-th/0607168)

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$

$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature: $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in dS_d with d even