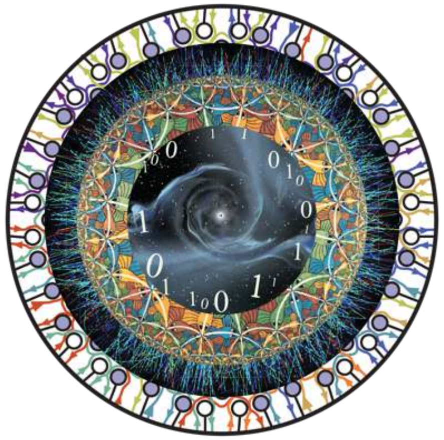
UC DAVIS
UNIVERSITY OF CALIFORNIA



EFFECTIVE FIELD THEORIES OF HYDRODYNAMICS

Mukund Rangamani
QMAP & Dept of Physics, UC Davis

Kavli Asian Winter School 2019
Sogang University, Seoul

Jan 8-11, 2019

Logistics

PLAN OF LECTURES

Lecture 1

- Axiomatic formulation of hydrodynamics
- Transport data
- Eightfold classification
- Realization in physical systems
- Long time tails and fluctuations

Lecture 2

- Microscopic picture of transport
- Schwinger-Keldysh observables
- Fluctuation-Dissipation relations
- Influence functionals
- Superspace encoding
- Beyond SK; OTOs

Lecture 3

- SK-KMS superalgebra
- Multiplets: gauge and matter
- 1d example: Langevin dynamics
- Fluid effective actions

Lecture 4

- Entropy inflow
- Jarzynski and the second law
- Classification of the hydro actions
- Open questions

PRIMARY REFERENCES

- Fluid/gravity correspondence
  - Bhattacharyya, Hubeny, Minwalla, MR [0712.2456]
  - Hubeny, Minwalla, MR [1107.5780]
- Review of relativistic hydrodynamics: MR [0905.4352]

- Classification of solutions to hydro axioms: Haehl, Loganayagam, MR [1412.1090] [1502.00636]
- The Fluid Manifesto (basic philosophy): Haehl, Loganayagam, MR [1510.02494]

- Schwinger-Keldysh effective field theory: Haehl, Loganayagam, MR [1610.01940]
- Thermal Equivariance: Haehl, Loganayagam, MR [1610.01941]
- Superspace embedding: Geracie, Haehl, Loganayagam, Narayan, Ramirez, MR [1712.04459]

PRIMARY REFERENCES

- Dissipative hydrodynamic actions:
  - Crossley, Glorioso, Liu [1511.03646] & [1701.07817]
  - Haehl, Loganayagam, MR [1511.07809] & [1803.11155]
  - Jensen, Pinzani-Fokeeva, Yarom [1701.07436]
  - Jensen, Marjieh, Pinzani-Fokeeva, Yarom [1804.04654]
- Overview of related works: [1701.07896]

- Second Law and entropy production:
  - Bhattacharyya [1312.0220] & [1403.7639]
  - Haehl, Loganayagam, MR [1511.07809]
  - Glorioso, Liu [1612.07705]
  - Haehl, Loganayagam, MR [1803.08490]
  - Jensen, Marjieh, Pinzani-Fokeeva, Yarom [1803.07070]

BACKGROUND MATERIAL

millennial developments in hydrodynamics

- Nickel, Son (2011)
- Dubovsky, Hui, Nicolis + Son (2011)
- Romatschke (2009), Bhattacharyya (2012)
- Banerjee, Bhattacharya, Bhattacharyya, Jain, Minwalla, Sharma (2012)
- Jensen, Kaminski, Kovtun, Meyer, Ritz, Yarom (2012)
- Jensen, Loganayagam, Yarom (2012-13)
- Bhattacharyya (2013-2014)

topological field theories and equivariance

- Cordes, Moore, Ramgoolam (1994)
- Blau, Thompson (1991, 1996)
- Vafa, Witten (1994)
- Dijkgraaf, Moore (1996)

stochastic and statistical dynamics

- Martin, Siggia, Rose (1973)
- Mallick, Moshe, Orland (2010)
- Kovtun, Moore, Romatschke (2014)
- Gaspard (2012)

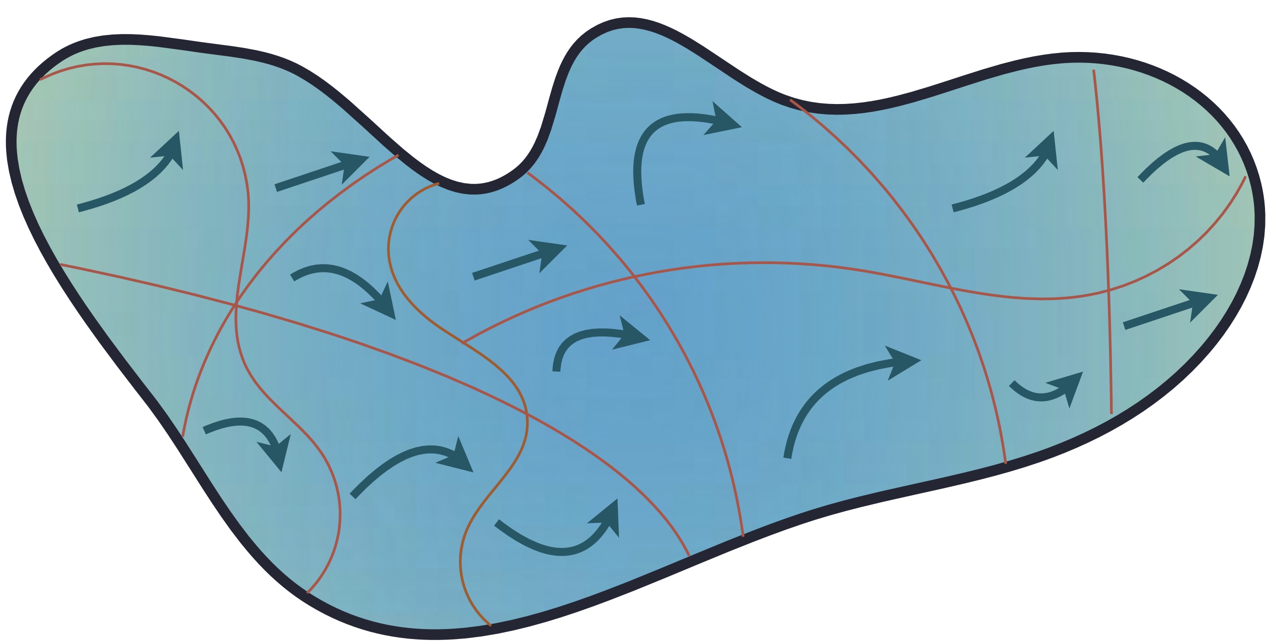
Lecture 1

Act I

Axiomatic Hydrodynamics

The hydrodynamic effective field theory

- Relativistic fluid dynamics is best thought of as an effective field theory for quantum systems in local, but not global, thermal equilibrium.
- The description in terms of fluid dynamics is valid when departures from equilibrium are on scales that are large compared to the characteristic mean free path of the underlying quantum dynamics.



$$\ell_{\rm mfp} \ll L\,, \qquad t_{\rm mfp} \ll t$$

- Local domains of equilibrated fluid can be characterized by the local temperature/energy density and conserved charges.
- Energy/charge flux exchanged across the domains: velocity field.

Axioms of Hydrodynamics I: Fields

- ✦ Hydrodynamics describes low-energy, near-equilibrium fluctuations of an equilibrium Gibbsian density matrix on scales large compared to the characteristic mean free path.
- ✦ The macroscopic description involves currents which capture energy-momentum and charge transport $T^{\mu\nu},\; J^{\mu}$ (and entropy current J^{μ}_{S}).
- ✦ The currents are functionals of the hydrodynamic fields, which are the intensive variables characterizing the density matrix and background sources.
  - ∗ temperature and chemical potential and a flux vector (fluid velocity) $\quad T, \mu, u^{\mu}, \qquad u^{\mu}\, u_{\mu} = -1$
  - ∗ background metric and electromagnetic potential $\quad g_{\mu\nu}, A_{\mu}$

Axioms of Hydrodynamics II: Data

* Repackage the dynamical degrees of freedom in a vector an scalar

thermal vector $\boldsymbol{\beta}^\mu \equiv \frac{u^\mu}{T}$, $\Lambda_{\boldsymbol{\beta}} \equiv \frac{\mu}{T} - \frac{u^\sigma}{T} A_\sigma$ thermal twist

* The currents of hydrodynamics are expressed as functionals of the hydrodynamical fields and the background sources.

- currents $T^{\mu\nu}, J^\mu, J^\mu_S$

- fields $\mathbf{\Psi} \equiv \{g_{\mu\nu}, A_\mu, \boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$

- constitutive relations

$$T^{\mu\nu} = T^{\mu\nu}[\mathbf{\Psi}] = T^{\mu\nu}[g_{\alpha\beta}, A_\alpha, \boldsymbol{\beta}^\alpha, \Lambda_{\boldsymbol{\beta}}]$$
$$J^\mu = J^\mu[\mathbf{\Psi}] = J^\mu[g_{\alpha\beta}, A_\alpha, \boldsymbol{\beta}^\alpha, \Lambda_{\boldsymbol{\beta}}]$$
$$J^\mu_S = J^\mu_S[\mathbf{\Psi}] = J^\mu_S[g_{\alpha\beta}, A_\alpha, \boldsymbol{\beta}^\alpha, \Lambda_{\boldsymbol{\beta}}] \ .$$

Axioms of Hydrodynamics III: Dynamics

- The dynamical content of hydrodynamics is the statement of conservation, modulo work done by sources and anomalies:

$$\nabla_\nu T^{\mu\nu} = J_\nu \cdot F^{\mu\nu} + \mathrm{T}^{\mu\perp}_H \qquad D_\nu J^\nu = \mathrm{J}^{\perp}_H$$

work term

covariant anomalies

- These are effectively *Ward identities* for the one-point functions of the conserved currents in the fluctuating Gibbs density matrix.

- The task of a hydrodynamicist is to specify the currents as a functional of the hydrodynamic fields, consistent with the dynamics, constructing a current algebra of sorts, but…

Axioms of Hydrodynamics IV: Constraints

- ✦ From a macroscopic, statistical viewpoint, one has to demand that a local form of the second law of thermodynamics is upheld.

$$\exists \;\; J_S^\mu[\boldsymbol{\Psi}]: \;\; \forall \;\; \boldsymbol{\Psi}_{\text{on-shell}} \;\; \nabla_\mu J_S^\mu[\boldsymbol{\Psi}] \geq 0$$

- ✦ This is required to be upheld on-shell, and complicates the analysis of hydrodynamics, for without it the current algebra can be analyzed purely in terms of representation theory.

- ✦ Note that usually one only requires the existence of *some* entropy current.

- ✦ From a microscopic viewpoint the entropy current is rather mysterious; it is not associated with any underlying symmetry per se.

- **Opportunity:** *Understand a Wilsonian hydrodynamic theory consistent with second law.*

Neutral fluids

- A neutral fluid is characterized by its energy-momentum stress tensor

$$T^{\mu\nu} = \epsilon(T)\, u^\mu\, u^\nu + p(T)\, P^{\mu\nu} - \eta(T)\, \sigma^{\mu\nu} - \zeta(T)\, \Theta\, P^{\mu\nu} + \cdots$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu\, u^\nu$$

spatial metric

$$\nabla_\mu u_\nu = \sigma_{(\mu\nu)} + \omega_{[\mu\nu]} + \Theta\, P_{\mu\nu} - u_\mu\, \mathfrak{a}_\nu$$

shear
$$\sigma_{\mu\nu} = P^\alpha_\mu\, P^\beta_\nu \left(\nabla_{(\alpha} u_{\beta)} - \frac{\Theta}{d-1}\, P_{\alpha\beta} \right)$$

vorticity
$$\omega_{\mu\nu} = P^\alpha_\mu\, P^\beta_\nu \nabla_{[\alpha} u_{\beta]}$$

expansion
$$\Theta = \nabla_\alpha u^\alpha$$

acceleration
$$\mathfrak{a}^\mu = u^\nu \nabla_\nu u^\mu$$

$$\mathcal{A}_{<\alpha\beta>} = \left(P_{\alpha\mu} P_{\beta\nu} - \frac{1}{d-1}\, P_{\alpha\beta}\, g_{\mu\nu} \right) \mathcal{A}^{\mu\nu}$$

Neutral fluids

✦ A neutral fluid is characterized by its energy-momentum stress tensor

$$T^{\mu\nu} = \epsilon(T)\, u^\mu\, u^\nu + p(T)\, P^{\mu\nu} - \eta(T)\, \sigma^{\mu\nu} - \zeta(T)\, \Theta\, P^{\mu\nu} + \cdots$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu\, u^\nu$$

spatial metric

$$\nabla_\mu u_\nu = \sigma_{(\mu\nu)} + \omega_{[\mu\nu]} + \Theta\, P_{\mu\nu} - u_\mu\, \mathfrak{a}_\nu$$

shear vorticity acceleration

expansion

✦ The second law forces some of the transport data to satisfy some inequalities, e.g., the viscosities are non-negative definite (friction)

$$J^\mu_S = s\, u^\mu + \cdots$$

$$\nabla_\mu J^\mu_S = \eta\, \sigma^2 + \zeta\, \Theta^2 + \cdots \qquad\qquad \eta, \zeta \geq 0$$

Benchmarking hydrodynamics

- ✦ To set the stage for our Wilsonian framework, we need to understand hydrodynamic constitutive relations compatible with second law.
- ✦ Ideally, this should data should be given to us off-shell, since we are aiming to construct effective action.

✦ **STRATEGY**

- ✳ Take the entropy current constraint off-shell.
- ✳ Classify all off-shell physical constitutive relations.
- ✳ Derive the resulting constitutive relation from an effective action.

Off-shell entropy production

- ✦ Take the statement of the second law off-shell Lagrange multipliers

$$\nabla_\mu J^\mu_S = \Delta \geq 0$$

$$\nabla_\mu J^\mu_S + \boldsymbol{\beta}_\mu \left(\nabla_\nu T^{\mu\nu} - J_\nu \cdot F^{\mu\nu} - \mathrm{T}^{\mu\perp}_H\right) + \left(\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\lambda A_\lambda\right) \cdot \left(D_\nu J^\nu - \mathrm{J}^\perp_H\right) = \Delta \geq 0$$

entropy production by dissipation

- ✦ The Lagrange multipliers are fixed to be the hydrodynamic fields exploiting field redefinition freedom.

- ✦ This off-shell formalism motivates separation of transport into:
  - ✳ dissipative (Class D)
  - ✳ adiabatic

$$\boldsymbol{\beta}^\mu \equiv \frac{u^\mu}{T}\,, \qquad \Lambda_{\boldsymbol{\beta}} \equiv \frac{\mu}{T} - \frac{u^\sigma}{T} A_\sigma$$

Free energy current

✦ Package the information in terms of a Gibbs free energy current, switching from a microcanonical to grand-canonical language:

$$\begin{aligned} J_S^\sigma &= (J_S^\sigma)_{\text{can}} + N^\sigma \\ &= -\left[\boldsymbol{\beta}_\nu\, T^{\nu\sigma} + (\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\nu A_\nu) \cdot J^\sigma\right] + N^\sigma \end{aligned}$$

$$\mathcal{G}^\mu = -T\, N^\sigma$$

free energy current

✦ The off-shell second law statement can be phrased now as

$$\nabla_\sigma N^\sigma - N^\perp = \frac{1}{2}\, T^{\mu\nu} \delta_{\mathscr{B}}\, g_{\mu\nu} + J^\mu \cdot \delta_{\mathscr{B}} A_\mu + \Delta$$

anomalous free energy

diffeomorphism

$$\begin{aligned} \delta_{\mathscr{B}}\, g_{\mu\nu} &\equiv \pounds_{\boldsymbol{\beta}} g_{\mu\nu} = \nabla_\mu \boldsymbol{\beta}_\nu + \nabla_\nu \boldsymbol{\beta}_\mu\,, \\ \delta_{\mathscr{B}}\, A_\mu &\equiv \pounds_{\boldsymbol{\beta}} A_\mu + \partial_\mu \Lambda_{\boldsymbol{\beta}} + [A_\mu, \Lambda_{\boldsymbol{\beta}}] \end{aligned}$$

flavour gauge transformation

Hydrodynamic taxonomy

- The off-shell formalism is quite powerful. One can classify hydrodynamic constitutive relations into eight distinct classes:

  - Class D: dissipative class
  - Class A: anomaly induced transport
  - Class B: Berry-like transport
  - Class C: conserved entropy

$$\mathcal{G}^\mu = \mathfrak{S}\,\boldsymbol{\beta}^\mu + \mathfrak{V}^\mu\,, \qquad \mathfrak{V}^\mu\,\boldsymbol{\beta}_\mu = 0$$

longitudinal vector

transverse vector (conserved)

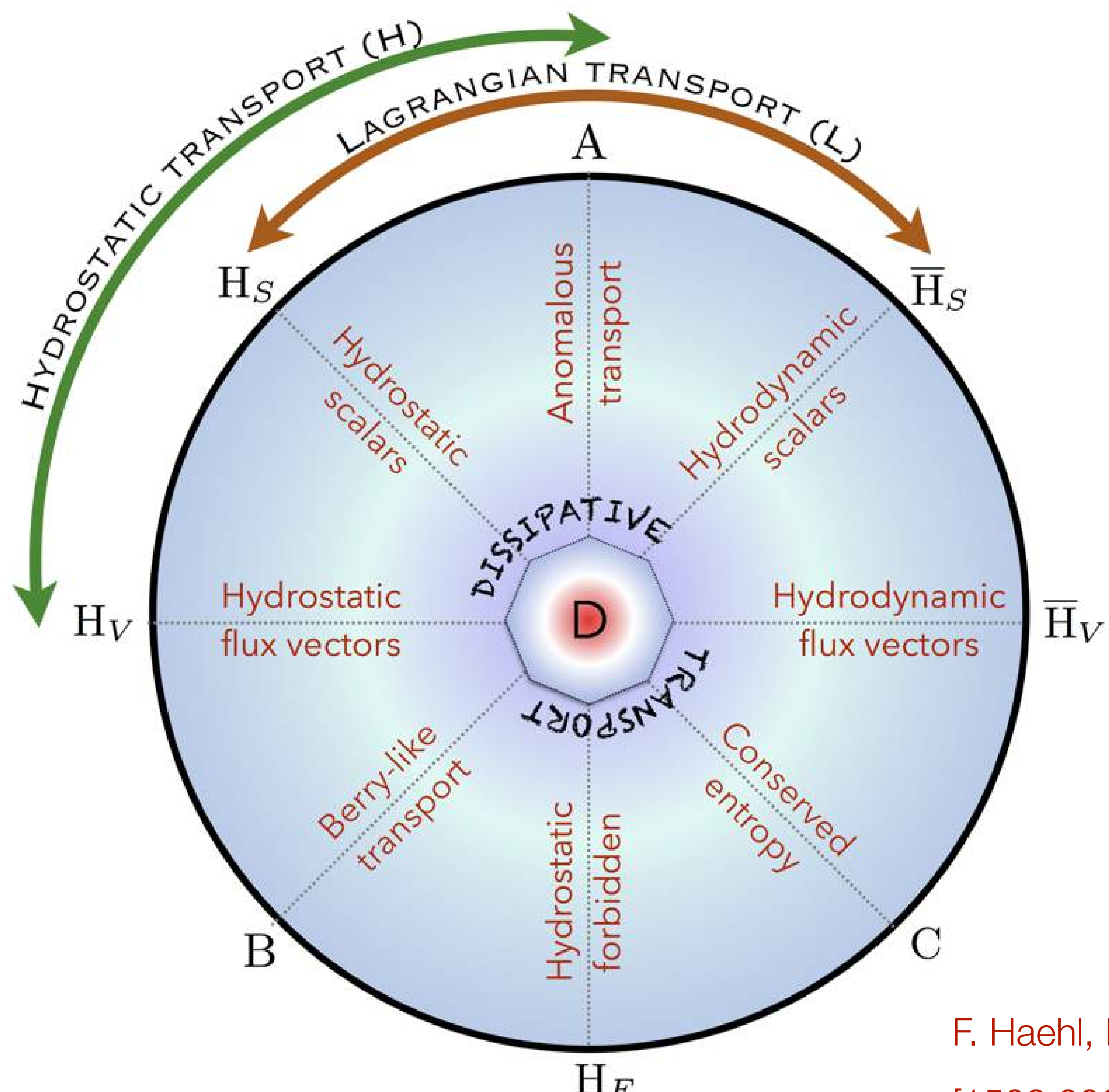
Free energy scalars

- Class H_S: Hydrostatic scalars
- Class $\overline{H}_S$: Landau-Ginzburg scalars

Free energy vectors

- Class H_V: Hydrostatic vectors
- Class $\overline{H}_V$: Gibbsian vectors

Eightfold classification of hydrodynamic transport



F. Haehl, R. Loganayagam, MR

[1502.00636], [1412.1090]

Class H: Hydrostatics ($H_S \cup H_V$)

- Hydrodynamic transport can be classified into two categories
  - Hydrostatic or thermodynamic response: fixed by equilibrium
  - Genuine hydrodynamic transport

- Hydrostatic data can be understood by time-independent configurations of the fluid in the presence of non-trivial (spatially varying) background sources.

- Can equivalently be encoded in a generating function, the equilibrium partition function which is a functional of stationary background sources.

$$\mathcal{K} \equiv \{K^\mu, \Lambda_K\}\,, \quad g_{\mu\nu}\, K^\mu\, K^\nu \leq 0 \;\longrightarrow\; \delta_{\mathcal{K}}\, g_{\mu\nu} = \delta_{\mathcal{K}} A_\mu = 0$$

Banerjee et. al. '12 Jensen et. al. '12

Class H: Hydrostatics

- The hydrostatic partition function is the integral of the (consistent) free energy current over the Wick rotated Euclidean manifold.

$$W_{\text{Hydrostatic}} = \left[\int_{\Sigma_E} N^\mu \, d^{d-1} S_\mu \right]_{\text{Hydrostatic}}$$

spatial integral!

- Since the free energy current is a vector field, it decomposes into

$$N^\mu = \mathfrak{S}\, \boldsymbol{\beta}^\mu + \mathfrak{V}^\mu , \qquad \mathfrak{V}^\mu \boldsymbol{\beta}_\mu = 0$$

longitudinal vector

transverse vector (conserved)

$$\mathrm{H} = \mathrm{H}_S \cup \mathrm{H}_V$$

partition fn scalars

partition fn vectors

Entropy constraint: Hydrostatic forbidden (HF)

- The scalars and vectors which do not vanish in equilibrium parameterize the free energy current and in turn generate the currents after varying with respect to the sources.

$$\delta W_{\text{Hydrostatic}} = \left[\int_{\Sigma_E} \left(\frac{1}{2} T^{\mu\nu}_{cons}\, \delta g_{\mu\nu} + J^{\mu}_{cons} \cdot \delta A_{\mu} \right) \boldsymbol{\beta}^{\alpha} d^{d-1} S_{\alpha} \right]_{\text{Hydrostatic}}$$

- At any given derivative order however, there are fewer scalars than the tensor structures in the currents.
- *Hydrostatics implies that certain constitutive relations are forbidden.*
- Intuitively think of hydrostatics as time-independent configurations; turning on time dependence one should find no linear term, for it can produce entropy of either sign.

Bhattacharyya ('13, '14)

Example: Ideal fluids

- For an ideal fluid, the hydrostatic partition function is generated by the a single function which is the free energy or pressure *p(T,μ)*. It completely fixes all pieces of transport:

$$T^{\mu\nu} = \epsilon\, u^\mu\, u^\nu + p\, P^{\mu\nu}\,, \qquad J_S^\mu = s\, u^\mu$$

$$\epsilon + p - T\, s = 0$$ ← Euler relation

$$\frac{d\epsilon}{dT} - T\,\frac{ds}{dT} = 0$$ ← Clausius relation

- At first order, only dissipative terms in the stress tensor, so we cannot learn anything new directly from the partition function.

Example: Second order neutral fluids

- At second order there are 15 tensor structures that can appear in the stress tensor. Of these only 8 survive when we restrict to hydrostatics (check that shear and expansion vanish in hydrostatics).
- There are only 3 scalars at second order which can enter the partition function

$$\mathfrak{R}, \quad \mathfrak{a}^\mu\, \mathfrak{a}_\mu, \quad \omega_{\mu\nu}\, \omega^{\mu\nu}$$

- This immediately implies 5 relations must hold between the 15 transport coefficients.
- Transport data that cannot appear from the hydrostatic analysis are the forbidden terms.

Bhattacharyya ('13, '14)

- Charged fluids: 51 pieces of transport data and 17 forbidden terms.

Lecture 2

Recap of Lecture 1

- Classification of allowed transport from axiomatic formulation of hydrodynamics captured by solution to adiabaticity equation (nb: off-shell)

$$\nabla_\sigma N^\sigma - N^\perp = \frac{1}{2} T^{\mu\nu} \delta_{\mathfrak{B}} g_{\mu\nu} + J^\mu \cdot \delta_{\mathfrak{B}} A_\mu + \Delta$$

- Useful to start with equilibrium as $\delta_{\mathfrak{B}} g, \delta_{\mathfrak{B}} A$ measure departures from equilibrium.

$$\delta_{\mathfrak{B}} g_{\mu\nu} \equiv \pounds_{\boldsymbol{\beta}} g_{\mu\nu} = \nabla_\mu \boldsymbol{\beta}_\nu + \nabla_\nu \boldsymbol{\beta}_\mu \,,$$
$$\delta_{\mathfrak{B}} A_\mu \equiv \pounds_{\boldsymbol{\beta}} A_\mu + \partial_\mu \Lambda_{\boldsymbol{\beta}} + [A_\mu, \Lambda_{\boldsymbol{\beta}}]$$

- Global equilbrium can be attained on a spatially curved geometry with a timelike Killing vector field (and corresponding stationary flavour sources, i.e., you can have magnetic fields but not electric fields)

Recap of Lecture 1

✦ General geometric background for equilibrium (nb: global Killing field, no ergosurfaces)

$$ds^2 = -e^{2\,\xi(x)}(dt + a_i(x)\,dx^i)^2 + \gamma_{ij}(x)\,dx^i\,dx^j$$

$$\boldsymbol{\beta}^\mu \propto \left(\frac{\partial}{\partial t}\right)^\mu \qquad \Longrightarrow \qquad \sigma^{\mu\nu} = 0\,,\Theta = 0$$

✦ Exercise: Fix normalization and check the above and also convince yourself that the partition function to second order in derivatives has to be:

$$\int d^{d-1}x\,\sqrt{-\gamma}\left[p(T) + \mathfrak{f}_{\mathfrak{a}}(T)\,\mathfrak{a}_\mu\mathfrak{a}^\mu + \mathfrak{f}_{\mathfrak{R}}(T)\,\mathfrak{R} + \mathfrak{f}_\omega(T)\,\omega_{\mu\nu}\omega^{\mu\nu}\right]$$

Class D: Dissipation

✦ Focus on positivity of Δ order by order in the gradient expansion. Deviations from equilibrium: $\delta_{\mathfrak{B}} g \,, \delta_{\mathfrak{B}} A$

- viscous dissipative terms $\quad \eta\, \sigma^{\mu\nu} + \zeta\, \Theta\, P^{\mu\nu} \Longrightarrow \Delta = \eta\, \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta\, \Theta^2 \sim (\delta_{\mathcal{B}} g)^2$
- descendant operators $\quad \delta_{\mathfrak{B}} g\, D\mathcal{O}_{k-2}$
- product composites $\quad (\delta_{\mathfrak{B}} g)^k \qquad (\delta_{\mathfrak{B}} A)^k$

→ sub-dissipative

✦ Sub-dissipative terms can be subsumed under viscous dissipative terms.

✦ **Theorem**: Entropy constraints operate only at leading order in the gradient expansion!

Bhattacharyya ('11, '13, '14)

✦ Useful restatement of the argument using tensor valued differential operators acting on $\delta_{\mathfrak{B}} g \,, \delta_{\mathfrak{B}} A$

Class D: Dissipation at 1st & 2nd order

1st order

$$T^{\mu\nu}_{(1)} = -2\,\eta\,\sigma^{\mu\nu} - \zeta\,\Theta\,P^{\mu\nu}\,, \qquad J^{\mu}_{S,(1)} = c_1\,\mathfrak{a}^{\mu} + c_2\,\Theta\,u^{\mu}$$

$$\Delta = \frac{1}{T}\left(2\eta\,\sigma_{\mu\nu}\,\sigma^{\mu\nu} + \zeta\,\Theta^2 + c_1\,\mathfrak{s}_1 + c_2\,\mathfrak{s}_2\right)$$

sign indefinite terms

$$\Delta \geq 0 \quad \Longrightarrow \quad \eta, \zeta \;\geq\; 0\,,\;\; \&\; c_1 = c_2 = 0$$

✦ product composite terms are always subleading

$$2\,\eta\,\sigma_{\mu\nu}\,\sigma^{\mu\nu} + \zeta\,\Theta^2 + \gamma_1\,\Theta^3 + \gamma_2\,\sigma^{\alpha\beta}\sigma_{\beta\gamma}\sigma^{\gamma}{}_{\alpha} + \gamma_3\,\Theta\,\sigma_{\mu\nu}\,\sigma^{\mu\nu} \geq 0$$

$$\Longrightarrow \qquad \eta, \zeta \geq 0\,, \text{and } \{\gamma_1, \gamma_2, \gamma_3\} \text{ unconstrained.}$$

2nd order

✦ descendants can be made subleading by completing squares

$$\Delta = \alpha_{2g}\,(\delta_{\scriptscriptstyle\mathcal{B}}\,g)^2 + \alpha_{2A}\,(\delta_{\scriptscriptstyle\mathcal{B}}\,A)^2 + \sum_{k=3}^{\infty}\left[\kappa_{kg}\;\delta_{\scriptscriptstyle\mathcal{B}}\,g\;D\mathcal{O}_{k-2} + \gamma_{kg}\,(\delta_{\scriptscriptstyle\mathcal{B}}\,g)^k + \cdots\right]$$

$$\sim \alpha_{2g}\left[\delta_{\scriptscriptstyle\mathcal{B}}\,g + \sum_{k=3}^{\infty}\frac{\kappa_{kg}}{2\,\alpha_{2g}}\;D\mathcal{O}_{k-2}\right]^2 + \sum_{k=3}^{\infty}\gamma_{kg}\,(\delta_{\scriptscriptstyle\mathcal{B}}\,g)^k + \cdots$$

General solution to Class D

- Class D data has at least one factor of $\delta_{\mathcal{B}} g\,,\delta_{\mathcal{B}} A$ as it signifies deviation from equilibrium.
- Keep one explicit factor and bury all other derivatives into a tensor valued differential operator $\boldsymbol{\eta}^{(\mu\nu)(\alpha\beta)} = \boldsymbol{\eta}^{(\alpha\beta)(\mu\nu)}$

$$T^{\mu\nu}_{\rm D} = -\frac{1}{2}\boldsymbol{\eta}^{(\mu\nu)(\alpha\beta)}\ \delta_{\mathcal{B}} g_{\alpha\beta} \qquad\qquad \Delta = \frac{1}{4}\boldsymbol{\eta}^{(\mu\nu)(\alpha\beta)}\ \delta_{\mathcal{B}} g_{\mu\nu}\ \delta_{\mathcal{B}} g_{\alpha\beta}$$

- At leading order in gradients we reproduce known results using:

$$\boldsymbol{\eta}^{\mu\nu\rho\sigma}_{(0)} = T\,\zeta\,P^{\mu\nu}P^{\rho\sigma} + 2\,T\,\eta\,P^{\rho<\mu}P^{\nu>\sigma}\,,$$

- At higher orders we can systematically add further gradient terms.

Class B: Berry-like transport

- This class of constitutive relations solves adiabaticity trivially. Non-equilibrium, non-dissipative data!

$$(T^{\mu\nu})_{\rm B} \equiv -\frac{1}{4}\left(\mathcal{N}^{(\mu\nu)(\alpha\beta)} - \mathcal{N}^{(\alpha\beta)(\mu\nu)}\right)\,\delta_{_\mathfrak{B}} g_{\alpha\beta} + \mathcal{X}^{(\mu\nu)\alpha}\cdot\delta_{_\mathfrak{B}} A_\alpha$$

$$(J^\alpha)_{\rm B} \equiv -\frac{1}{2}\mathcal{X}^{(\mu\nu)\alpha}\delta_{_\mathfrak{B}} g_{\mu\nu} - \mathcal{S}^{[\alpha\beta]}\cdot\delta_{_\mathfrak{B}} A_\beta$$

- The entropy current is canonical (given just by projections of energy-momentum and charge currents)

Hall Transport in 3 dimensions

$$(T^{\mu\nu})_B = -\tilde{\eta}_{_H}\, u_\rho\, \left(\varepsilon^{\rho\mu\alpha}\,\sigma^\nu_\alpha + \varepsilon^{\rho\nu\alpha}\,\sigma^\mu_\alpha\right)$$

$$(J^\alpha)_B = \tilde{\sigma}_{_H}\cdot u_\rho\,\varepsilon^{\rho\alpha\beta}\left[E_\beta - T\,D_\beta\left(\frac{\mu}{T}\right)\right]$$

Neutral fluids in arbitrary dimensions

$$(T^{\mu\nu})_B = -\lambda_\sigma\left(\Theta\,\sigma^{\mu\nu} - \sigma^2\,P^{\mu\nu}\right) - \lambda_\omega\left(\omega^{\mu\alpha}\sigma^\nu_\alpha + \omega^{\nu\alpha}\sigma^\mu_\alpha\right)$$

Class C: Conserved entropy

- AE can be solved by considering an exactly conserved entropy current.

$$(J_S^\mu)_{\rm C} = \mathsf{J}^\mu\,, \qquad (T^{\mu\nu})_{\rm C} = 0\,, \qquad (J^\mu)_{\rm C} = 0$$

- Currents must be cohomologically non-trivial (non-Komar terms) for them to be physically interesting.

- Eg., Wen-Zee current in 3 spacetime dimensions (more generally Euler currents in odd spacetime dimensions.

$$\mathsf{J}^\sigma_{\rm Euler} = \frac{1}{2} c_{\rm Euler}\ \varepsilon^{\sigma\alpha\beta}\ \varepsilon^{\mu\nu\lambda}\ u_\mu \left(\nabla_\alpha u_\nu \nabla_\beta u_\lambda - \frac{1}{2} R_{\nu\lambda\alpha\beta}\right)$$

- These currents count the degeneracy of topological states in the thermal density matrix and can be realized holographically (eg., Gauss-Bonnet contribution to black hole entropy in ABJM like theories).

Class $\overline{\text{H}}$v: Gibbsian vectors

- Just as hydrostatic vectors entered into parameterization of the free energy current, there are non-trivial hydrodynamic vectors which lead to adiabatic constitutive relations.

- These are parameterized by tensor valued differential operators with an explicit vector index

$$(T^{\mu\nu})_{\overline{\mathrm{H}}_V} = \frac{1}{2}\, D_\rho \mathfrak{C}^{\rho(\mu\mu)(\alpha\beta)}\, \delta_{\mathscr{B}} g_{\alpha\beta}$$

- No explicit data on such transport, but they do appear in charged fluids at second order in gradients.

Class L = H$_S$ ∪ $\overline{\text{H}}_S$

- Consider diffeomorphism and gauge invariant scalar Lagrangian densities which are functionals of hydrodynamic fields $\mathbf{\Psi} \equiv \{g_{\mu\nu}, A_\mu, \boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$

$$S_{\text{hydro}} = \int d^d x \sqrt{-g}\; \mathcal{L}[\mathbf{\Psi}]$$

- The basic variational principle of this theory defines currents:

$$\begin{aligned}\frac{1}{\sqrt{-g}}\delta\left(\sqrt{-g}\;\mathcal{L}\right) - \nabla_\mu(\not{\delta}\Theta_{\text{PS}})^\mu \\ = \frac{1}{2}\,T^{\mu\nu}\,\delta g_{\mu\nu} + J^\mu\cdot\delta A_\mu + T\,V_\sigma\,\delta\boldsymbol{\beta}^\sigma + T\,\zeta\cdot(\delta\Lambda_{\boldsymbol{\beta}} + A_\sigma\,\delta\boldsymbol{\beta}^\sigma)\end{aligned}$$

- Entropy density is defined as in thermodynamics

$$s \equiv \left(\frac{1}{\sqrt{-g}}\,\frac{\delta}{\delta T}\,\int\sqrt{-g}\;\mathcal{L}[\mathbf{\Psi}]\,\right)\bigg|_{\{u^\sigma,\,\mu,\,g_{\alpha\beta},\,A_\alpha\}=\text{fixed}} \qquad\qquad J^\mu_S = s\,u^\mu$$

Class L = $H_S \cup \overline{H}_S$

- Second order neutral fluid Lagrangian

Removable by field redefinition

$$\int d^{d-1}x\,\sqrt{-g}\,\Big[p(T) + \boldsymbol{\beta}^\sigma \nabla_\sigma \mathfrak{f}_1(T) + \mathfrak{f}_{\mathfrak{a}}(T)\,\mathfrak{a}_\mu \mathfrak{a}^\mu + \mathfrak{f}_{\mathfrak{R}}(T)\,\mathfrak{R} + \mathfrak{f}_\omega(T)\,\omega_{\mu\nu}\omega^{\mu\nu} + \mathfrak{f}_\sigma(T)\,\sigma_{\mu\nu}\,\sigma^{\mu\nu} + \mathfrak{f}_\Theta(T)\,\Theta^2 + \mathfrak{f}_{2a}(T)\,(\nabla T)^2 + \mathfrak{f}_{2b}(T)\,\Theta\,\boldsymbol{\beta}\cdot\nabla T + \mathfrak{f}_{2c}(T)\,\Theta\,\mathfrak{a}\cdot\nabla T + \cdots\Big]$$

Non-hydrostatic

- Variational principle gives all the necessary currents.

- Exercise: Generalize to charged fluids (classification of necessary scalars available).

Class L adiabaticity

Now diffeomorphism and flavour gauge symmetries of the Lagrangian imply a set of Bianchi identities:

$$\nabla_\nu T^{\mu\nu} = J_\nu \cdot F^{\mu\nu} + \frac{g^{\mu\nu}}{\sqrt{-g}} \delta_{\mathcal{B}} \left(\sqrt{-g}\; T V_\nu\right) + g^{\mu\nu}\, T\, \zeta \cdot \delta_{\mathcal{B}} A_\nu$$

$$D_\sigma J^\sigma = \frac{1}{\sqrt{-g}} \delta_{\mathcal{B}} \left(\sqrt{-g}\; T\zeta\right)$$

Together with the identity and an off-shell Euler relation

$$\nabla_\sigma J_S^\sigma = \nabla_\sigma (T\, s\, \boldsymbol{\beta}^\sigma) = \frac{1}{\sqrt{-g}} \delta_{\mathcal{B}} \left(\sqrt{-g}\; T s\right) \qquad T\, s + \mu \cdot \zeta + u^\sigma\, V_\sigma = 0$$

we get the off-shell adiabaticity equation (with zero entropy production):

$$\nabla_\mu J_S^\mu + \boldsymbol{\beta}_\mu \left(\nabla_\nu T^{\mu\nu} - J_\nu \cdot F^{\mu\nu}\right) + \left(\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\lambda A_\lambda\right) \cdot D_\nu J^\nu = 0$$

Dynamics in Class L

- The dynamics in Class L is supposed to reduce to the conservation of energy-momentum and charge currents.
- Naive variation with respect to $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$ does not respect this requirement, since it would lead to vanishing of the adiabatic heat/charge currents.
- Constrained variational principle: vary the hydrodynamic fields along a family related by Lie transport.

$$\mathbb{\delta}: \quad \mathbb{\delta}\boldsymbol{\beta}^\mu = \delta_\chi \boldsymbol{\beta}^\mu\,, \qquad \mathbb{\delta}\Lambda_{\boldsymbol{\beta}} = \delta_\chi \Lambda_{\boldsymbol{\beta}}\,, \qquad \mathbb{\delta} g_{\mu\nu} = \mathbb{\delta} A_\mu = 0$$

- This variation leads to equations of motion which when combined with the Bianchi identities leads to conservation

$$\frac{1}{\sqrt{-g}}\delta_{\mathbb{B}}\left(\sqrt{-g}\; T\, V_\mu\right) + T\,\zeta\cdot\delta_{\mathbb{B}} A_\mu \simeq 0$$

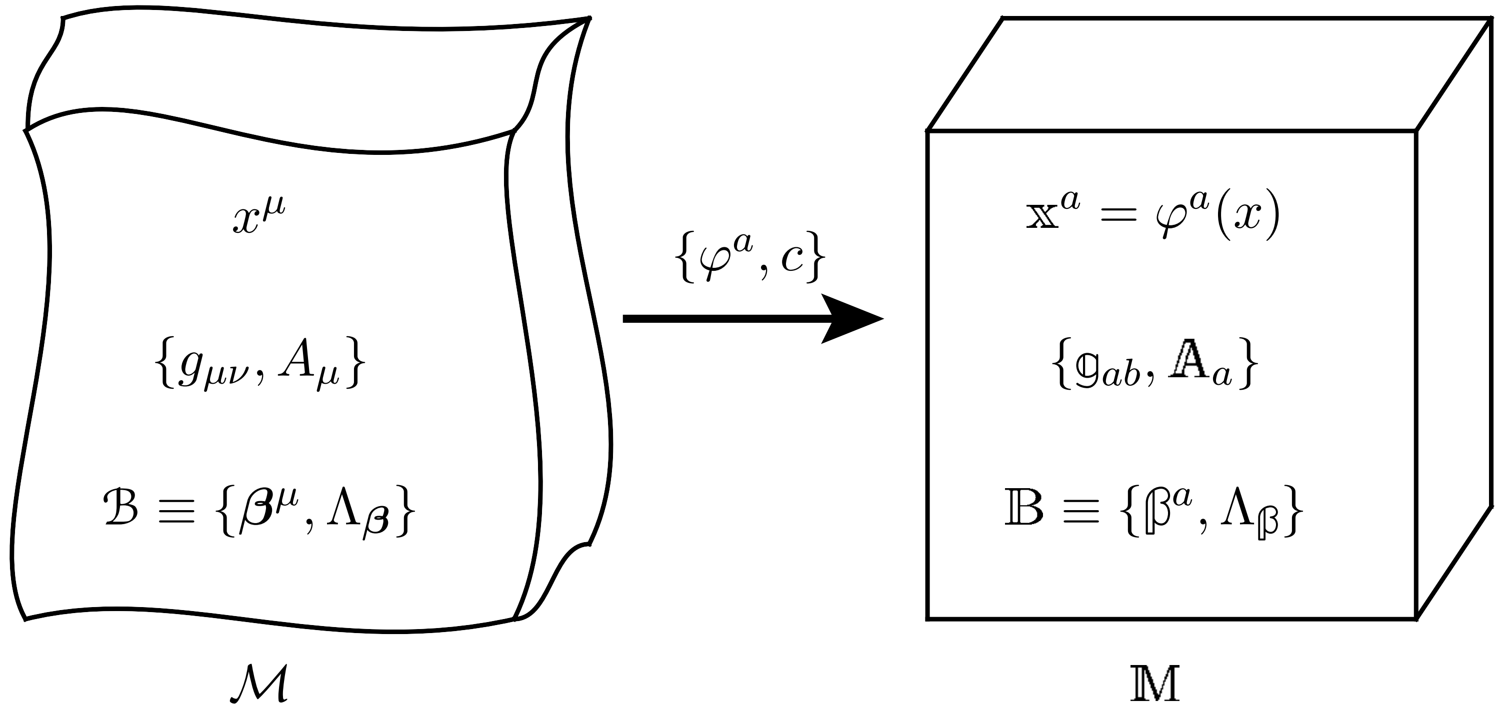
$$\frac{1}{\sqrt{-g}}\delta_{\mathbb{B}}\left(\sqrt{-g}\; T\zeta\right) \simeq 0$$

$$\xRightarrow{\text{+ Bianchi}}$$

$$\nabla_\nu T^{\mu\nu} \simeq 0$$

$$D_\nu J^\nu \simeq 0$$

Reference fields for Class L



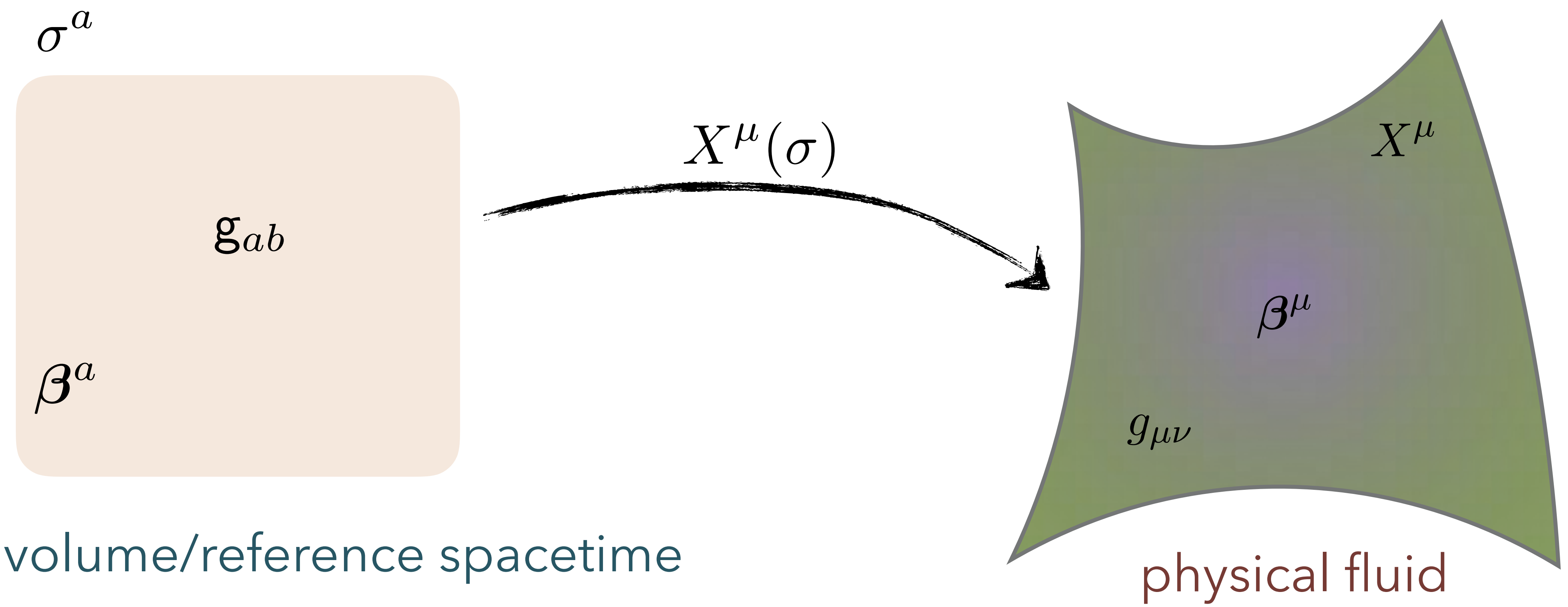
The constrained variational principle can be alternately phrased as fixing a reference configuration and varying along the pull-back maps by diffeos and gauge transformations.

$$e^\mu_a\, \partial_\nu \varphi^a = \delta^\mu_\nu \ , \qquad e^\mu_a\, \partial_\mu \varphi^b = \delta^a_b \ .$$

$$\boldsymbol{\beta}^\mu = e^\mu_a(x)\ \mathbb{\beta}^a[\varphi(x)]$$

$$\Lambda_{\boldsymbol{\beta}} = c(x)\ \Lambda_{\mathbb{\beta}}[\varphi(x)]\ c^{-1}(x) + \boldsymbol{\beta}^\sigma(x)\, \partial_\sigma c(x)\ c^{-1}(x)$$

Sigma model viewpoint



$$\mathrm{g}_{ab} = g_{\mu\nu} \partial_a X^\mu \, \partial_b X^\nu$$

$$\boldsymbol{\beta}^\mu = \boldsymbol{\beta}^a \partial_a X^\mu$$

★ Worldvolume with fixed reference thermal vector $\boldsymbol{\beta}^a$

★ Target space diffeomorphisms become field variations on the worldvolume.

★ Physical equations of motion then amount to conservation equation.

Entropy as a Noether current

- Following the variational principle one can give a simple expression for the free energy current:

$$N^\mu = \boldsymbol{\beta}^\mu \mathcal{L} - (\not{\delta}_{\mathfrak{B}} \Theta_{\text{PS}})^\mu$$

$$\nabla_\mu N^\mu = \frac{1}{2} T^{\mu\nu} \delta_{\mathfrak{B}} g_{\mu\nu} + J^\mu \cdot \delta_{\mathfrak{B}} A_\mu$$

- This of course holds when we restrict to hydrostatic configurations and signals a nice consistency check between fluid entropy and black hole entropy in equilibrium in the holographic context.

Act II

Microscopic perspective on hydrodynamics

Response Theory

- Hydrodynamics is a theory of response functions. System prepared in equilibrium, disturbed by external sources, and then causal response measured.

- Linear response (small amplitude deformations): $\mathcal{S} = \mathcal{S}_0 + \int d^d x \, \mathcal{J}_a(x) \, \mathcal{O}_a(x)$

$$\langle \, \mathcal{O}_a(x) \, \rangle = - \int d^d y \, G^R_{ab}(x,y) \, \mathcal{J}_b(y) \qquad G^R_{ab}(x,y) = -i \, \theta(x^0 - y^0) \, \langle \, [\mathcal{O}_a(x), \mathcal{O}_b(y)] \, \rangle$$

- Non-conserved quantities die off exponentially after transient behaviour (cf., quasinormal modes). Hydrodynamic response captured by retrarded Green's functions of conserved currents;

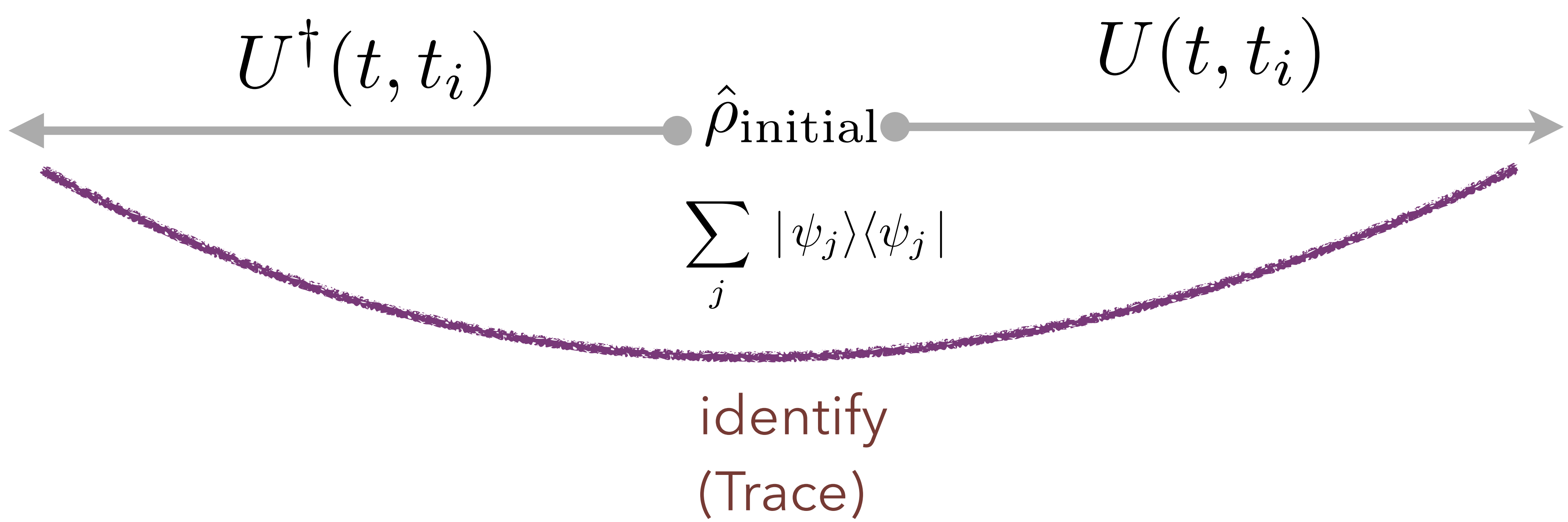
$$G_{\mu\nu,\alpha\beta}(x,y) \;\; = \;\; -i \, \mathrm{Tr} \left(\rho \, [T_{\mu\nu}(x), T_{\alpha\beta}(y)] \right) \equiv -i \, \theta(x^0 - y^0) \, \langle \, [T_{\mu\nu}(x), T_{\alpha\beta}(y)] \, \rangle_\beta$$

$$\eta = - \lim_{\omega \to 0} \frac{1}{\omega} \, \mathrm{Im} \left(G^R_{xy,xy}(\omega, \mathbf{0}) \right) .$$

Kubo formula for shear viscosity

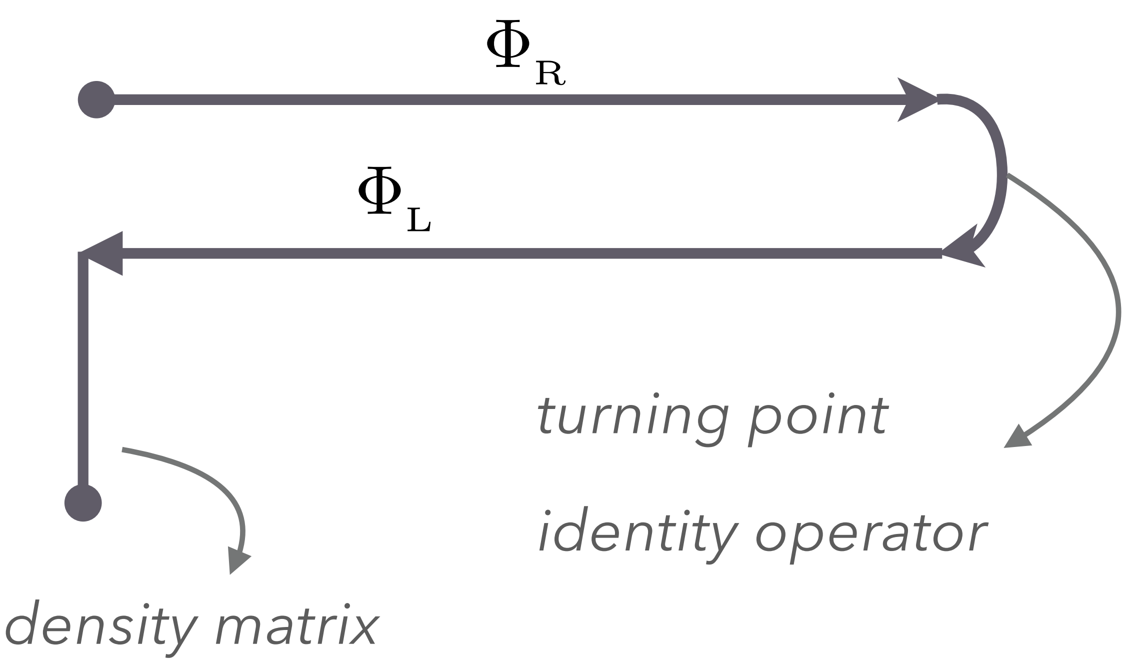
Retarded Response and Schwinger-Keldysh

- Causal response requires being agnostic of future evolution. Achieved explicitly using the closed-time-path, in-in, or Schwinger-Keldysh complex time functional integral contours.



- Think of evolving the disturbed system with the time-dependent Hamiltonian, and then reversing back the evolution to land on the initial state (which you know). Equivalent picture by thinking about phases accrued in evolution (normalization of functional integral).

Schwinger-Keldysh contour



Generating functional (time ordered)

$$\mathcal{Z}_T[\mathcal{J}_{\rm R}, \mathcal{J}_{\rm L}] = \text{Tr}\left(U[\mathcal{J}_{\rm R}]\,\hat{\rho}_T\,(U[\mathcal{J}_{\rm L}])^\dagger\right)$$

$$\int [\mathcal{D}\Phi_{\rm R}][\mathcal{D}\Phi_{\rm L}]\, e^{i\left(S_{SK}+\delta S_{SK}\right)}$$

$$S_{SK} = S[\Phi_{\rm R}] - S[\Phi_{\rm L}] \qquad\qquad \delta S_{SK} = \int d^dx\,\sqrt{-g}\,(\mathcal{J}_{\rm R}\,\mathbb{O}_{\rm R} - \mathcal{J}_{\rm L}\,\mathbb{O}_{\rm L})$$

✦ Convenient to work in doubled Hilbert space (space of bras and kets) with two sets of operators. Physical observables of single-copy theory have natural realization in this space.

SK Time-ordering rules

✦ Time-ordering = contour ordering (R = time-ordered, L=anti-time-ordered)

$$\langle\ \overline{\mathcal{T}}\left[\widehat{\mathbb{O}}^{(1)}\widehat{\mathbb{O}}^{(2)}\ldots\widehat{\mathbb{O}}^{(p)}\right]\ \mathcal{T}\left[\widehat{\mathbb{O}}^{(p+1)}\widehat{\mathbb{O}}^{(p+2)}\ldots\widehat{\mathbb{O}}^{(p+q)}\right]\ \rangle$$
$$\equiv \langle\ \mathcal{T}_{SK}\ \mathbb{O}_{\mathrm{L}}^{(1)}\mathbb{O}_{\mathrm{L}}^{(2)}\ldots\mathbb{O}_{\mathrm{L}}^{(p)}\ \mathbb{O}_{\mathrm{R}}^{(p+1)}\mathbb{O}_{\mathrm{R}}^{(p+2)}\ldots\mathbb{O}_{\mathrm{R}}^{(p+q)}\ \rangle\,,$$

Keldysh (light-cone) basis: $\mathbb{O}_{dif} \equiv \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}}\ , \qquad \mathbb{O}_{av} \equiv \frac{1}{2}\left(\mathbb{O}_{\mathrm{R}} + \mathbb{O}_{\mathrm{L}}\right)$

Response functions computed directly from the contour-ordered objects simply in Keldysh basis

$$\langle\ \mathcal{T}_{SK}\ \mathbb{A}_{av}(x)\ \mathbb{B}_{av}(y)\ \rangle = \langle\ \left\{\widehat{\mathbb{A}}(x),\widehat{\mathbb{B}}(y)\right\}\ \ \rangle\,,$$
$$\langle\ \mathcal{T}_{SK}\ \mathbb{A}_{av}(x)\ \mathbb{B}_{dif}(y)\ \rangle = \Theta_{\mathbb{AB}}\,\langle\ \left[\widehat{\mathbb{A}}(x),\widehat{\mathbb{B}}(y)\right]\ \ \rangle\,,$$
$$\langle\ \mathcal{T}_{SK}\ \mathbb{A}_{dif}(x)\ \mathbb{B}_{av}(y)\ \rangle = -\Theta_{\mathbb{BA}}\ \langle\ \left[\widehat{\mathbb{A}}(x),\widehat{\mathbb{B}}(y)\right]\ \ \rangle\,,$$
$$\langle\ \mathcal{T}_{SK}\ \mathbb{A}_{dif}(x)\ \mathbb{B}_{dif}(y)\ \rangle = 0\,.$$

HYDRODYNAMICS: TRANSPORT, FLUCTUATIONS

- Hydrodynamics: low energy dynamics of conserved currents in near equilibrium situations.
- Transport is captured by response functions: these are the first non-trivial correlators involving 1-average and rest difference operators.

- KMS relations relate response functions to fluctuations, e.g., and embody the fluctuation-dissipation theorem:

$$\hat{\rho}_T = e^{-\beta H}$$

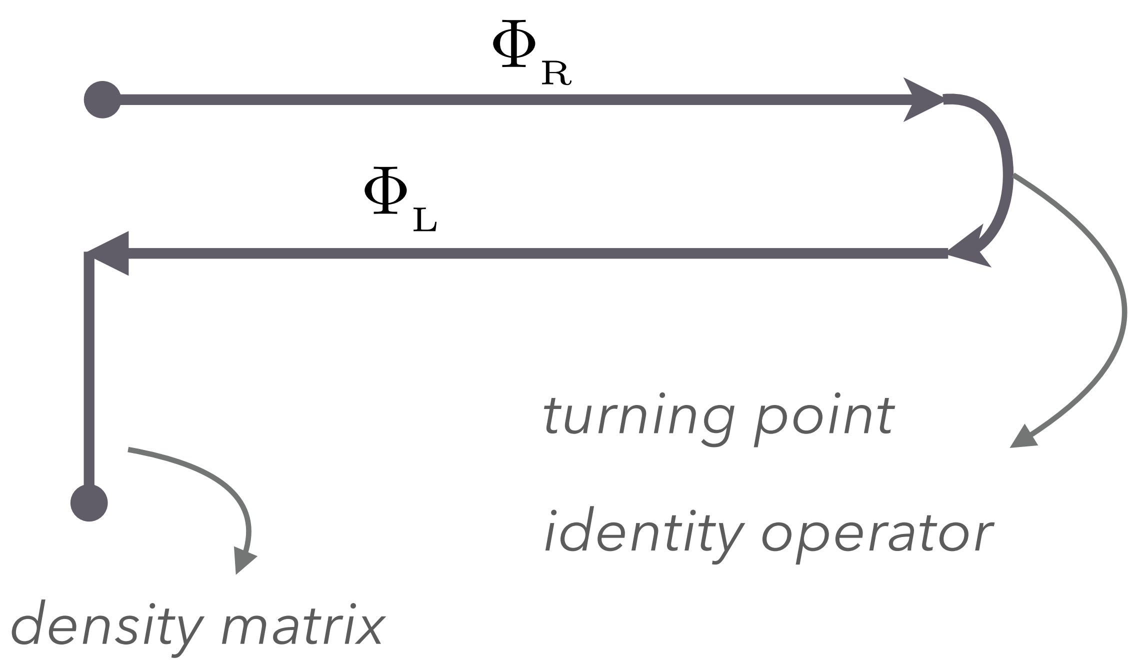
$$\mathrm{Tr}\left(\widehat{\mathbb{A}}(t_A)\,\widehat{\mathbb{B}}(t_B)\hat{\rho}_T\right) = \mathrm{Tr}\left(\widehat{\mathbb{B}}(t_B - i\,\beta)\,\widehat{\mathbb{A}}(t_A)\hat{\rho}_T\right)$$

$$\Longrightarrow \quad \langle\{\widehat{\mathbb{A}},\widehat{\mathbb{B}}\}\rangle = -\coth\left(\frac{1}{2}\,\beta\,\omega_B\right)\,\langle[\widehat{\mathbb{A}},\widehat{\mathbb{B}}]\rangle$$

- Look to constructing an effective field theory that captures all hydrodynamic transport & attendant fluctuations.

Lecture 3

Schwinger-Keldysh contour: recap



Generating functional (time ordered)

$$\mathcal{Z}_T[\mathcal{J}_{\rm R}, \mathcal{J}_{\rm L}] = \text{Tr}\left(U[\mathcal{J}_{\rm R}]\,\hat{\rho}_T\,(U[\mathcal{J}_{\rm L}])^\dagger\right)$$

$$\int [\mathcal{D}\Phi_{\rm R}][\mathcal{D}\Phi_{\rm L}]\, e^{i\,\left(S_{SK}+\delta S_{SK}\right)}$$

$$S_{SK} = S[\Phi_{\rm R}] - S[\Phi_{\rm L}] \qquad \delta S_{SK} = \int d^dx\,\sqrt{-g}\,\left(\mathcal{J}_{\rm R}\,\mathbb{O}_{\rm R} - \mathcal{J}_{\rm L}\,\mathbb{O}_{\rm L}\right)$$

- ✦ Convenient to work in doubled Hilbert space (space of bras and kets) with two sets of operators. Physical observables of single-copy theory have natural realization in this space.

SK effective field theory

- Initial state correlations inherent in the density matrix lead to non-trivial LR correlation functions.

  - Therefore, integrating out high energy modes starting from microscopic Schwinger-Keldysh leads to coupling between L and R encoded in influence functionals.

- What influence functionals are consistent with microscopic unitarity?

Feynman, Vernon '63

- Examine Schwinger-Keldysh Ward identities and encode them in terms of symmetry principles.

SK topological limit

- Lorentz signature inner product in R-L basis from forward/backward evolution implies:

$$\mathcal{Z}_{SK}[\mathcal{J}_{\rm R} = \mathcal{J}_{\rm L} = \mathcal{J}] = {\rm Tr}\Big\{\hat{\rho}_{\rm initial}\Big\}$$

- Equal sources on L-R collapses to a theory of initial conditions.

$$\langle \mathcal{T}_{SK} \prod_k \left(\mathbb{O}_{\rm R}^{(k)} - \mathbb{O}_{\rm L}^{(k)}\right)\rangle \equiv \langle \mathcal{T}_{SK} \prod_k \ \mathbb{O}_{dif}^{(k)}\rangle = 0$$

- Keldysh (light-cone) basis $\qquad \mathbb{O}_{dif} \equiv \mathbb{O}_{\rm R} - \mathbb{O}_{\rm L} \ , \qquad \mathbb{O}_{av} \equiv \frac{1}{2}\left(\mathbb{O}_{\rm R} + \mathbb{O}_{\rm L}\right)$

- Rather remarkable statement, which is agnostic of microscopic dynamics.

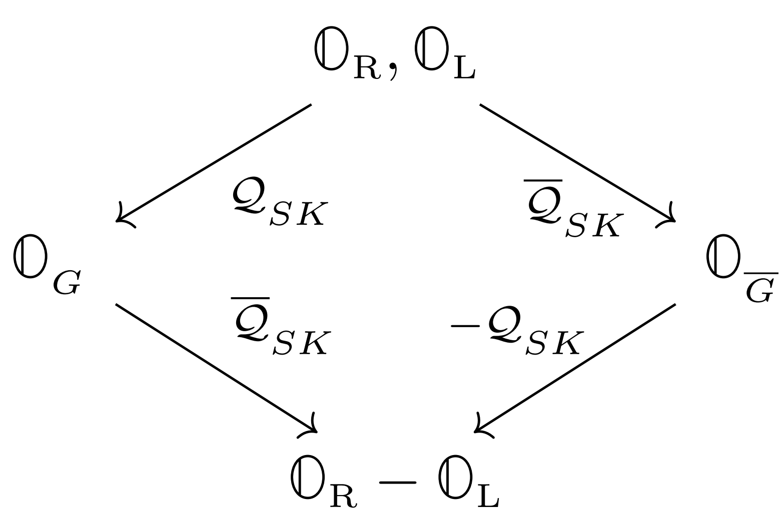
- It is a Ward identity arising from field redefinition symmetry inherent in doubling; rephrase as a BRST symmetry.

- *Largest time equation*: Difference operator cannot be futuremost.

The Schwinger-Keldysh quartet

- Difference operator correlation functions would vanish if they were trivial elements of a BRST cohomology.
- There exists a pair of BRST charges which annihilate the BRST charges. They are Grassmann odd implying that we have ghost SK-partners.
- The SK theory is covariantly expressed in terms of a quartet of fields, which usual doubled formalism being a gauge fixed version (ghosts =0).

$$
\begin{aligned}
&[\mathcal{Q}_{SK}, \mathbb{O}_{\rm L}]_\pm = [\mathcal{Q}_{SK}, \mathbb{O}_{\rm R}]_\pm = \mathbb{O}_G, \quad [\mathcal{Q}_{SK}, \mathbb{O}_G]_\pm = 0, \quad \left[\mathcal{Q}_{SK}, \mathbb{O}_{\overline{G}}\right]_\pm = -\left(\mathbb{O}_{\rm R} - \mathbb{O}_{\rm L}\right), \\
&\left[\overline{\mathcal{Q}}_{SK}, \mathbb{O}_{\rm L}\right]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \mathbb{O}_{\rm R}\right]_\pm = \mathbb{O}_{\overline{G}}, \quad \left[\overline{\mathcal{Q}}_{SK}, \mathbb{O}_{\overline{G}}\right]_\pm = 0, \quad \left[\overline{\mathcal{Q}}_{SK}, \mathbb{O}_G\right]_\pm = \left(\mathbb{O}_{\rm R} - \mathbb{O}_{\rm L}\right).
\end{aligned}
$$



$$
\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{SK}\right]_\pm = 0
$$

$$
\left[\mathcal{Q}_{SK}, \mathbb{O}_{dif}\right]_\pm = 0 \ , \qquad \left[\overline{\mathcal{Q}}_{SK}, \mathbb{O}_{dif}\right]_\pm = 0
$$

Thermal density matrices and KMS condition

- ✦ Thermal density matrices $\hat{\rho}_T = e^{-\beta\,(\widehat{\mathbb{H}} - \mu_I\,\widehat{\mathbb{Q}}^I)}$ define stationary equilibrium configurations.

- ✦ Correlation functions have analyticity properties which allows for a Euclidean (Matsubara) formulation, cf., $\mathcal{Z}_T(\beta, \mu_I) = \mathrm{Tr}\left(\hat{\rho}_T\right)$

$$\mathcal{Z}_T[\mathcal{J}_{\mathrm{R}}, \mathcal{J}_{\mathrm{L}}] = \mathrm{Tr}\left(U[\mathcal{J}_{\mathrm{R}}]\,\hat{\rho}_T\,(U[\mathcal{J}_{\mathrm{L}}])^\dagger\right)$$

- ✦ KMS condition asserts that the correlation functions are analytic in the time strip $0 < \Im(t) < \beta$.

- ✦ Equivalently within correlation functions, operators and their KMS conjugates (or thermal translates) are equivalent.

KMS conjugates & thermal sum rules

- To extract the physical content of the KMS condition, let us define the KMS conjugate operator:

$$\tilde{\mathbb{O}}_{\rm L}(t) = \mathbb{O}_{\rm L}(t - i\beta)$$

- One corollary of the KMS condition and the structure of the SK correlation functions discussed earlier is the sum rule

$$\langle \mathcal{T}_{SK} \prod_{k=1}^{n} \left(\mathbb{O}_{\rm R}^{(k)} - \tilde{\mathbb{O}}_{\rm L}^{(k)} \right) \rangle = 0$$

- It is useful to rephrase these sum rules by passing from the L-R basis to the advanced-retarded basis $e^{-i\delta_\beta}\mathbb{O}(t) = \mathbb{O}(t - i\beta)$

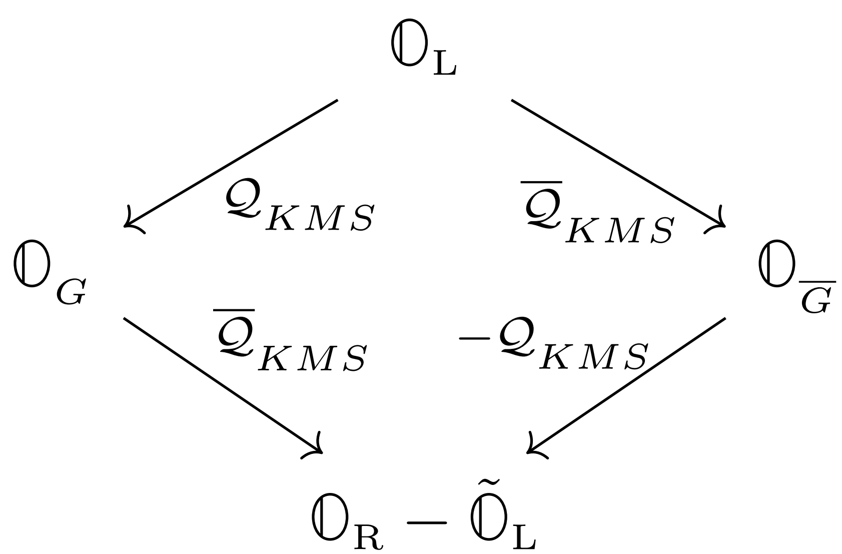
$$\mathbb{O}_{adv} = \mathbb{O}_{\rm R} - \mathbb{O}_{\rm L}\,, \qquad \mathbb{O}_{ret} = \frac{1}{1 - e^{-i\,\delta_\beta}} \left(\mathbb{O}_{\rm R} - e^{-i\delta_\beta}\mathbb{O}_{\rm L} \right)$$

- *Thermal smallest time* equation: retarded operator cannot be pastmost.

The KMS charges

- The KMS conditions can be implemented by requiring the presence of a second pair of BRST charges which refer to the thermally translated KMS conjugate operators

$$i\,\Delta_\beta = 1 - e^{-i\,\delta_\beta}$$



$$[\mathcal{Q}_{KMS}, \mathbb{O}_{ret}]_\pm = 0, \qquad \left[\mathcal{Q}_{KMS}, \mathbb{O}_{\overline{G}}\right]_\pm = i\Delta_{\boldsymbol{\beta}}\mathbb{O}_{ret}\,,$$

$$[\mathcal{Q}_{KMS}, \mathbb{O}_G]_\pm = 0, \qquad [\mathcal{Q}_{KMS}, \mathbb{O}_{adv}]_\pm = i\Delta_{\boldsymbol{\beta}}\mathbb{O}_G\;,$$

$$\left[\overline{\mathcal{Q}}_{KMS}, \mathbb{O}_{ret}\right]_\pm = 0, \qquad \left[\overline{\mathcal{Q}}_{KMS}, \mathbb{O}_{\overline{G}}\right]_\pm = 0\,,$$

$$\left[\overline{\mathcal{Q}}_{KMS}, \mathbb{O}_G\right]_\pm = i\Delta_{\boldsymbol{\beta}}\mathbb{O}_{ret}, \qquad \left[\overline{\mathcal{Q}}_{KMS}, \mathbb{O}_{adv}\right]_\pm = i\Delta_{\boldsymbol{\beta}}\mathbb{O}_{\overline{G}}\;.$$

- While deceptively simple, the expressions above hide the fact that the KMS BRST charges are non-local (they relate operators across a thermal period).

The SK-KMS superalgebra

- The SK and KMS BRST charges generate an interesting superalgebra:

$$\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \mathcal{Q}_{KMS}^2 = \overline{\mathcal{Q}}_{KMS}^2 = 0 \,,$$
$$[\mathcal{Q}_{SK}, \mathcal{Q}_{KMS}]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{SK}\right]_\pm = \left[\overline{\mathcal{Q}}_{KMS}, \mathcal{Q}_{KMS}\right]_\pm = 0 \,,$$
$$\left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = -\left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{KMS}\right]_\pm = i\Delta_{\boldsymbol{\beta}} \,.$$

- This algebra is well known in some circles, and forms part of the $\mathcal{N}_T = 2$ extended equivariant cohomology algebra.

Vafa, Witten '94
Dijkgraaf, Moore '96

- The $\mathcal{N}_T = 1$ algebra is realized as the standard Weil algebra satisfied by the de Rham complex involving exterior derivatives, Lie derivative and interior contraction.

THE SCHWINGER-KELDYSH SUPER-QUARTET

- We encode the content of the sum rules into a BRST superalgebra, with the idea that the topological structure is robust against the coarse-graining RG transformation.
- This allows us to identify influence functionals in the low energy theory, which arise from integrating out the high energy modes, and crucially couple the two segments of the Schwinger-Keldysh contour.

Feynman Vernon '63

- Put differently, the macroscopic theory will have a knowledge of the two sets of degrees of freedom
    - the average or classical fields (which dissipate)
    - the difference or quantum/stochastic fluctuation fields
- Consistent couplings are dictated by the ghost fields that are now part of the BRST multiplet. All of this structure can be nicely captured in a superspace that is locally $\mathbb{R}^{(d-1,1)|2}$ coordinatized by $z^I = \{\sigma^a, \theta, \bar{\theta}\}$.

THE WEIL MODEL

- Gauge structure can be captured by a Grassmann odd gauge potential G (fermions = differential forms) and its field strength ϕ

$$dG^i + \frac{1}{2} f^i_{jk}\, G^j\, G^k = \phi^i$$
$$d\phi^i + f^i_{jk}\, G^j\, \phi^k = 0$$

Cartan equations for gauge structure

$$\delta^i_j + \mathcal{I}_j G^i = 0\,, \quad \mathcal{I}_j \phi^i = 0$$

interior contractions pick out gauge directions

$$\mathcal{L}_j = \{d, \mathcal{I}_j\}$$

Lie derivations follow from above

Weil superalgebra

$$\{\mathcal{I}_i, \mathcal{I}_j\} = 0 \qquad \{d, \mathcal{I}_j\} = \mathcal{L}_j$$
$$[\mathcal{L}_i, \mathcal{I}_j] = -f^k_{ij}\, \mathcal{I}_k \qquad [d, \mathcal{L}_j] = 0$$
$$[\mathcal{L}_i, \mathcal{L}_j] = -f^k_{ij}\, \mathcal{L}_k \qquad \{d, d\} = 0$$

- *invariant horizontal forms are polynomial functions of field strengths.*

EXTENDED EQUIVARIANCE I

- One can extend the algebraic constructions to situations with more than one differential. We will focus on the case with 2 generators of the cohomology and swiftly pass to superspace: $\mathbb{d}_{\mathrm{W}} = \partial_{\bar\theta}(\ldots)| \,, \qquad \bar{\mathbb{d}}_{\mathrm{W}} = \partial_{\theta}(\ldots)| \,.$

- The Weil model closes on 6 generators: 2 derivations, 3 interior contraction, and one Lie derivation

$$\mathbb{d}_{\mathrm{W}}^2 = \bar{\mathbb{d}}_{\mathrm{W}}^2 = \left[\mathbb{d}_{\mathrm{W}}, \bar{\mathbb{d}}_{\mathrm{W}}\right]_{\pm} = 0$$

$$\left[\mathbb{d}_{\mathrm{W}}, \bar{\mathcal{I}}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{I}_j\right]_{\pm} = \mathcal{L}_j \,, \qquad \left[\mathbb{d}_{\mathrm{W}}, \mathcal{I}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \bar{\mathcal{I}}_j\right]_{\pm} = 0$$

$$\left[\mathbb{d}_{\mathrm{W}}, \mathcal{I}^0_j\right]_{\pm} = \mathcal{I}_j \,, \qquad \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{I}^0_j\right]_{\pm} = -\bar{\mathcal{I}}_j$$

$$\left[\mathbb{d}_{\mathrm{W}}, \mathcal{L}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{L}_j\right]_{\pm} = 0$$

$$\left[\bar{\mathcal{I}}_i, \mathcal{I}_j\right]_{\pm} = f^k_{ij}\, \mathcal{I}^0_k$$

$$\left[\mathcal{L}_i, \bar{\mathcal{I}}_j\right]_{\pm} = -f^k_{ij}\, \bar{\mathcal{I}}_k \,, \qquad \left[\mathcal{L}_i, \mathcal{I}_j\right]_{\pm} = -f^k_{ij}\, \mathcal{I}_k \,, \qquad \left[\mathcal{L}_i, \mathcal{I}^0_j\right]_{\pm} = -f^k_{ij}\, \mathcal{I}^0_k \,.$$

Blau, Thompson '91
Cordes, Moore, Ramgoolam '94

Vafa, Witten '94
Dijkgraaf, Moore '96

SUPERSPACE

- We have two cohomology generators which we assign equal and opposite ghost charge ± 1
- Introduce two new Grassmann coordinates $\theta, \bar{\theta}$ with assignments $\text{gh}(\theta) = 1, \ \text{gh}(\bar{\theta}) = -1$
- All fields are promoted to superfields which have a finite expansion in the Grassmann coordinates. The bottom component will turn out to be the average fields and the top components the difference fields.

$$\mathring{\mathfrak{F}} = \mathfrak{F}(\sigma, \theta, \bar{\theta}) = \mathfrak{F} + \theta\, \mathfrak{F}_{\overline{\psi}} + \bar{\theta}\, \mathfrak{F}_{\psi} + \bar{\theta}\, \theta\, \tilde{\mathfrak{F}}$$

- Much of the discussion can be carried out by switching off the ghost fields, except for a few source terms which we will do when we return to hydrodynamics.
- CPT symmetry nicely implementing as a swap of the two Grassmann coordinates (difference operators are CPT odd).

THE GAUGE SECTOR

- While the SK functional integral gives us the BRST symmetry which can be captured by working in superspace we also have the gauge sector.
- We can parameterize the gauge sector quite universally, and then return to the actual gauge symmetry transformations.
- Package the universal data into a set of gauge superfield 1-form which we assume lives on a worldvolume with coordinates σ^a.

$$\mathring{\mathcal{A}} = \mathring{\mathcal{A}}_I \, dz^I = \mathring{\mathcal{A}}_a \, d\sigma^a + \mathring{\mathcal{A}}_\theta \, d\theta + \mathring{\mathcal{A}}_{\bar{\theta}} \, d\bar{\theta} \,.$$

- Covariant derivatives fields strengths are defined as usual, except that we now have superspace directions as well:

$$\mathring{\mathcal{D}}_I = \partial_I + [\mathring{\mathcal{A}}_I, \ \cdot \ ] \,, \qquad \mathring{\mathfrak{F}}_{IJ} \equiv (1 - \frac{1}{2}\,\delta_{IJ}) \left(\partial_I \, \mathring{\mathcal{A}}_J - (-)^{IJ} \, \partial_J \, \mathring{\mathcal{A}}_I + [\mathring{\mathcal{A}}_I, \mathring{\mathcal{A}}_J]\right)$$

GAUGE SECTOR II

- Analyzing the field strength, Bianchi identities etc., the eight components of the gauge superfield 1-form with legs in superspace be captured into 3 gauge non-invariant potentials, 3 field strengths, and 2 derivatives of field strengths

ghost charge	Faddeev-Popov ghost triplet	Vafa-Witten ghost of ghost quintet	Vector quartet
2		ϕ	
1	G	η	λ_a
0	B	ϕ^0	$\mathcal{A}_a$ $\mathcal{F}_a$
-1	$\bar{G}$	$\bar{\eta}$	$\bar{\lambda}_a$
-2		$\bar{\phi}$	

$$\phi = \mathring{\mathfrak{F}}_{\bar{\theta}\bar{\theta}}|$$
$$\phi^0 = \mathring{\mathfrak{F}}_{\theta\bar{\theta}}|$$
$$\bar{\phi} = \mathring{\mathfrak{F}}_{\theta\theta}|$$

- Cartan charges are gauge-covariant super derivations and obey:

$$\mathring{\mathcal{D}}_{\bar{\theta}}^2 = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\bar{\theta}\bar{\theta}}}\,, \qquad \mathring{\mathcal{D}}_{\theta}^2 = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\theta\theta}}\,, \qquad \left[\mathring{\mathcal{D}}_{\bar{\theta}}, \mathring{\mathcal{D}}_{\theta}\right]_{\pm} = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\theta\bar{\theta}}}\,.$$

SK-KMS THERMAL EQUIVARIANCE

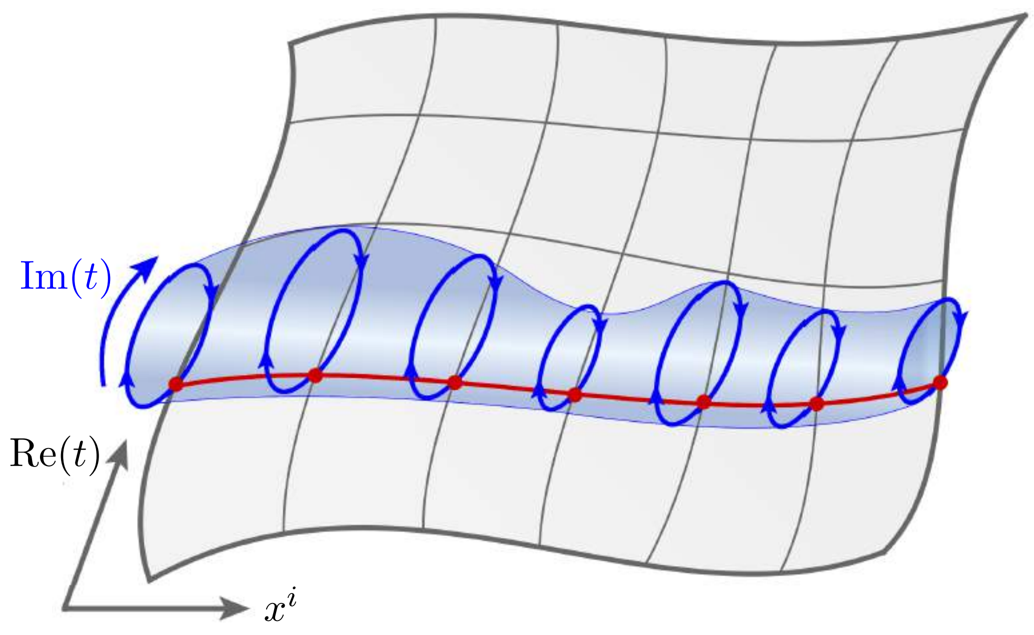
- SK charges are akin to Weil differentials, while the KMS charges fill out the interior contractions.
- The Lie derivation takes operators around the thermal circle.

$$
\begin{array}{rcl}
\underline{\mathcal{N}_{\mathsf{T}} = 2 \text{ algebra}} & | & \underline{\text{SK-KMS symmetries}} \\
\{\mathbb{d}_{\rm W}, \overline{\mathbb{d}}_{\rm W}\} & \leftrightarrow & \{\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{SK}\}\,, \\
\{\mathcal{I}_k, \overline{\mathcal{I}}_k\} & \leftrightarrow & \{\mathcal{Q}_{KMS}, \overline{\mathcal{Q}}_{KMS}\}\,, \\
\{\mathcal{L}_k, \mathcal{I}^0_k\} & \leftrightarrow & \{\mathcal{L}_{KMS}, \mathcal{Q}^0_{KMS}\}\,.
\end{array}
$$

- The algebraic structure for arbitrary temperature is complicated by non-locality of thermal translations.
- Some form of deformation of the group of circle diffeomorphisms…

SK-KMS THERMAL EQUIVARIANCE

- Life is simpler at high temperatures when thermal circle is small.



- Literally implement thermal translations as diffeomorphisms along the thermal circle and demand equivariance with this symmetry. $\mathring{\mathcal{L}}^{\text{KMS}} \mathring{\mathbb{O}} = \Delta_{\boldsymbol{\beta}} \mathring{\mathbb{O}}$

$$\mathring{\mathcal{A}}_I \mapsto \mathring{\mathcal{A}}_I + (\mathring{\Lambda}, \mathring{\mathcal{A}}_I)_{\boldsymbol{\beta}} - \partial_I \mathring{\Lambda}\,, \qquad (\mathring{\Lambda}, \mathring{\Lambda}')_{\boldsymbol{\beta}} = \mathring{\Lambda} \pounds_{\boldsymbol{\beta}} \mathring{\Lambda}' - \mathring{\Lambda}' \pounds_{\boldsymbol{\beta}} \mathring{\Lambda}\,.$$

- The algebra is non-abelian since it involves diffeomorphisms along the thermal vector.

THERMAL CARTAN AND WEIL MODELS

- ✦ The gauge covariant Cartan charges (supercovariant derivations) can be mapped to the basic building blocks as follows:

$$\mathcal{Q} \equiv \mathcal{Q}_{SK} + \phi_{\mathsf{T}}\, \overline{\mathcal{Q}}_{KMS} + \phi^0_{\mathsf{T}}\, \overline{\mathcal{Q}}_{KMS} + \eta_{\mathsf{T}}\, \mathcal{Q}^0_{KMS}\,,$$
$$\overline{\mathcal{Q}} \equiv \overline{\mathcal{Q}}_{SK} + \overline{\phi}_{\mathsf{T}}\, \mathcal{Q}_{KMS}\,.$$

- ✦ The superalgebra structure can then be captured by the anti-commutation relation among the Cartan charges as

$$\mathcal{Q}^2 = (\mathring{\mathfrak{F}}_{\bar\theta\bar\theta}|_{\bar\theta=\theta=0})\,\mathcal{L}_{KMS}\,, \qquad \overline{\mathcal{Q}}^2 = (\mathring{\mathfrak{F}}_{\theta\theta}|_{\bar\theta=\theta=0})\,\mathcal{L}_{KMS}\,, \qquad [\mathcal{Q},\overline{\mathcal{Q}}]_\pm = (\mathring{\mathfrak{F}}_{\theta\bar\theta}|_{\bar\theta=\theta=0})\,\mathcal{L}_{KMS}$$

➤ Assume: dynamically consistent in dissipative systems to set all but the zero ghost number element of the Vafa-Witten quintet to zero: $\langle \mathring{\mathcal{F}}_{\theta\bar\theta}|\rangle = -i$

$$\mathcal{Q}^2 = 0\,, \qquad \overline{\mathcal{Q}}^2 = 0\,, \qquad [\mathcal{Q},\overline{\mathcal{Q}}]_\pm = -i\,\mathcal{L}_{KMS} \mapsto i\pounds_{\boldsymbol{\beta}}$$

CONSTRAINTS ON LOW ENERGY DYNAMICS

- Topological (BRST super) symmetries are efficient ways to encode SK + KMS constraints.

HLR '15
Crossley, Glorioso, Liu '15

Unitarity → SK BRST supercharges

KMS/FDT → thermal diffeomorphism gauge symmetry

$$\mathcal{Q}^2 \sim 0\,, \qquad \bar{\mathcal{Q}}^2 \sim 0\,, \qquad \{\mathcal{Q}, \bar{\mathcal{Q}}\} \sim i\,\beta\,\partial_t \equiv i\,\pounds_\beta$$

- Precedent: Langevin dynamics of a Brownian particle

Martin, Siggia, Rose (1973) *Parisi Sourlas (1982)*

- Direct implementation of MSR type logic in hydrodynamics

Kovtun, Moore, Romatschke (2014)

LOW ENERGY CONSTRAINTS II

- Effective dynamics constrained by

$$\mathcal{Q}^2 = -\mathring{\mathcal{F}}_{\bar\theta\bar\theta} | \pounds_{\boldsymbol\beta} \,, \qquad \bar{\mathcal{Q}}^2 = -\mathring{\mathcal{F}}_{\theta\theta} | \pounds_{\boldsymbol\beta} \,, \qquad \{\mathcal{Q}, \bar{\mathcal{Q}}\} = -\mathring{\mathcal{F}}_{\theta\bar\theta} | \pounds_{\boldsymbol\beta}$$

HLR [1510.02494]

Crossley, Glorioso, Liu [1511.03646]

* Basic BRST charges $\mathcal{Q}_{SK}$, $\bar{\mathcal{Q}}_{SK}$ are nilpotent and arise from SK unitarity & are CPT conjugates.

HLR + Geracie, Narayan, Ramirez [1712.04459]

* KMS conditions are implemented by gauge these BRST charges (equivariance).

$$\mathcal{Q}^2 = 0 \,, \quad \bar{\mathcal{Q}}^2 = 0 \,, \quad \{\mathcal{Q}, \bar{\mathcal{Q}}\} = i \, \pounds_{\boldsymbol\beta}$$

* Single nilpotent BRST supercharge δ from SK unitarity.

* KMS condition is an involution (after combining with CPT) and gives another supercharge $\bar\delta$.

$$\delta^2 = \bar\delta^2 = 0, \qquad \{\delta, \bar\delta\} = 2 \tanh\left(\frac{i}{2} \beta \, \partial_t\right) \simeq i \, \beta \, \partial_t$$

- This algebra is well known in the statistical mechanics literature in the context of stochastic Langevin dynamics.

Mallick, Moshe, Orland [1009.4800]

Lecture 4

CONSTRAINTS ON LOW ENERGY DYNAMICS

- Topological (BRST super) symmetries are efficient ways to encode SK + KMS constraints.

HLR '15

Crossley, Glorioso, Liu '15

Unitarity → SK BRST supercharges

KMS/FDT → thermal diffeomorphism gauge symmetry

$$\mathcal{Q}^2 \sim 0\,, \qquad \bar{\mathcal{Q}}^2 \sim 0\,, \qquad \{\mathcal{Q}, \bar{\mathcal{Q}}\} \sim i\,\beta\,\partial_t \equiv i\,\pounds_\beta$$

- Precedent: Langevin dynamics of a Brownian particle

Martin, Siggia, Rose (1973) *Parisi Sourlas (1982)*

- Direct implementation of MSR type logic in hydrodynamics

Kovtun, Moore, Romatschke (2014)

TOY MODEL: LANGEVIN DYNAMICS

- Point particle in external potential subject to external forcing and noise

$$m\frac{d^2x}{dt^2} + \frac{\partial U}{\partial x} + \nu\, \Delta_{\boldsymbol{\beta}} x = \mathbb{N}$$

- One can write down a SK effective action for this dissipative dynamics

$$x = -i\,\Delta_{\boldsymbol{\beta}}^{-1}\left(x_{\rm R} - e^{-i\,\delta_{\boldsymbol{\beta}}}\, x_{\rm L}\right)\,, \qquad \tilde{x} = x_{\rm R} - x_{\rm L}$$

Martin, Siggia, Rose 1973

$$\mathcal{L}_{SK} = \left[\tilde{x}\ \frac{\partial U}{\partial x} + \bar{\psi}\frac{\partial^2 U}{\partial x^2}\psi\right] - m\left[\tilde{x}\ \frac{d^2x}{dt^2} + \bar{\psi}\ \frac{d^2\psi}{dt^2}\right] - \nu\left[\tilde{x}\ \Delta_{\boldsymbol{\beta}} x - \bar{\psi}\ \Delta_{\boldsymbol{\beta}}\psi\right] + i\,\nu\,\tilde{x}^2\,.$$

- The dissipative part of the action is controlled by ghosts and is related to the fluctuation terms difference fields - fluctuation/dissipation relation.
- Convergence of the path integral fixes the sign of dissipative terms.
- Simplest realization of the extended equivariant cohomology algebra.

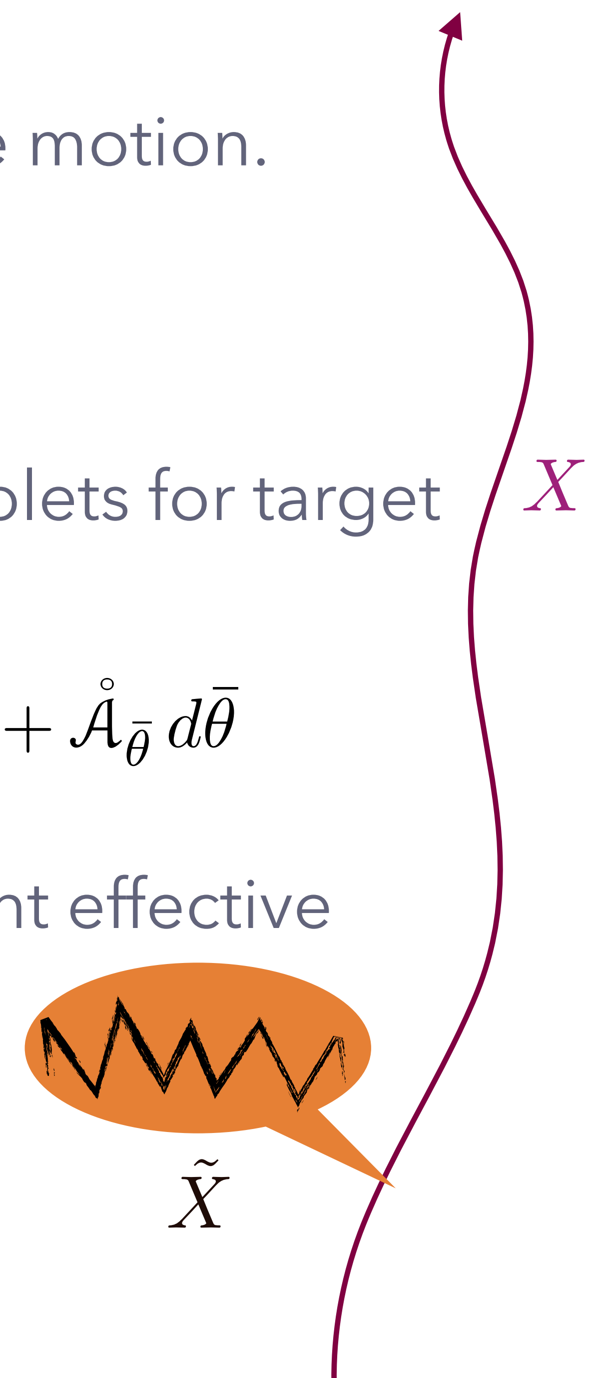
BROWNIAN BRANES

- Brownian particle immersed in a fluid undergoes dissipative motion.
- Langevin effective action: worldvolume B0-brane theory.
- Data for the worldvolume theory: thermal equivariant multiplets for target space coordinate map and thermal gauge field data.

$$\mathring{X} = X + \theta X_{\overline{\psi}} + \bar{\theta}\, X_{\psi} + \bar{\theta}\,\theta\,\tilde{X} \qquad\qquad \mathring{\mathcal{A}} \equiv \mathring{\mathcal{A}}_t\, dt + \mathring{\mathcal{A}}_\theta\, d\theta + \mathring{\mathcal{A}}_{\bar{\theta}}\, d\bar{\theta}$$

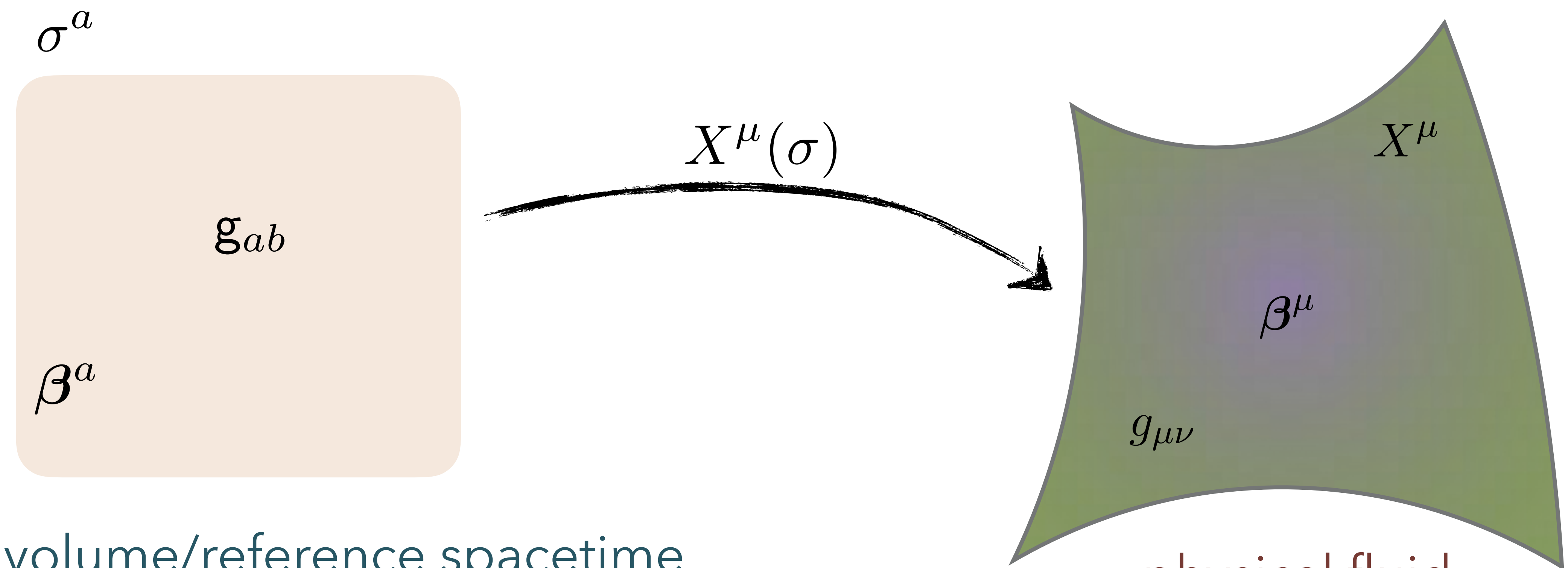
- MSR action follows as the basic thermal *U(1)$_T$* gauge invariant effective action of the worldline theory

$$S_{\text{B0}} = \int dt\, d\theta\, d\bar{\theta} \left\{ \frac{m}{2} \left(\mathring{\mathcal{D}}_t \mathring{X}\right)^2 - U(\mathring{X}) - i\,\nu\, \mathring{\mathcal{D}}_\theta \mathring{X} \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{X} \right\}$$



$$(\mathring{\Lambda}, \mathring{X})_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \pounds_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \Delta_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \boldsymbol{\beta} \frac{d}{dt} \mathring{X} \qquad\qquad \mathring{\mathcal{D}}_I = \partial_I + [\mathring{\mathcal{A}}_I,\ \cdot\ ]$$

Hydrodynamic sigma models recap



worldvolume/reference spacetime

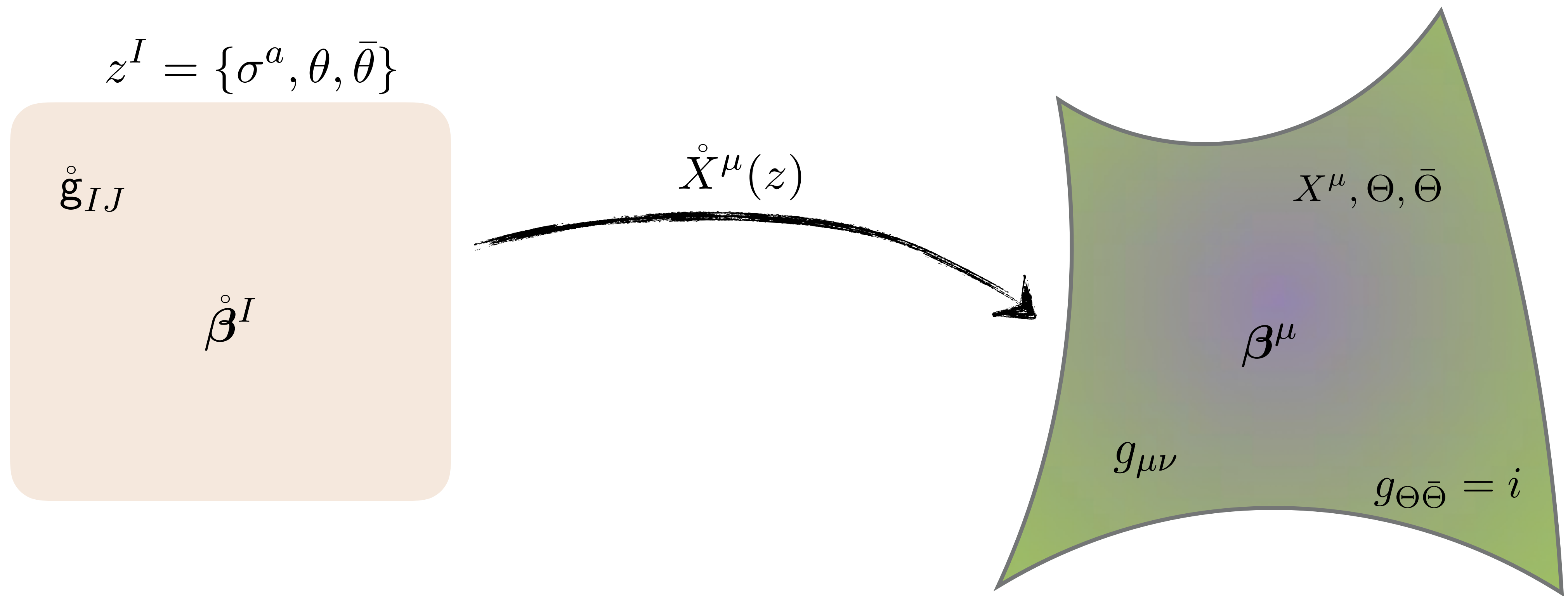
$$\mathrm{g}_{ab} = g_{\mu\nu}\partial_a X^\mu\, \partial_b X^\nu$$

physical fluid

$$\boldsymbol{\beta}^\mu = \boldsymbol{\beta}^a \partial_a X^\mu$$

- ⋆ Worldvolume with fixed reference thermal vector $\boldsymbol{\beta}^a$
- ⋆ Physical degrees of freedom are the target space maps, leading to conservation equation as dynamics
- ⋆ Upgrade to include SK+KMS constraints

Topological sigma models: Fields



worldvolume superspace

gauge fixed target supergeometry

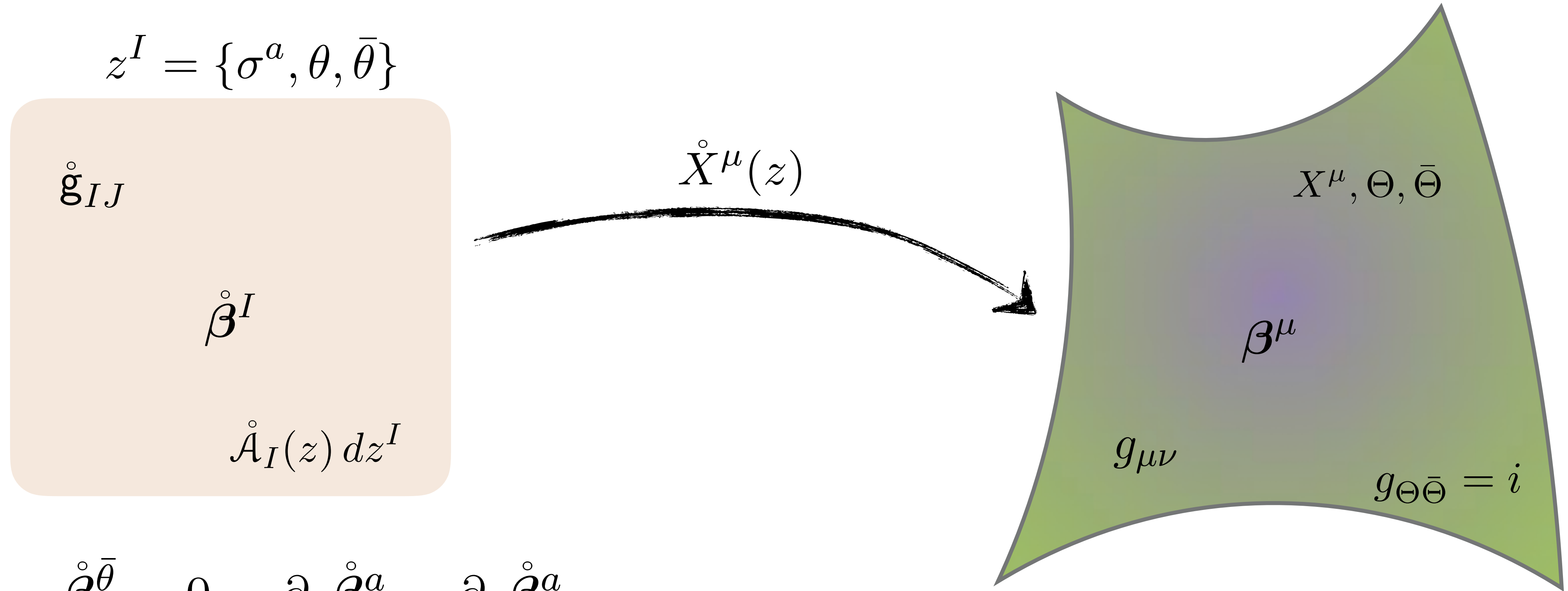
$$\mathring{X}^{\breve\mu}(z) = \{\mathring{X}^\mu(z), \mathring{\Theta}(z), \mathring{\bar\Theta}(z)\}$$

$$\mathring{\Theta} = \theta\,, \qquad \mathring{\bar\Theta} = \bar\theta\,,$$

$$g_{\mu\Theta} = g_{\mu\bar\Theta} = 0\,, \qquad g_{\Theta\bar\Theta} = -g_{\bar\Theta\Theta} = i\,.$$

$$\mathring{X}^\mu = X^\mu + \theta\, X^\mu_{\bar\psi} + \bar\theta\, X^\mu_\psi + \bar\theta\theta\left(\tilde{X}^\mu - \Gamma^\mu_{\rho\sigma}\, X^\rho_{\bar\psi}\, X^\sigma_\psi\right)$$

Topological sigma models: Gauge Symmetry



$$\mathring{\boldsymbol{\beta}}^\theta = \mathring{\boldsymbol{\beta}}^{\bar{\theta}} = 0 = \partial_\theta \mathring{\boldsymbol{\beta}}^a = \partial_{\bar{\theta}} \mathring{\boldsymbol{\beta}}^a$$

$$(\mathring{\Lambda}, \mathring{X}^\mu)_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \mathring{\boldsymbol{\beta}}^I \partial_I \mathring{X}^\mu$$

- ⋆ KMS or fluctuation/dissipation data encoded by the thermal diffeomorphism gauge invariance.
- ⋆ Gauged topological sigma model with a topological (BF-type) gauge kinetic term captures consistently all constraints on the low energy influence functionals.

Gauge+matter degrees of freedom

ghost charge	Faddeev-Popov ghost triplet	Vafa-Witten ghost of ghost quintet	Vector quartet	Position multiplet
2		ϕ_{T}		
1	G_{T}	η_{T}	λ_a	X^{μ}_{ψ}
0	B_{T}	ϕ^0_{T}	$\mathcal{A}_a$ $\mathcal{F}_a$	X^{μ} $\tilde{X}^{\mu}$
-1	$\overline{G}_{\mathsf{T}}$	$\bar{\eta}_{\mathsf{T}}$	$\bar{\lambda}_a$	$X^{\mu}_{\bar{\psi}}$
-2		$\overline{\phi}_{\mathsf{T}}$		

Source deformed topological sigma models

★ Physical fluids are of course not captured by the topological data alone. We need to deform the sigma model to get physical dynamics.

★ Topological sector is the theory of all difference correlators in a convenient form (no dynamics in this sector).

★ Source deformation by a difference source, will insert an average operator into the functional integral as necessary to compute hydrodynamic response functions.

★ Easily implemented in the sigma model by

$$\mathring{\mathrm{g}}_{IJ}(z) \;\to\; \mathring{\mathrm{g}}_{IJ}(z) + \bar{\theta}\,\theta\; h_{IJ}(\sigma)\,.$$

Time reversal symmetry breaking

★ Our discussion thus far is time-reversal symmetric (has to be since we are keeping all constraints from microscopics).

★ Time-reversal is broken dynamically by a choice of vacuum.

★ We assume that there exists dynamics for thermal diffeomorphisms consistent with a gauge invariant ghost number zero field strength getting a vev.

$$\langle \mathcal{F}_{\bar{\theta}\theta} \rangle = -i$$

★ Note this is consistent with CPT action exchanging $\theta \leftrightarrow \bar{\theta}$

★ If we implement this a-priori we end up with the stochastic BRST algebra as discussed.

$$\langle \mathring{\mathcal{A}}_a \rangle = 0 \,, \qquad \langle \mathring{\mathcal{A}}_{\bar{\theta}} \rangle = 0 \,, \qquad \langle \mathring{\mathcal{A}}_\theta \rangle = \bar{\theta}\,(-i)$$

MMO limit

$$\mathcal{Q}^2 = \overline{\mathcal{Q}}^2 = 0 \,, \qquad \{\mathcal{Q}, \overline{\mathcal{Q}}\} = i\,\pounds_{\boldsymbol{\beta}}$$

The hydrodynamic effective action

$$S_{wv} = \int d^d\sigma \; \mathcal{L}_{wv} \,, \qquad \mathcal{L}_{wv} = \int d\theta \, d\bar{\theta} \; \frac{\sqrt{-\mathring{\mathrm{g}}}}{\mathring{\mathbf{z}}} \; \mathring{\mathcal{L}}[\mathring{\mathrm{g}}_{IJ}, \boldsymbol{\beta}^a, \mathring{\mathfrak{D}}_I, \mathring{\mathfrak{g}}^{(\overline{\psi})}_{IJ}, \mathring{\mathfrak{g}}^{(\psi)}_{IJ}] \,,$$

- ★ Superspace action manifestly respects topological and thermal diffeomorphism symmetry.
- ★ Physical target space superdiffeomorphisms should be respected, so we cannot have potentials in the target coordinates.
- ★ Allowed worldvolume diffeomorphisms are of the form: $z^I \mapsto z^I + f^I(\sigma^a)$
- ★ Anti-linear CPT involution (broken perhaps by choice of vacuum).
- ★ Ghost number conservation.

The hydrodynamic effective action

⋆ Target space superdiffeomorphisms are manifestly respected by working with the covariant pullback data.

$$S_{wv} = \int d^d\sigma \, \mathcal{L}_{wv} \,, \qquad \mathcal{L}_{wv} = \int d\theta \, d\bar{\theta} \, \frac{\sqrt{-\mathring{\mathfrak{g}}}}{\mathring{\mathbf{z}}} \, \mathring{\mathcal{L}}[\mathring{\mathfrak{g}}_{IJ}, \boldsymbol{\beta}^a, \mathring{\mathfrak{D}}_I, \mathring{\mathfrak{g}}^{(\overline{\psi})}_{IJ}, \mathring{\mathfrak{g}}^{(\psi)}_{IJ}] \,,$$

⋆ Covariant pullback of target implies measure is modified on worldvolume

$$\mathring{\mathbf{z}} = 1 + \mathring{\boldsymbol{\beta}}^I \, \mathring{\mathcal{A}}_I \qquad \mathring{\mathfrak{D}}_I \mathring{X}^\mu = \mathring{\mathcal{D}}_I \mathring{X}^\mu = \partial_I \mathring{X}^\mu + (\mathring{\mathcal{A}}_I, \mathring{X}^\mu)_{\boldsymbol{\beta}} = (-)^J \left(\delta_I{}^J + \mathring{\mathcal{A}}_I \, \mathring{\boldsymbol{\beta}}^J\right) \partial_J \mathring{X}^\mu$$

⋆ Some parts of the worldvolume metric's covariant derivative are dynamical

$$\mathring{\mathfrak{g}}^{(\psi)}_{IJ} \equiv \mathring{\mathcal{D}}_\theta \mathring{\mathfrak{g}}_{IJ} \,, \qquad \mathring{\mathfrak{g}}^{(\psi)}_{IJ} \equiv \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{\mathfrak{g}}_{IJ} \,.$$

dynamical data

⋆The worldvolume geometry+thermal gauge covariant derivative is $\mathfrak{D}$

Superadiabaticity Bianchi identity

- ⋆ Consider the Bianchi identity for the thermal diffeomorpshism gauge symmetry

Super energy-momentum
$$\mathring{\mathbf{T}}^{IJ} \equiv \frac{2}{\sqrt{-\mathring{\mathrm{g}}}} \frac{\delta}{\delta \mathring{\mathrm{g}}_{IJ}} \left(\sqrt{-\mathring{\mathrm{g}}}\ \mathring{\mathcal{L}}\right),$$

Free energy current
$$\mathring{\mathbf{N}}^I \equiv -\frac{\mathring{\mathbf{z}}}{\sqrt{-\mathring{\mathrm{g}}}} \frac{\delta}{\delta \mathring{\mathcal{A}}_I} \left(\frac{\sqrt{-\mathring{\mathrm{g}}}}{\mathring{\mathbf{z}}}\ \mathring{\mathcal{L}}\right).$$

$$\begin{aligned}
\delta_{\mathring{\Lambda}} S_{\mathrm{wv}} &= \int d^d\sigma \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathrm{g}}}}{\mathring{\mathbf{z}}} \left\{\frac{1}{2}\mathring{\mathbf{T}}^{IJ}\, (\mathring{\Lambda}, \mathring{\mathrm{g}}_{IJ})_{\boldsymbol{\beta}} + \mathring{\mathcal{D}}_I(\mathring{\Lambda})\, \mathring{\mathbf{N}}^I\right\} \\
&= \int d^d\sigma \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathrm{g}}}}{\mathring{\mathbf{z}}}\, \mathring{\Lambda} \left\{\frac{1}{2}\, \mathring{\mathbf{T}}^{IJ}\, \pounds_{\boldsymbol{\beta}} \mathring{\mathrm{g}}_{IJ} - \mathring{\mathfrak{D}}_I \mathring{\mathbf{N}}^I\right\}.
\end{aligned}$$

- ⋆ Off-shell super-adiabaticity equation:

$$\mathring{\mathfrak{D}}_I \mathring{\mathbf{N}}^I - \frac{1}{2}\, \mathring{\mathbf{T}}^{IJ}\, \pounds_{\boldsymbol{\beta}} \mathring{\mathrm{g}}_{IJ} = 0\,.$$

$$(\mathring{\Lambda}, \mathring{X}^\mu)_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \mathring{\boldsymbol{\beta}}^I \partial_I \mathring{X}^\mu$$

Superadiabacity & Entropy Inflow

★ We can examine component form of the superadiabaticity equation. Note first the terms that will push-forward to familiar terms in target space:

$$\mathring{\mathfrak{D}}_a \mathring{\mathbf{N}}^a \Big| = \nabla_\mu N^\mu + \text{ghost bilinears} + \text{fluctuations}$$
$$\mathring{\mathbf{T}}^{ab} \pounds_{\boldsymbol{\beta}} \mathring{\mathfrak{g}}_{ab} \Big| = T^{\mu\nu} \pounds_{\boldsymbol{\beta}} g_{\mu\nu} + \text{ghost bilinears} + \text{fluctuations}$$

★ Separating out the components along the Grassmann directions:

$$\underbrace{\left(\mathring{\mathfrak{D}}_a \mathring{\mathbf{N}}^a - \frac{1}{2}\,\mathring{\mathbf{T}}^{ab}\, \pounds_{\boldsymbol{\beta}}\, \mathring{\mathfrak{g}}_{ab}\right)\Bigg|}_{\text{classical + fluctuations}} = \underbrace{-\left(\mathring{\mathfrak{D}}_\theta \mathring{\mathbf{N}}^\theta + \mathring{\mathfrak{D}}_{\bar{\theta}} \mathring{\mathbf{N}}^{\bar{\theta}} + \mathring{\mathbf{T}}^{a\theta}\, \pounds_{\boldsymbol{\beta}}\, \mathring{\mathfrak{g}}_{a\theta} + \mathring{\mathbf{T}}^{a\bar{\theta}}\, \pounds_{\boldsymbol{\beta}}\, \mathring{\mathfrak{g}}_{a\bar{\theta}} + \mathring{\mathbf{T}}^{\theta\bar{\theta}}\, \pounds_{\boldsymbol{\beta}}\, \mathring{\mathfrak{g}}_{\theta\bar{\theta}}\right)\Bigg|}_{\text{entropy inflow}}$$

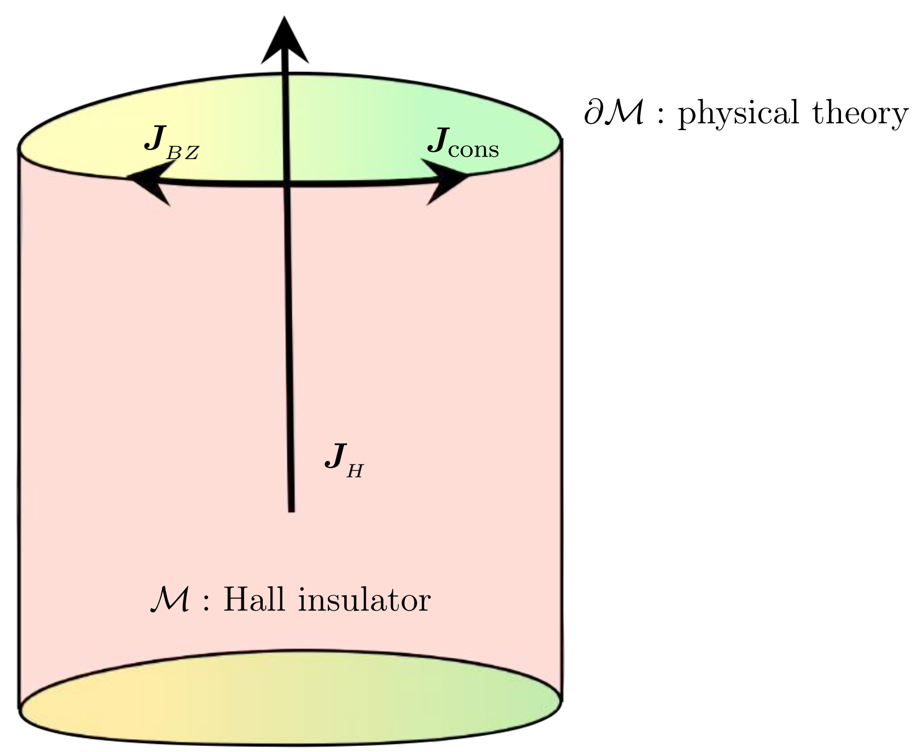
Entropy production

✦ Switching off the fluctuation fields leads to physical entropy flowing from superspace:

$$\Delta = -\left(\mathfrak{D}_\theta \mathbf{N}^\theta + \mathfrak{D}_{\bar{\theta}} \mathbf{N}^{\bar{\theta}}\right) + \text{ghost bilinears}\,.$$

Inflow analogy

- While the inflow mechanism for entropy arises from the superspace, it is morally similar to the manner in which the inflow mechanism operates in the context of Hall insulators & chiral edge states ('t Hooft anomalies).



Callan, Harvey (1985)

anomaly inflow: coupling to a topological sector
with physical entropy being sourced in superspace

No Energy-Momentum Inflow

- Target space diffeomorphisms ensure that the dynamical content of the effective action is simply super-energy momentum conservation.

$$\mathring{\mathfrak{D}}_I \left(\mathring{\mathbf{T}}^{IJ} \, \mathring{\mathfrak{D}}_J \mathring{X}^\mu \right) = 0$$

- This by itself would be problematic, since we would learn that the equations are contaminated by the presence of super-components which turn out to include physical degrees of freedom (not ghosts or fluctuations).

- However, superspace components of energy-momentum tensor conspire to mutually cancel out and do not modify dynamical equations.

$$= \nabla_\mu T^{\mu\nu} + \text{ghost bilinears} + \text{fluctuations}$$

Dissipative terms & second law

- The set of dissipative terms in the effective action can be captured by the superspace upgrade of our 4-tensor structure:

$$\mathcal{L}_{\text{wv, diss}} = \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathfrak{g}}}}{\mathring{\mathbf{z}}} \left(-\frac{i}{4}\right) \mathring{\boldsymbol{\eta}}^{IJKL}\, \mathring{\mathfrak{g}}^{(\overline{\psi})}_{IJ}\, \mathring{\mathfrak{g}}^{(\psi)}_{KL}\,.$$

$$\mathring{\boldsymbol{\eta}}^{(IJ)(KL)} = (-)^{IJ}\, \mathring{\boldsymbol{\eta}}^{(JI)(KL)} = (-)^{KL}\, \mathring{\boldsymbol{\eta}}^{(IJ)(LK)}$$
$$\mathring{\boldsymbol{\eta}}^{(IJ)(KL)}_{(\text{D})} = (-)^{(I+J)(K+L)}\, \mathring{\boldsymbol{\eta}}^{(KL)(IJ)}_{(\text{D})}\,.$$

- Computing the variations etc., we can check what the entropy production is and obtain the by now familiar constraint from Bhattacharyya's theorem:

$$\Delta = \frac{1}{4}\, \boldsymbol{\eta}^{abcd}\, \pounds_{\boldsymbol{\beta}} \mathfrak{g}_{ab}\, \pounds_{\boldsymbol{\beta}} \mathfrak{g}_{cd} + \text{fluctuations} + \text{ghost-bilinears}\,,$$

- A clean way to understand this is to demand that the imaginary part of the effective action is positive definite (this implies the second law).

Glorioso, Liu [1612.07705]

Fluctuation-dissipation & Jarzynski

- The spontaneous CPT symmetry breaking in dissipative dynamical systems leads to a susy Ward identity that implies the Jarzynski relation.

Mallick, Moshe, Orland ’10; Gaspard ‘12

- Jarzynski is a non-equilibrium fluctuation dissipation relation that relates work done on the system out of equilibrium to the free energy difference.

$$\langle e^{-\frac{W}{\mathsf{T}}} \rangle = e^{-\frac{1}{\mathsf{T}}\left(G_f - G_i\right)}$$

Jarzynski ’97; Crooks ‘98

- Using Jensen’s inequality, or convexity of exponential one arrives at

$$\langle W \rangle \geq G_f - G_i$$

Fluctuation-Dissipation and Time reversal breaking

- Stochasticity and dissipation arises because of spontaneous CPT symmetry breaking.

- The Ward identities following from CPT convolved with a thermal gauge transformation results in the Jarzynski work relation for the Brownian particle

$$S_{\text{B0}} \mapsto S_{\text{B0}} - i\,\langle \mathring{\mathcal{F}}_{\theta\bar{\theta}}|\rangle\,\boldsymbol{\beta}\,(\Delta G + W) \quad \Longrightarrow \quad \langle e^{-\beta W}\rangle = e^{-\beta\,\Delta G}$$

Mallick, Moshe, Orland (2010)

- The CPT symmetry in our construction is implemented as R-parity in superspace and its breaking encoded in the vev for the ghost number zero field strength: $\langle \mathring{\mathcal{F}}_{\theta\bar{\theta}}|\rangle = -i$

- Expect similar statements to hold in hydrodynamic effective field theories.

Exemplifying effective actions

- ✦ Ideal fluid is obviously captured by the pressure super-potential functional

$$\mathring{\mathcal{L}}^{(\text{ideal})} = \frac{\sqrt{-\mathring{\mathrm{g}}}}{\mathring{\mathbf{z}}}\, \mathring{\mathfrak{f}}(\mathring{T})$$

- ✦ Dissipative terms are captured by a an appropriate 4-tensor inspired coupling that involves the superderivatives of the metric:

$$\mathcal{L}_{\text{wv, diss}} = \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathrm{g}}}}{\mathring{\mathbf{z}}} \left(-\frac{i}{4}\right) \mathring{\boldsymbol{\eta}}^{IJKL}\, \mathring{\mathfrak{g}}^{(\overline{\psi})}_{IJ}\, \mathring{\mathfrak{g}}^{(\psi)}_{KL}$$

$$\mathring{\boldsymbol{\eta}}^{IJKL} = \mathring{\zeta}(\mathring{T})\, \mathring{T}\, \mathring{P}^{IJ}\, \mathring{P}^{KL} + 2\, \mathring{\eta}(\mathring{T})\, \mathring{T}\, (-)^{K(I+J)}\, \mathring{P}^{K\langle I}\, \mathring{P}^{J\rangle L}$$

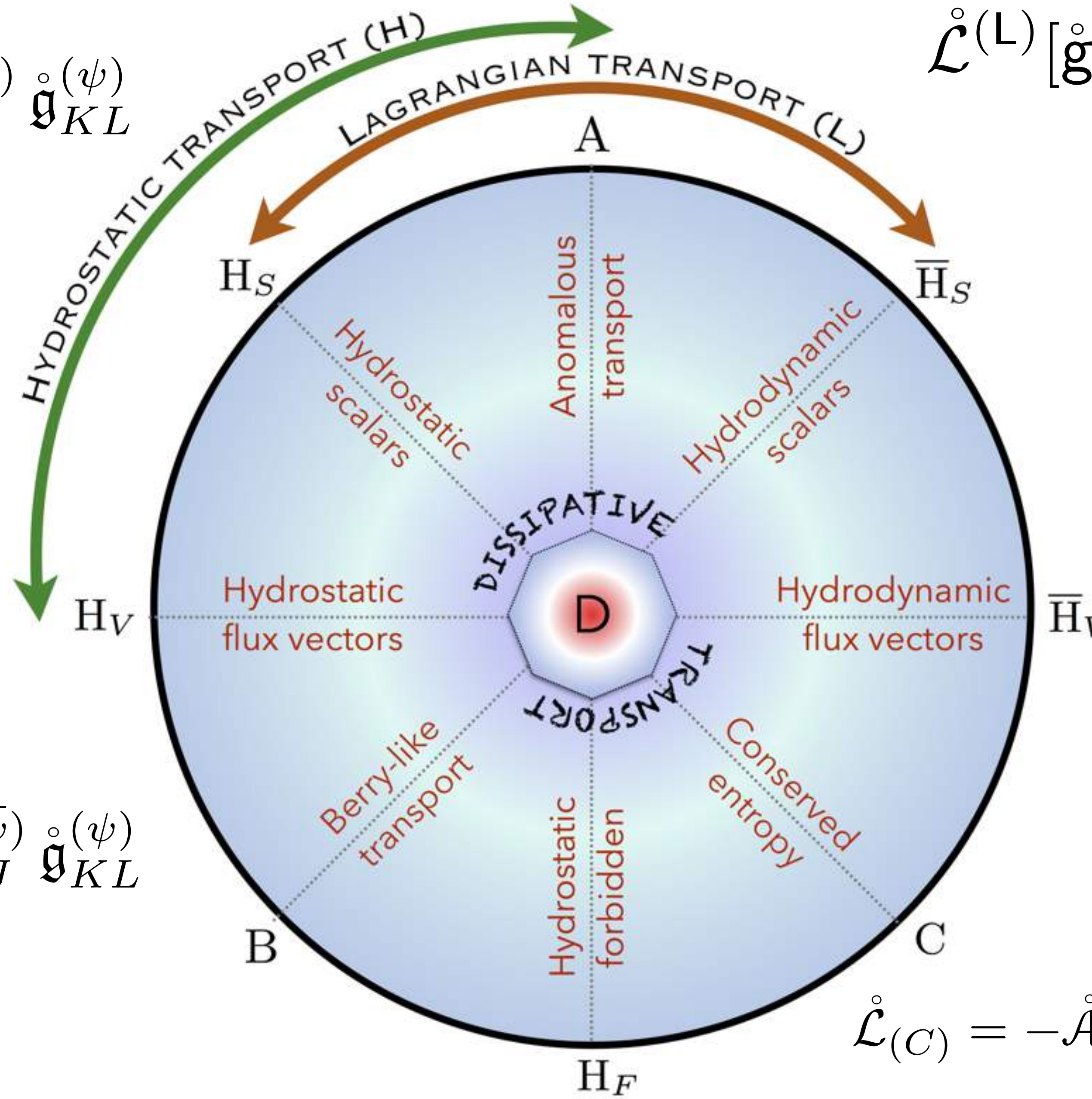
- ✦ Positivity of entropy production follows on demanding the imaginary part is positive definite (which reduces us back to the remit of Bhattacharyya's theorem).

$$\Delta \;= \frac{1}{4}\, \boldsymbol{\eta}^{abcd}\, \pounds_{\boldsymbol{\beta}} \mathrm{g}_{ab}\, \pounds_{\boldsymbol{\beta}} \mathrm{g}_{cd} + \text{fluctuations} + \text{ghost-bilinears}$$

Eightfold Classification revisited

$$\mathring{\mathcal{L}}_{(D)} = -\frac{i}{4}\,\mathring{\boldsymbol{\eta}}^{((IJ)|(KL))}\;\mathring{\mathfrak{g}}^{(\bar\psi)}_{IJ}\;\mathring{\mathfrak{g}}^{(\psi)}_{KL}$$

$$\mathring{\mathcal{L}}^{(\mathsf{L})}[\mathring{\mathfrak{g}}_{IJ}, \mathring{\boldsymbol{\beta}}^I, \mathring{\mathfrak{D}}_J]$$



$$\mathring{\mathcal{L}}_{(B)} = \frac{1}{4}\,\mathring{\mathfrak{F}}_{\bar\theta\theta}\,\mathring{\boldsymbol{\eta}}^{[(IJ)|(KL)]}_{(B)}\;\mathring{\mathfrak{g}}^{(\bar\psi)}_{IJ}\;\mathring{\mathfrak{g}}^{(\psi)}_{KL}$$

$$\mathring{\mathcal{L}}_{(C)} = -\mathring{\mathcal{A}}_I\,\mathring{\mathbf{N}}^I_{(C)}\,,\quad \mathring{\mathfrak{D}}_I\,\mathring{\mathbf{N}}^I_{(C)} = 0$$

$$\mathring{\mathcal{L}}_{(\overline{\mathrm{H}}_V)} = \frac{i}{4}\,\mathring{\mathfrak{c}}^{M(IJ)(KL)}\;\Big[\Big(\mathring{\mathfrak{D}}_M\mathring{\mathfrak{g}}^{(\bar\psi)}_{IJ} + (\mathring{\mathfrak{F}}_{\theta M}, \mathring{\mathfrak{g}}_{IJ})_{\boldsymbol{\beta}}\Big)\,\mathring{\mathfrak{g}}^{(\psi)}_{KL} - \mathring{\mathfrak{g}}^{(\bar\psi)}_{IJ}\;\Big(\mathring{\mathfrak{D}}_M\mathring{\mathfrak{g}}^{(\psi)}_{KL} + (\mathring{\mathfrak{F}}_{\bar\theta M}, \mathring{\mathfrak{g}}_{KL})_{\boldsymbol{\beta}}\Big)\Big]$$

Epilogue

The fluid/gravity correspondence

❖ The fluid/gravity correspondence establishes a correspondence between Einstein's equations with a negative cc and those of relativistic conformal fluids.

Einstein's eqn with negative cosmological constant (cc)

$$E_{MN} = R_{MN} - \frac{1}{2} G_{MN} R - \frac{d(d-1)}{2} G_{MN} = 0$$

$$\updownarrow$$

Relativistic ideal fluid equations and beyond…

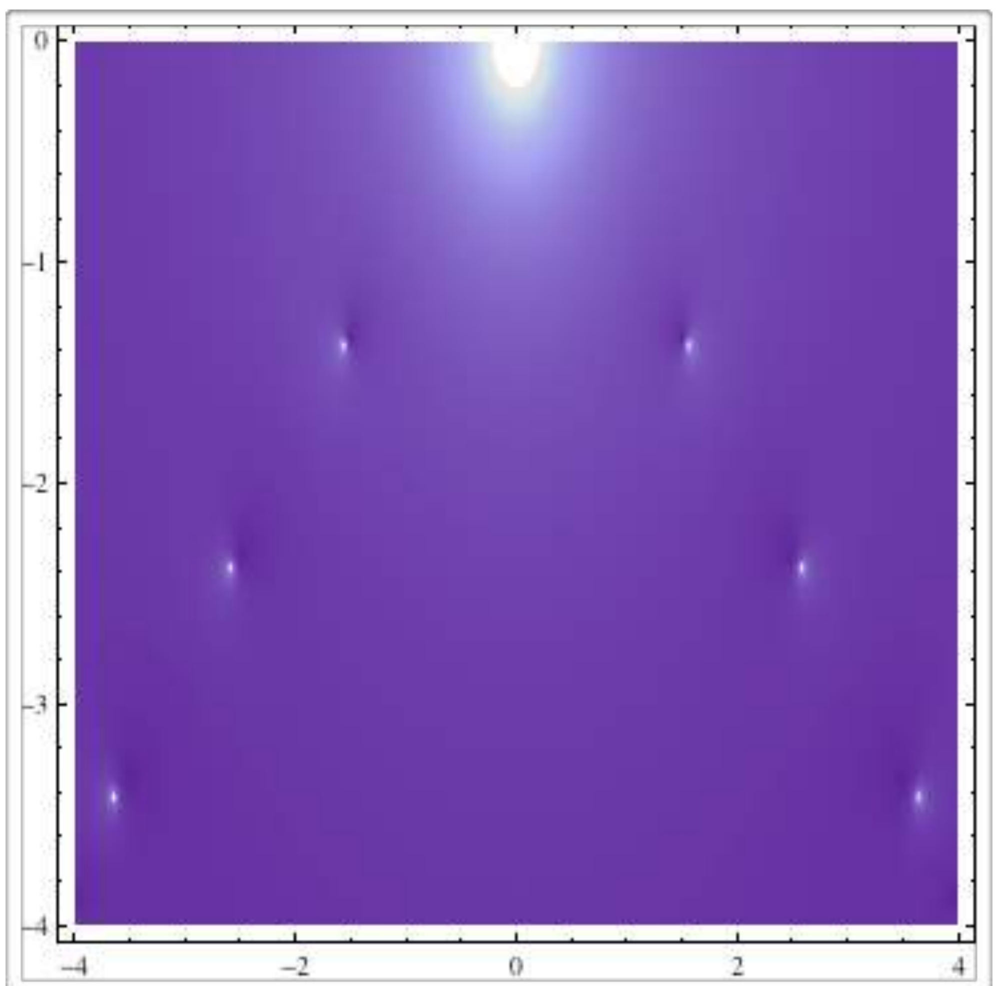
$$(\epsilon + p)\,\Theta + u^\mu \nabla_\mu \epsilon + \cdots = 0$$
$$P_\alpha{}^\mu \nabla_\mu p + (\epsilon + p)\,\mathfrak{a}_\alpha + \cdots = 0$$

✦ Given any solution to the hydrodynamic equations, one can construct, in a gradient expansion, an approximate *inhomogeneous, dynamical black hole* solution in an asymptotically AdS spacetime.

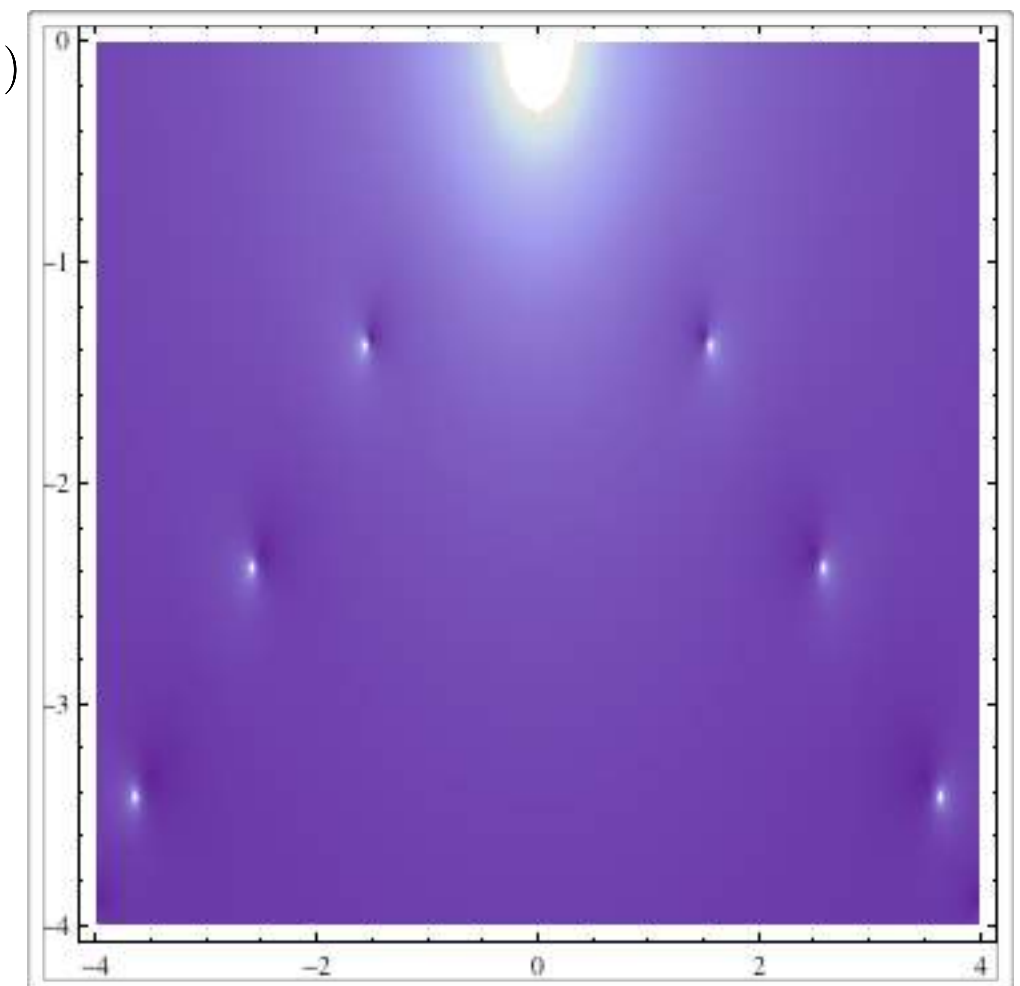
Bhattacharyya, Hubeny, Minwalla, MR '07

Long-wavelength in gravity

- How do we `derive' fluid dynamics from gravity?
- Intuition: perturbations of planar AdS black holes reveal long-wavelength quasinormal modes having hydrodynamic character: $\omega(k) \to 0 \;\text{ as }\; k \to 0$



scalar/sound channel

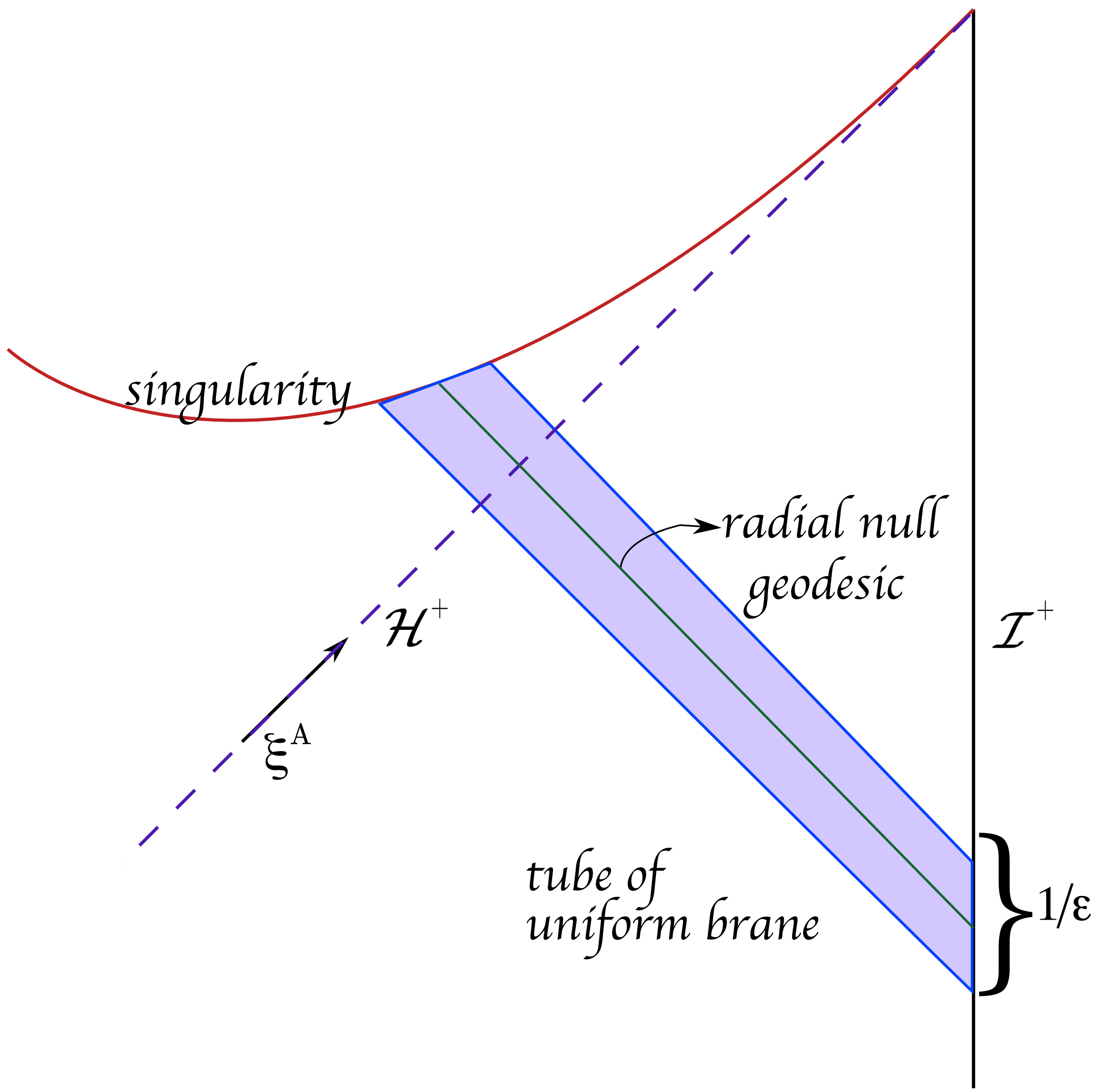


vector/shear channel

Horowitz, Hubeny '98; Policastro, Son, Starinets '02

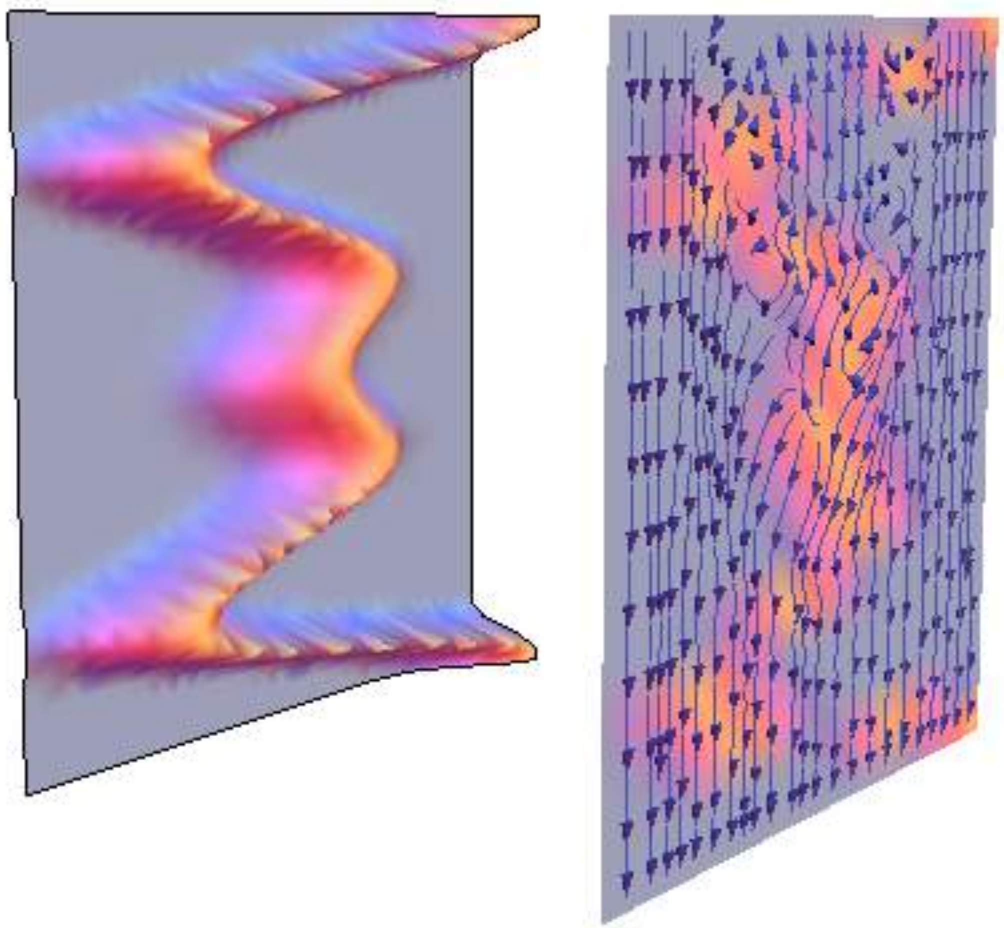
Nonlinear fluids from gravity

- ✳ Treat the light quasinormal modes as moduli of the gravitational problem.
- ✳ Hydrodynamics is the collective field theory of these modes and can be constructed systematically in a perturbation expansion.
- ✳ Intuition: patch together different bulk black hole spacetimes, along tubes of locally equilibrated fluid.



Black holes as lumps of fluid

- Black holes really behave as lumps of fluid in the low energy limit.
- In the fluid/gravity correspondence, the fluid lives at the end of the universe, on the asymptotic boundary of the spacetime where the black hole resides.
- Here the fluid is a hologram, honestly capturing all the low energy physics of the entire geometry.



Classification of Weyl invariant fluids

- Weyl invariant neutral (and to some extent charged) fluids have been well studied from both

  - kinetic theory (weak coupling) — York, Moore '08

  - holography via fluid/gravity (strong coupling) — Baier et. al.; Bhattacharyya et. al., '07

- All of the known data can be neatly compiled into the eightfold classification scheme.

Classification of Weyl invariant fluids

✦ The stress tensor for a conformal fluid can be expressed in the eightfold basis as:

$$T^{\mu\nu} = p\,(d\,u^\mu u^\nu + g^{\mu\nu}) - \eta\,\sigma^{\mu\nu}$$
$$+ (\lambda_1 - \kappa)\,\sigma^{<\mu\alpha}\sigma_\alpha^{\ \nu>} + (\lambda_2 + 2\,\tau - 2\kappa)\ \sigma^{<\mu\alpha}\omega_\alpha^{\ \nu>}$$
$$+ \tau\left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\,\sigma^{<\mu\alpha}\omega_\alpha^{\ \nu>}\right) + \lambda_3\,\omega^{<\mu\alpha}\omega_\alpha^{\ \nu>}$$
$$+ \kappa\ \left(C^{\mu\alpha\nu\beta}\,u_\alpha\,u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha^{\ \nu>} + 2\,\sigma^{<\mu\alpha}\omega_\alpha^{\ \nu>}\right).$$

Hs

D

B

$\overline{\text{H}}$s

Holographic fluids: 2nd order transport

- For holographic fluids with Einstein gravity dual, shear viscosity is related to entropy density and the second order transport data is explicitly known:

$$\eta = \frac{s}{4\pi}$$
$$\zeta = 0$$

$$\begin{aligned}
\tau &= -\left(2\,(d-2)\,k_R + 2\,k_\sigma\right)\, T^{d-2}\,, \\
\kappa &= -2\,(d-2)\,k_R\, T^{d-2}\,, \\
\lambda_1 &= -2\,(d-2)\,k_R\, T^{d-2}\,, \\
\lambda_2 &= 4\,k_\sigma T^{d-2}\,, \\
\lambda_3 &= -2\,((d-2)\,k_R - 2\,k_\omega)\, T^{d-2}\,.
\end{aligned}$$

$$\begin{aligned}
k_R &= -\frac{c_{\text{eff}}}{d-2}\left(\frac{4\pi}{d}\right)^{d-2}\,, \\
k_\omega &= \frac{d-2}{2}\,k_R\,, \\
k_\sigma &= \frac{c_{\text{eff}}}{2\,d}\left(\frac{4\pi}{d}\right)^{d-2}\,\text{Harmonic}\left(\frac{2}{d}-1\right)\,,
\end{aligned}$$

Holographic fluids effective action?

- Known second order transport of holographic fluids follows from the action (note we have dropped some superspace terms for simplicity)

$$\mathcal{L}_{\rm wv} = c_{\rm eff} \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathsf{g}}}}{1+\boldsymbol{\beta}^e\, \mathring{\mathcal{A}}_e} \Bigg\{ \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^d \left(1 - \frac{i\, d}{8\pi}\, \mathring{\mathsf{P}}^{c\langle a}\mathring{\mathsf{P}}^{b\rangle d}\; \mathring{\mathcal{D}}_\theta \mathring{\mathsf{g}}_{ab}\; \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{\mathsf{g}}_{cd}\right) - \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^{d-2} \left[\frac{{}^{w}\mathring{\mathsf{R}}}{d-2} + \frac{1}{d}\,\text{Harmonic}\left(\frac{2}{d}-1\right) \mathring{\sigma}^2 + \frac{1}{2}\mathring{\omega}^2\right]\Bigg\}$$

- How does the bulk gravity theory realize this effective action?

Thank You