Anomalies and topological phases in relativistic QFT

Overview

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These lectures are about symmetries and anomalies in quantum field theory (QFT).

Recent developments are stimulated by topological considerations in condensed matter physics: Classification of topological phases of matter.

However, I will just focus on relativistic QFT relevant for high energy physics (particles physics & strings).

This lecture:

Brief overview (but biased by my own preference)

Later lectures:

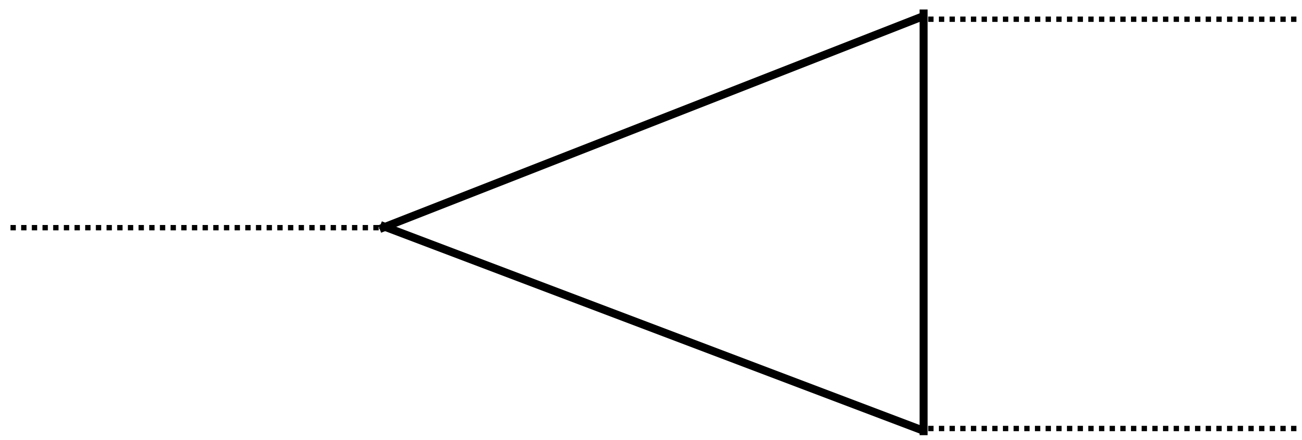
More specific topics

(Some topological properties of gauge theories)

Contents

Symmetry and anomaly

Usual symmetries and perturbative anomalies are well-treated in standard textbooks in QFT and string theory.



famous triangle diagram in 4-dim.

Anomalies: examples

Anomaly of Dynamical Gauge symmetry:
Need to be cancelled for a consistent theory

- Anomaly cancellation in $SU(3) \times SU(2) \times U(1)$
- Selection of possible gauge non-abelian groups $E_8 \times E_8$, $SO(32)$ in 10d supergravity

Anomaly of Global symmetry: ['t Hooft, 1979]
Very convenient for the study of dynamics

- Chiral symmetry in QCD: $SU(N_f)_L \times SU(N_f)_R$

Remark:
These two types of anomalies will not be distinguished.

Symmetry and anomaly

What is symmetry?

A textbook answer

$\phi(x)$: fields $\qquad$ $S[\phi]$: action

Symmetry means that the action is invariant under tranformation

$$\phi(x) \to g \cdot \phi(x)$$

$$S[g \cdot \phi] = S[\phi]$$

Symmetry and anomaly

What is anomaly?

A textbook answer

The classical action $S[\phi]$ is invariant under transformation of fields $\phi(x) \to g \cdot \phi(x)$

But quantum mechanically it is violated (e.g. by path integral measure).

Symmetry and anomaly

The concepts of symmetry and anomaly are organized and generalized more and more in recent years.

I will review some of those developments.

Contents

Symmetry

What is symmetry in modern understanding?

Actually I don’t know how to treat them most generally.

The terminology “symmetry” is not appropriate in some of generalizations.

I feel more abstract language is necessary for a unified treatment of several generalizations.

Symmetry

- **Usual symmetry (continuous, discrete, spacetime)**
- **Higher form symmetry** [Kapustin-Seiberg 2014, Gaiotto-Kapustin-Seiberg-Willett 2014]
- **2-group** [Kapustin-Thorngren 2013, Tachikawa 2017, Cordova-Dumitrescu-Intriligator-2018, Benini-Cordova-Hsin2018]
- **Duality group** [Seiberg-Tachikawa-KY 2018]
- **Topological defect operator** [Bhardwaj-Tachikawa 2017, Chan-Lin-Shao-Wang-Yin 2018]
- **…**

Some properties

Let me describe some properties of some of them.

Some properties: Topology

Let us recall the case of usual continuous symmetry.

J^μ : conserved current

$\nabla_\mu J^\mu = 0$: conservation equation

In the language of diffential forms, it is more beautiful.

$$J := J_\mu dx^\mu$$

$$*J = \frac{1}{(d-1)!} J^\mu \epsilon_{\mu\mu_1\cdots\mu_{d-1}} dx^{\mu_1} \wedge \cdots dx^{\mu_{d-1}}$$

$d(*J) = 0$: conservation equation

Some properties: Topology

$Q(\Sigma) = \int_{\Sigma} *J$: charge operator

Σ : codimension-1 (dimension $d-1$) surface

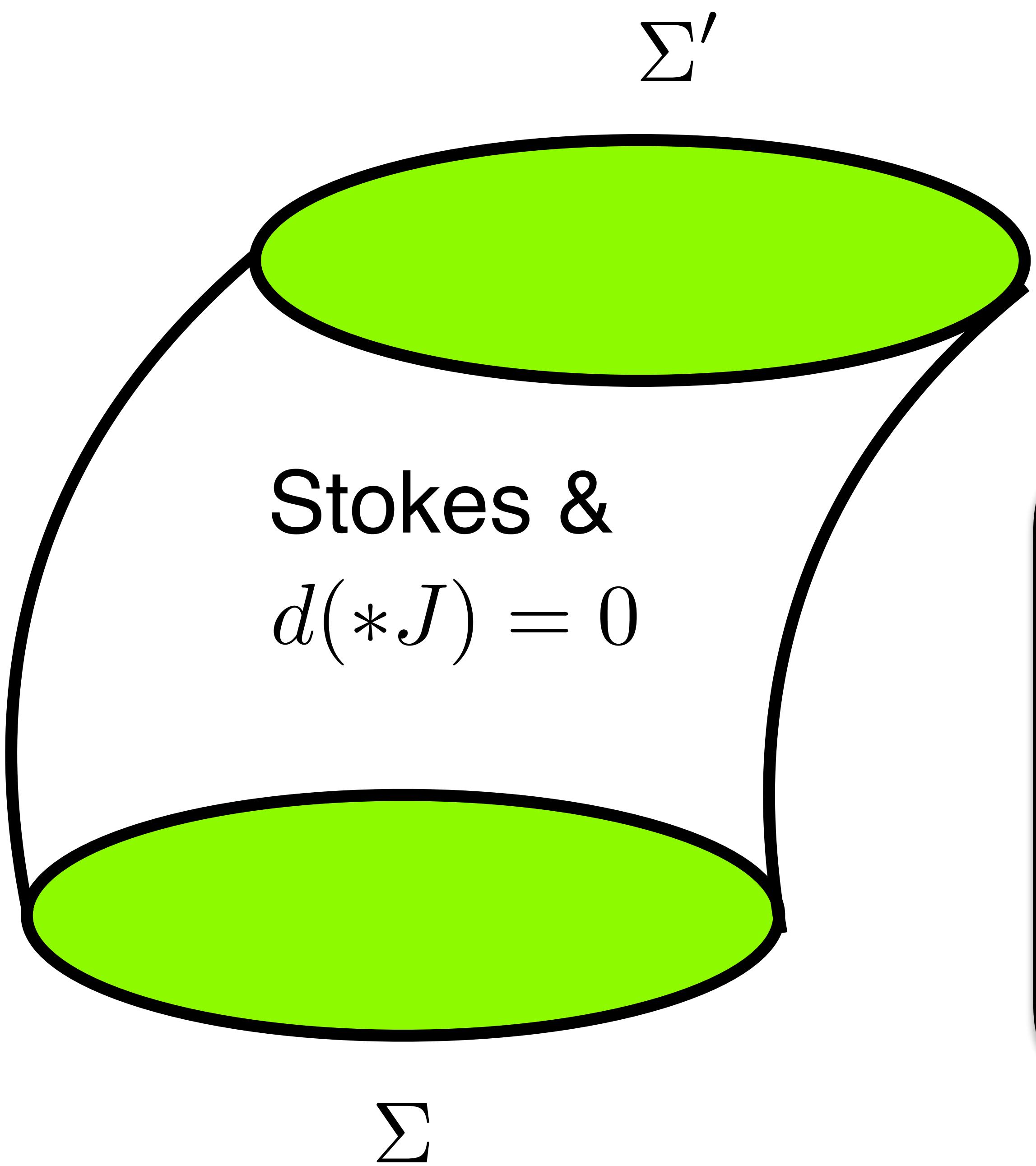
This is total charge on the surface Σ

$Q(\Sigma)$ is invariant under continuous deformation of Σ by

- Stokes theorem
- $*J$ is closed: $d(*J) = 0$

In this sense, $Q(\Sigma)$ is topological.

Some properties: Topology



$$Q(\Sigma) = \int_\Sigma *J$$

$$Q(\Sigma) = Q(\Sigma')$$

topological
(charge conservation)

Some properties: Topology

Symmetry operator:

$$U(\Sigma, \alpha) = \exp(i\alpha Q(\Sigma)) = \exp(i\alpha \int_\Sigma *J)$$

- It is topological in the sense that it is invariant under continuous deformation of the surface Σ

- The operator $U(\Sigma, \alpha)$ exists for each group element

$$e^{i\alpha} = g \in G$$

g : element of group
α : element of Lie algebra

Some properties: Topology

So the usual symmetry is implemented by operators

$U(\Sigma, g)$: topological operator

Σ : surface

g : "label" of the operator (group element)

Some properties: Topology

$U(\Sigma, g)$: topological operator

Some properties: Topology

$U(\Sigma, g)$: topological operator

Q: Does it need to be an exponential of Q ?
A: No. Discrete symmetry has U without Q .

Some properties: Topology

$U(\Sigma, g)$: topological operator

Q: Does it need to be an exponential of Q ?
A: No. Discrete symmetry has U without Q .

Q: Does the surface Σ need to be codimension-1?
A: No. Higher form symmetry uses higher codimension Σ .

Some properties: Topology

$U(\Sigma, g)$: topological operator

Q: Does it need to be an exponential of Q ?
A: No. Discrete symmetry has U without Q .

Q: Does the surface Σ need to be codimension-1?
A: No. Higher form symmetry uses higher codimension Σ .

Q: Do we need group elements $g \in G$?
A: No. Topological defect operator is just topological without any group.

Some properties: Topology

Remarks:

- Sometimes these operators cannot be written explicitly in “elementary ways” by using fields.
- These operators are sometimes described by abstract mathematical concepts such as fiber bundles, algebraic topology, and so on.

Some properties: Background

Quite generally, operators can be coupled to background fields.

The most basic case of an operator $O(x)$ coupled to a background field $A(x)$

$$Z[A] = \langle \exp(i \int A(x) O(x)) \rangle$$

: called generating functional or partition function.

I will use the terminology partition function.

Some properties: Background

For the current operator $J^\mu(x)$, we have a background gauge field $A_\mu(x)$. The coupling between them is

$$\int d^d x A_\mu J^\mu = \int A \wedge *J$$

In a similar (but more abstract) way, the symmetry corresponding to $U(\Sigma, \alpha)$ can be coupled to background fields.

Some properties: Background

I will write

A : abstract background fields for the symmetry U

$Z[A]$: partition function in the presence of the background fields A

Some properties: Background

Examples of background field A :

- For discrete symmetry G ,

  A : principal G bundle

- For higher form symmetry such as p-form symmetry with abelian group $G = \mathbb{Z}_N$

  $A \in H^{p+1}(M, \mathbb{Z}_N)$ (cohomology group)

- Parity, Time-reversal symmetry

  A : non-orientable manifold (e.g. Klein-bottle)

Some properties: Background

Remarks:

- As the previous examples show, abstract description of the background fields requires mathematical concepts from topology and geometry.
- But if you don't like mathematics, some simple cases can be treated in elementary ways.
- For example, p-form field = BF theory.
  E.g. [Banks-Seiberg, 2010]

Contents

Anomaly

What is anomaly in modern understanding?

There is now a way which is believed to describe almost all anomaly.

Anomaly

A : abstract background fields for the symmetry

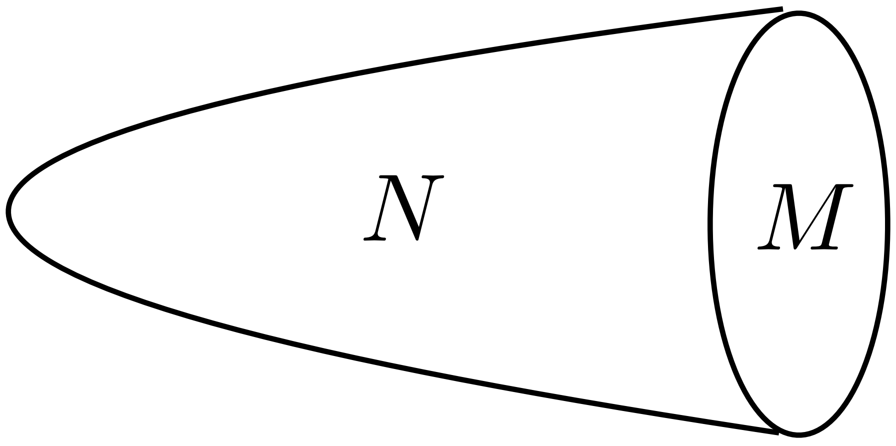
$Z[A]$: partition function in the presence of the background field

M : d-dimensional spacetime manifold (part of background fields for spacetime symmetry)

Anomaly

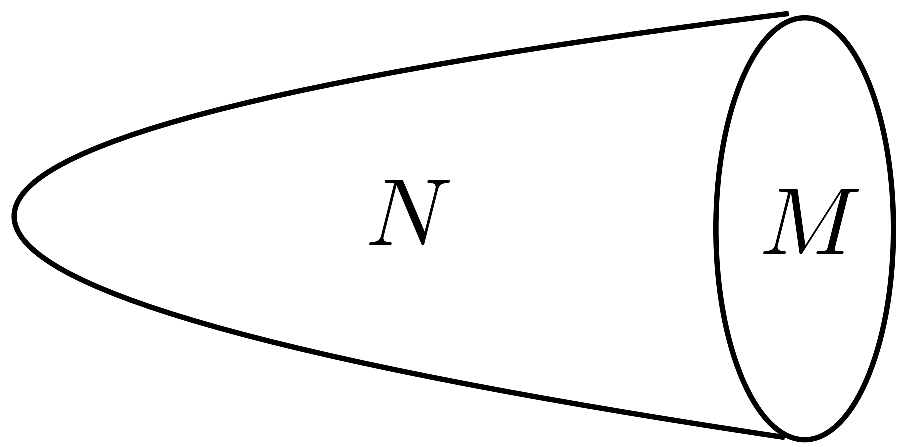
- First of all, anomaly means that the partition function $Z[A]$ is ambiguous.

- However, if we take a (d+1)-dimensional manifold N whose boundary is the spacetime M and on which the background fields A are extended, then the partition function is fixed without any ambiguity.



$$\partial N = M$$

Anomaly



$$\partial N = M$$

$Z[N, A]$: no ambiguity, if we are given N and extension of A into N

The dependence on N : anomaly

Anomaly

This description of anomaly may look quite abstract, but there is a very natural motivation from condensed matter physics / domain wall fermion.

Topological phase

Some material:
Topological phase
(Manifold N)

← **Anomalous theory on the surface/domain wall**
(Manifold M)

- Cond-mat/Lattice systems are not anomalous as a whole.
- It is anomalous if we only look at surface/domain-wall.
- The bulk topological phase is gapped.
  No degrees of freedom. Almost trivial theory.

Example: quantum Hall system

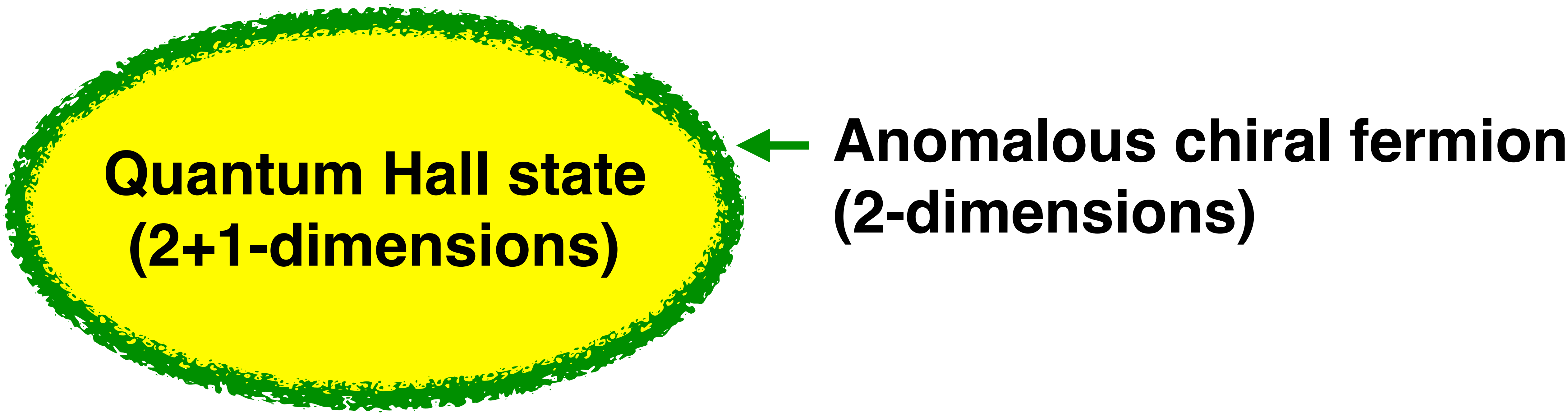
The simplest example is given by quantum Hall system.

$$d = 2, \quad d + 1 = 3$$

symmetry: U(1) (QED)

Anomalous theory: chiral fermion in $d = 2$

Example: quantum Hall system



On the surface of matter exhibiting quantum Hall effect, there appears chiral fermions which have anomaly under U(1) symmetry.

$A = A_\mu dx^\mu$: U(1) background gauge field

Example: quantum Hall system

The action of the quantum Hall system in 3-dimensions

$$S = i\frac{k}{4\pi}\int_{\text{bulk}} AdA$$

(A : background. No dynamical degrees of freedom.)

Gauge transformation $\delta A = d\alpha$

$$\delta_\alpha S = i\frac{k}{4\pi}\int_{\text{bulk}} d\alpha dA = i\frac{k}{4\pi}\int_{\text{boundary}} \alpha dA$$

Cancels against boundary fermion anomaly

Example: quantum Hall system

$$Z[N, A] = \quad \underbrace{\det(P_L \gamma^\mu D_\mu)}_{} \cdot \underbrace{\exp(\frac{ik}{4\pi}\int_{\text{bulk}} AdA)}_{}$$

boundary fermion partition function

bulk topological phase partition function

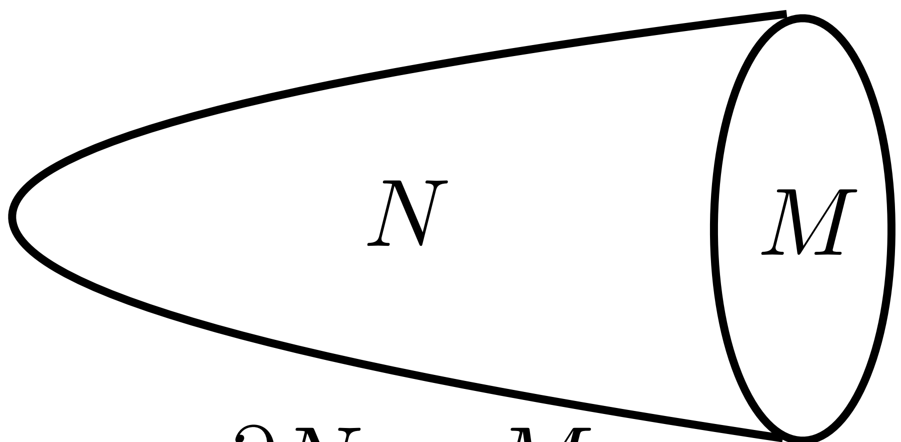
Product of bulk and boundary partition functions: gauge invariant

Characterization of anomaly

More generally, anomalies are completely characterized by (d+1)-dimensional topological phases.

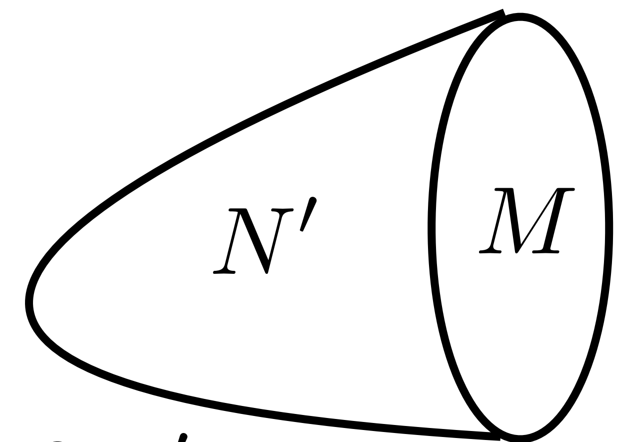
Characterization of anomaly

A manifold:



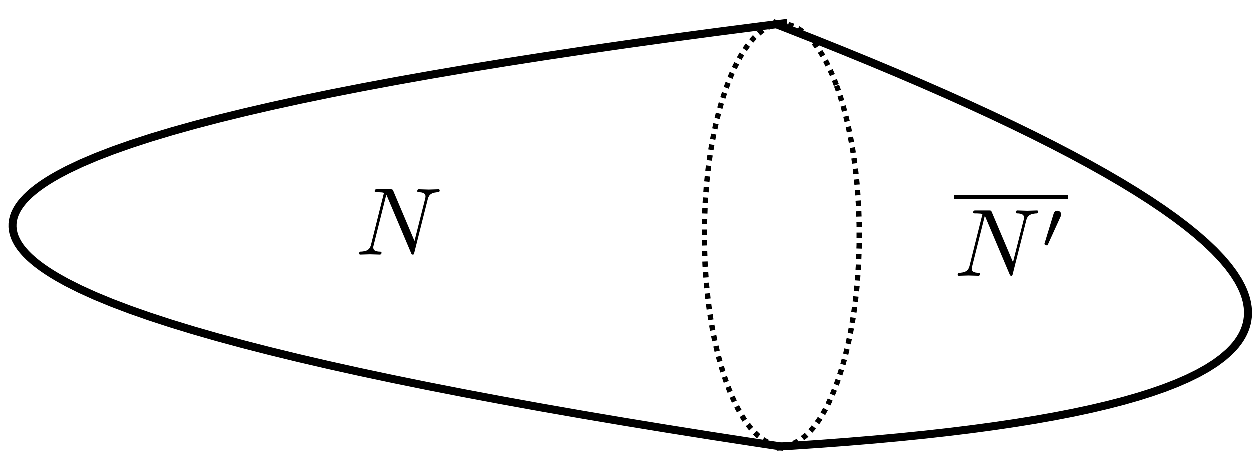
$\partial N = M$

Another manifold:



$\partial N' = M$

Gluing the two manifold:



Closed manifold

$X = N \cup \overline{N'}$

Characterization of anomaly

$$\frac{Z[N]}{Z[N']} = Z[X] \qquad (\ X = N \cup \overline{N'} \ )$$

Anomaly is characterized by (d+1)-dimensional partition function $Z[X]$ on the closed manifold X

- If $Z[X] = 1$, there is no anomaly because $Z[N] = Z[N']$ means that the partition function is independent of N

- $Z[X]$ is really the partition function of (d+1)-dim. theory. This (d+1)-dim theory is called symmetry protected topological phases (SPT phases) or invertible field theory

Perturbative anomaly

Example : Perturbative anomaly

Usual perturbative anomaly is described by the so-called descent equation for the gauge field $F = dA + A \wedge A$

$$I_{d+2} \sim \mathrm{tr}\ F^{(\frac{d+2}{2})}$$: anomaly polynomial in (d+2)-dimensions

$$I_{d+2} = dI_{d+1}$$ I_{d+1} : Chern-Simons

$$Z[X] = \exp(i \int_X I_{d+1})$$: Chern-Simons

Quantum Hall system: $I_{2+2} \sim F^2 = dAdA$

$$I_{2+1} \sim AdA$$

General fermion anomaly

More general fermion anomalies

I_{d+1} can be used only for perturbative anomalies of continuous symmetries.

More generally: global anomaly

Examples of global anomaly:

- SU(2) with a doublet Weyl fermion in 4-dimensions
- All anomalies of discrete symmetries ($\mathbb{Z}_N$, time-reversal,..)

General fermion anomaly

General fermion anomaly formula [Witten, 2015]

$$Z[X] = \exp(-2\pi i \eta)$$

η : Atiyah-Patodi-Singer η invariant

[Atiyah-Patodi-Singer 1975]

Perturbatively $\eta \sim I_{d+1}$, but the η invariant contains more information. (Details omitted)

All global anomalies

All global anomalies (not restricted to fermions)

Classified by the cobordism groups.

Conjectured in
[Kapustin 2014, Kapustin-Thorngren-Turzillo-Wang 2014]

Essentially proved in
[Freed-Hopkins 2016, Yonekura 2018] Details omitted.

By looking at the cobordism groups, we can see what anomaly is possible in a given dimension with a given symmetry group.

Applications

In the rest of this lecture, I will give a sketch of applications of the refined concepts of symmetries and anomalies.

There are many applications.
I sketch only a few of them (related to my own works).

Contents

Anomalies in string theory

It is well-known how perturbative anomalies are cancelled in string theory.

This does not mean that we have the complete picture of anomaly cancellation in string theory.

More subtle global anomalies and more subtle topological structures in string theory, as I now discuss.

Flux quantization

String theory contains several higher form gauge fields, such as RR p-form field C :

$$C = C_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

Their fluxes, $F = dC$, are often said to have integral periods due to Dirac quantization condition:

$$\int_{(p+1)\text{-cycle}} F \in \mathbb{Z}$$

O-plane charges

Another well-known fact:

Op-plane RR charges are given by 2^{p-5} (sign neglected)

This means that the integral of $F = dC$ around the O-plane is given by

$$\int_{\text{around } \mathrm{O}p} F = 2^{p-5}$$

This is not integer for $p < 5$

Dirac quantization

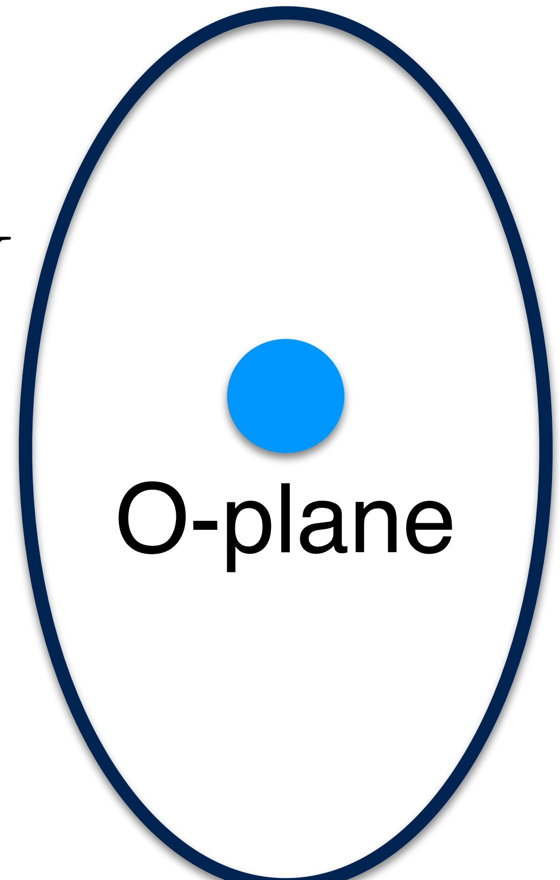
It seems that Dirac quantization condition is violated.

Is string theory inconsistent?

Let us recall the argument of Dirac quantization.

Dirac quantization

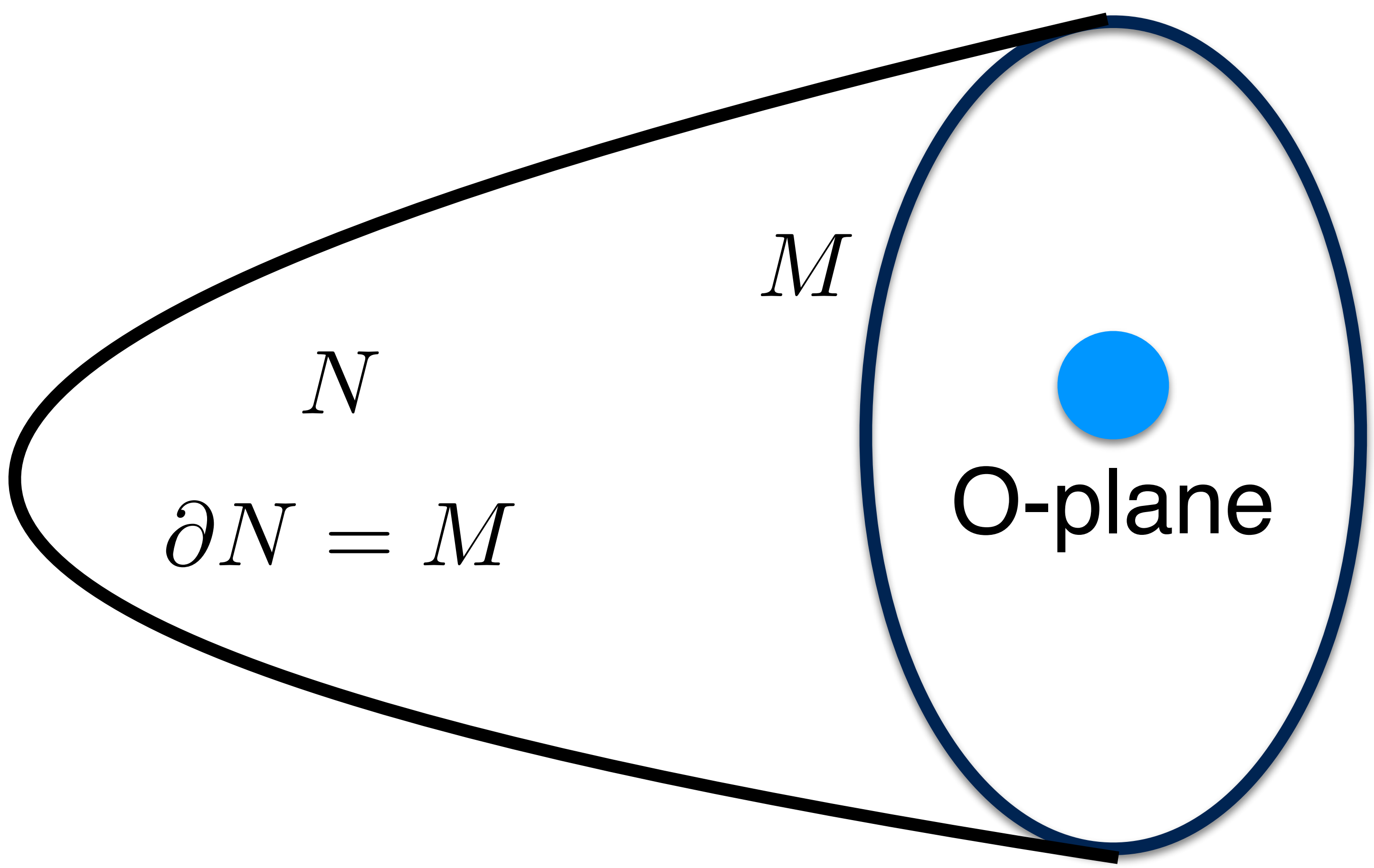
D-brane
worldvolume M



Coupling to RR field $\exp(i \int_M C)$

C : RR-field

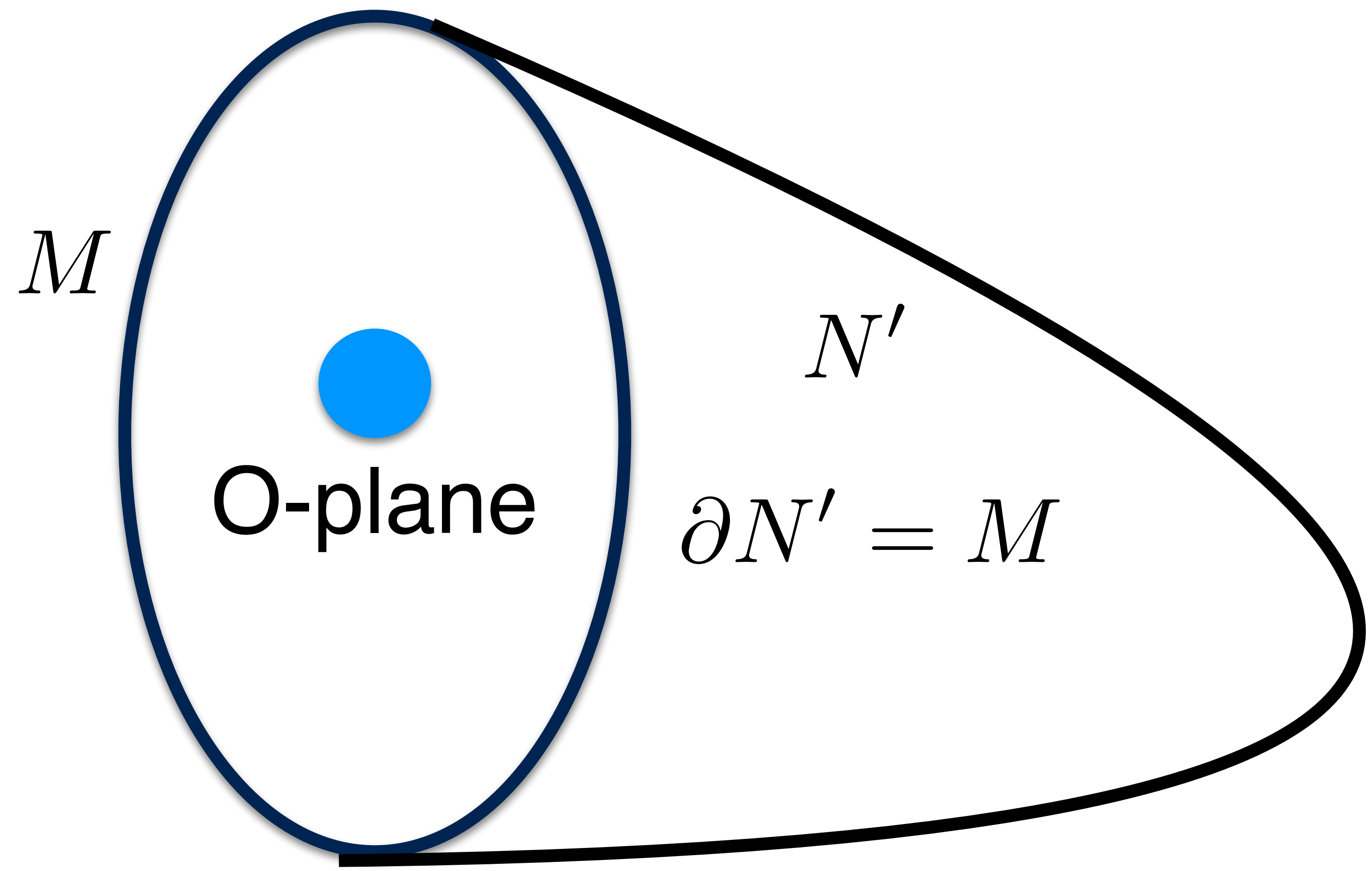
Dirac quantization



Coupling to RR field $\exp(i \int_M C) = \exp(i \int_N F)$

C : RR-field $F = dC$

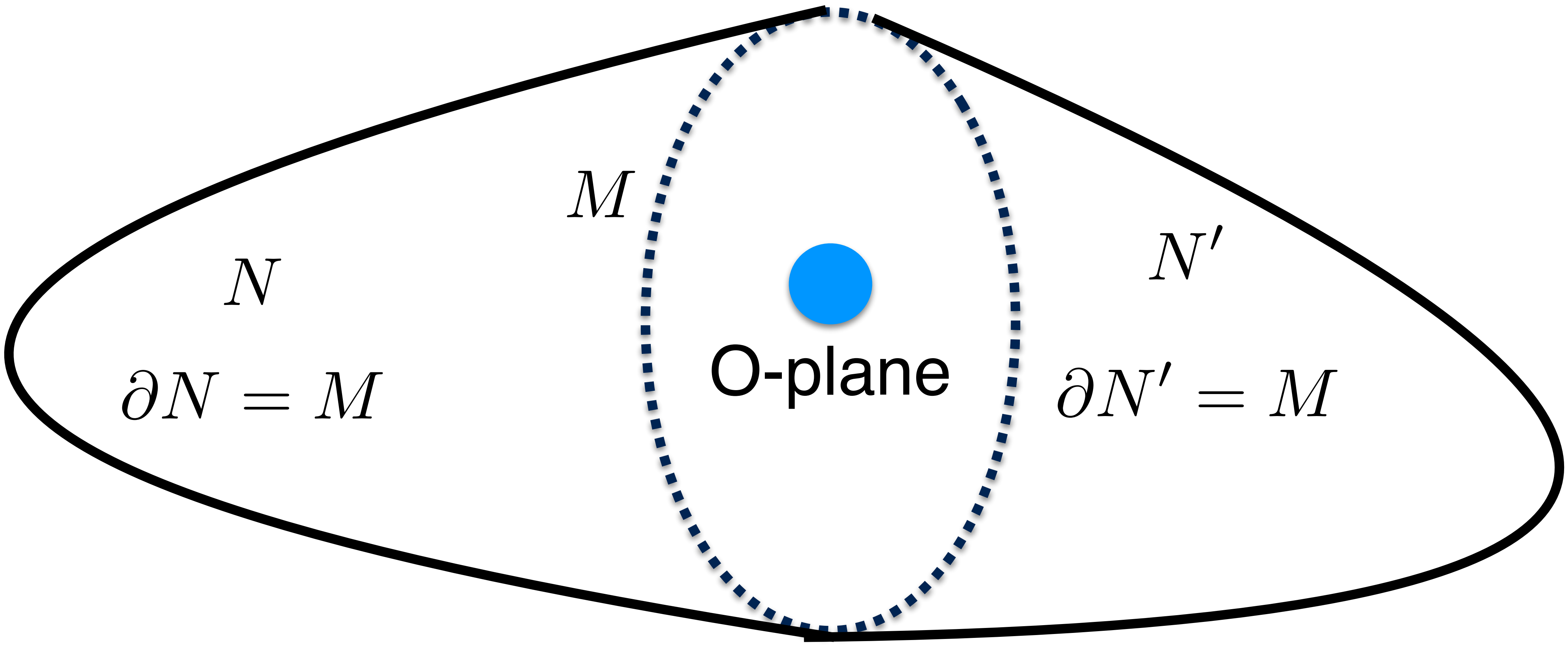
Dirac quantization



Coupling to RR field $\exp(i \int_M C) = \exp(i \int_{N'} F)$

C : RR-field $\quad F = dC$

Dirac quantization



$$X = N \cup \overline{N'}$$

$$\frac{\exp(i\int_N F)}{\exp(i\int_{N'} F)} = \exp(i\int_X F) \neq 1$$

If the charge is not integer.

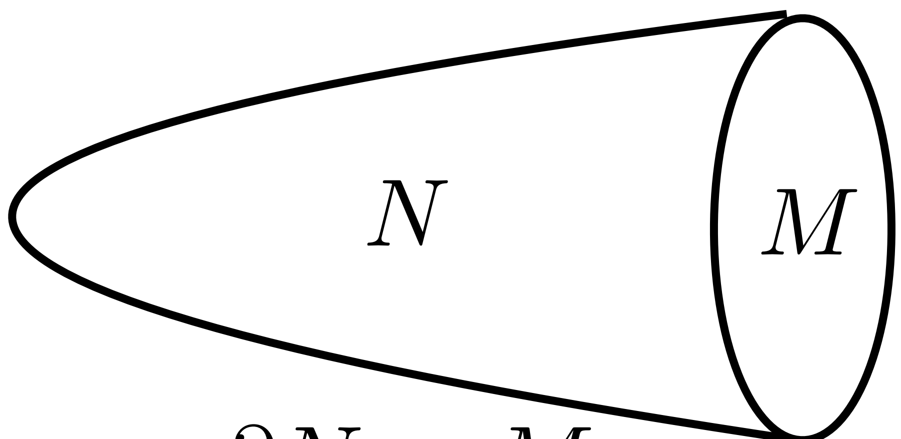
Dirac quantization

Didn't we see similar figures before?

Maybe you have forgotten, so let me repeat it.

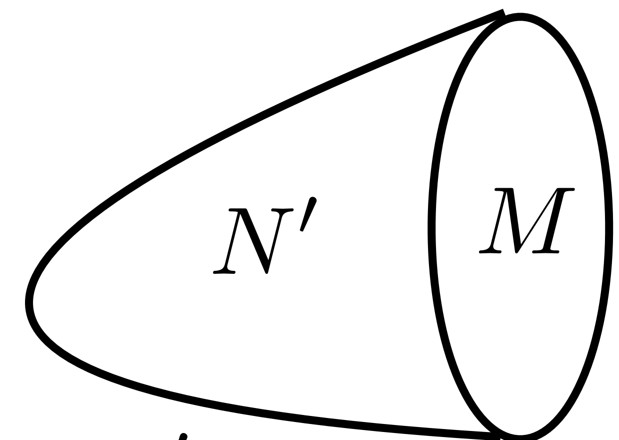
Characterization of anomaly

A manifold:



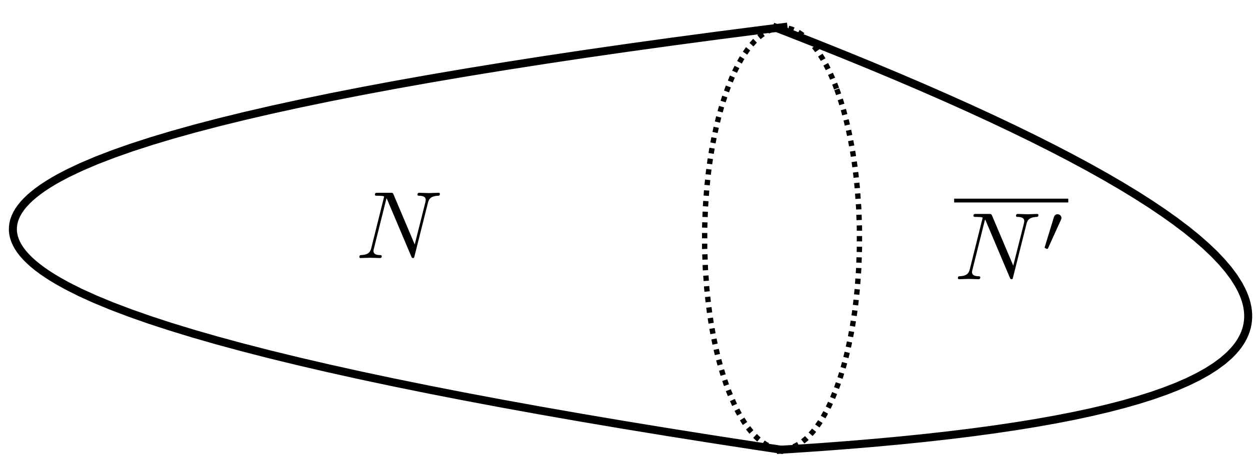
$\partial N = M$

Another manifold:



$\partial N' = M$

Gluing the two manifold:



Closed manifold

$$X = N \cup \overline{N'}$$

Characterization of anomaly

$$\frac{Z[N]}{Z[N']} = Z[X] \qquad (\; X = N \cup \overline{N'} \;)$$

Anomaly is characterized by (d+1)-dimensional partition function $Z[X]$

Anomaly means $Z[X] \neq 1$

Characterization of anomaly

The total partition function of D-brane worldvolume is

$$Z[N]\exp(i\int_N F)$$

The partition function of worldvolume fields (fermions, gauge fields) — $Z[N]$

Coupling to RR-field — $\exp(i\int_N F)$

$$\frac{Z[N]\exp(i\int_N F)}{Z[N']\exp(i\int_{N'} F)} = Z[X]\exp(i\int_X F)$$

Characterization of anomaly

The (in)consistency of the worldvolume of D-branes is controlled not by $Z[X]$ or $\int F$, but by the combination

$$Z[X]\exp(i\int_X F)$$

Anomaly free condition:

$$Z[X]\exp(i\int_X F) = 1$$

[Moore-Witten,1999]
[Freed-Hopkins,2000]
[Witten,2016]
[Tachikawa-Yonekura,2018 & work in progress]

Shifted flux quantization

Conclusion:

Fluxes are not quantized to be integers, but

$$\int F \in q + \mathbb{Z}$$

q : quantity controlled by the anomaly of worldvolume theory of D-brane.

A complete story is not yet understood.
More work is necessary.

Contents

’t Hooft anomaly matching

Anomalies are very useful because of ’t Hooft matching:

’t Hooft anomaly matching

UV: Some UV theory with global symmetry F

↓

IR: ???

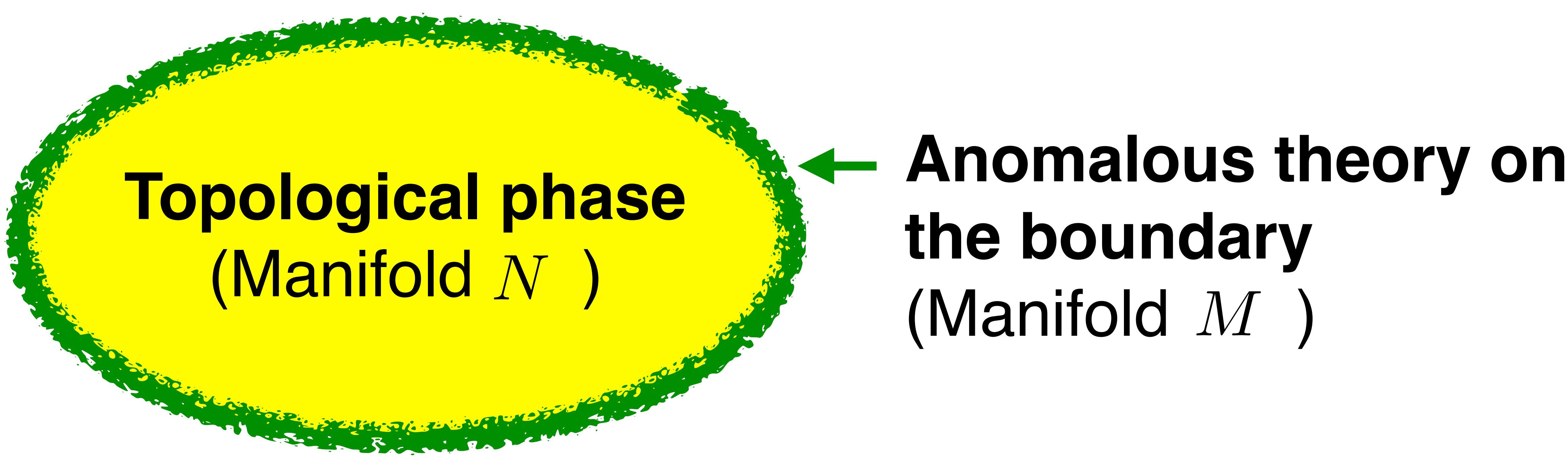
Anomaly of F in UV = Anomaly of F in IR

’t Hooft anomaly matching

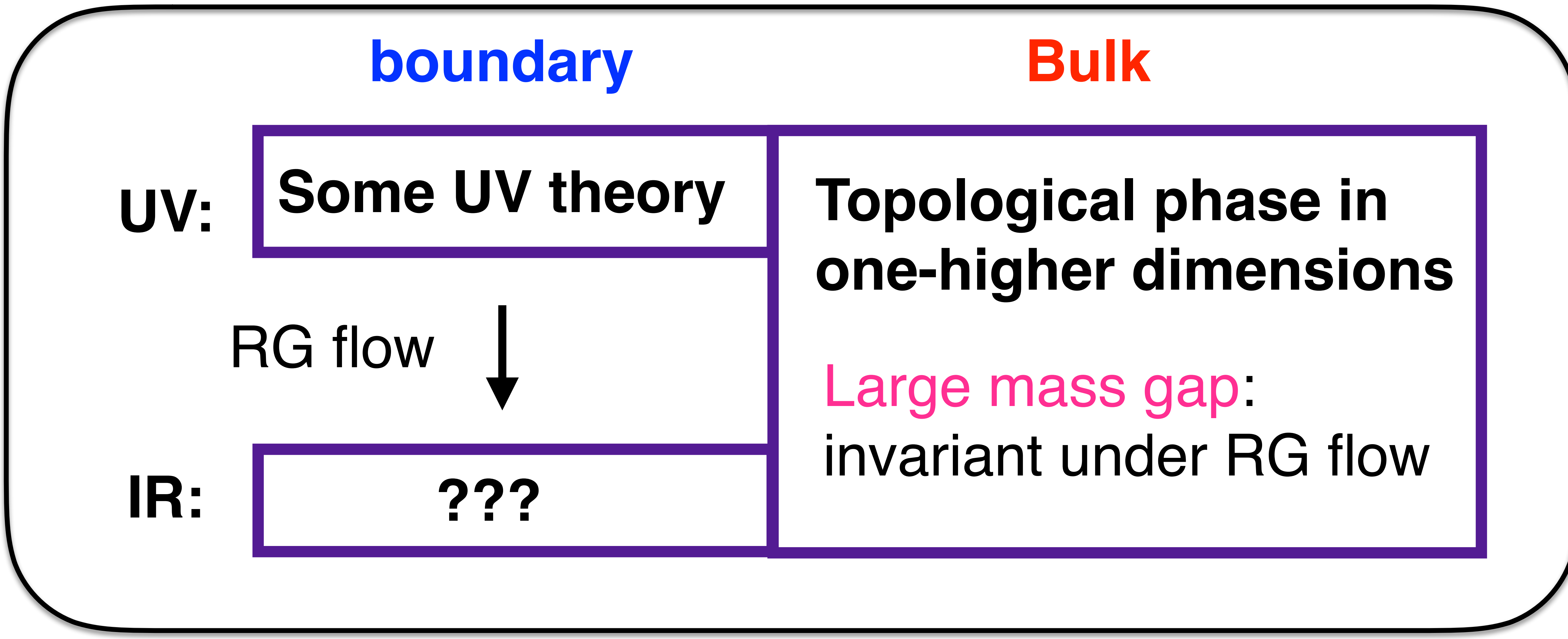
In other words:
Anomalies are conserved under RG flows.

Why?

Let’s recall that anomalous theories are realized on a boundary of topological phase.



't Hooft anomaly matching



UV and IR anomalies are controlled by the same bulk topological phase. Therefore, anomalies are conserved.

’t Hooft anomaly in QCD

’t Hooft anomaly matching is very useful in studying strong coupling dynamics such as QCD.

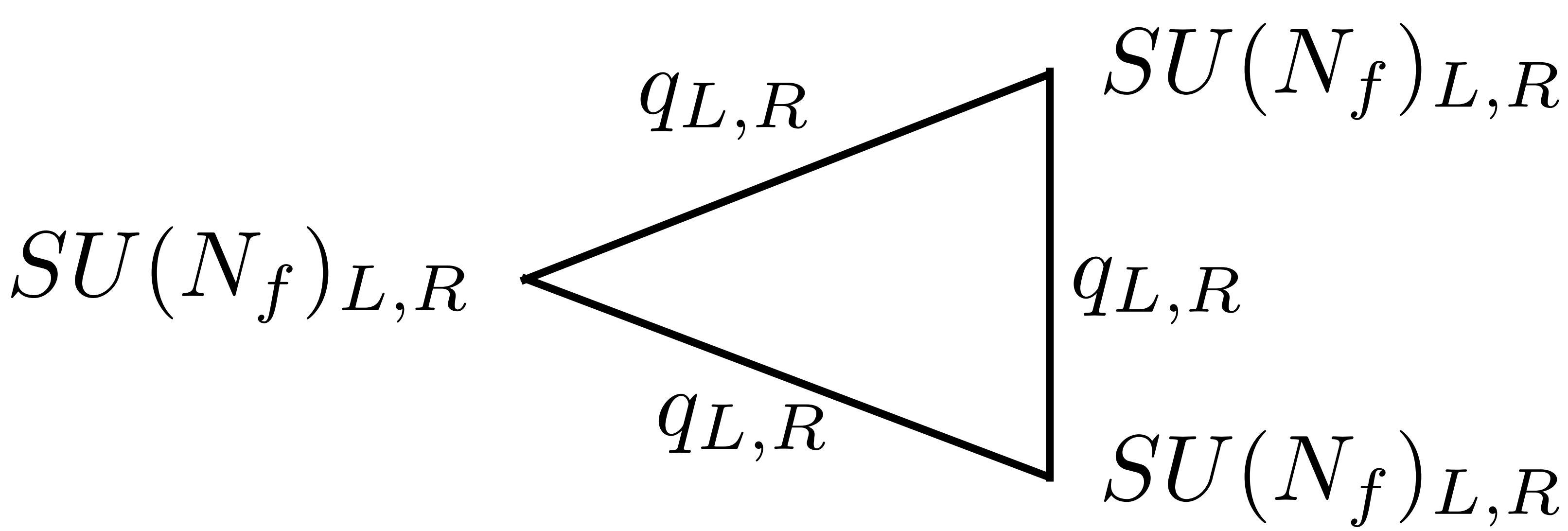
't Hooft anomaly in QCD

't Hooft anomaly matching in QCD

In QCD, there exist perturbative triangle anomalies of chiral symmetry $SU(N_f)_L \times SU(N_f)_R$

q_L : left-handed quarks, rotated by $SU(N_f)_L$

q_R : right-handed quarks, rotated by $SU(N_f)_R$



’t Hooft anomaly in QCD

Implications of ’t Hooft anomaly matching in QCD

UV: The quarks have the ’t Hooft anomaly

↓ confinement

IR: If there is no chiral fermion to match the anomaly, the chiral symmetry must be spontaneously broken.

't Hooft anomaly in QCD

Simple case: $N_c = \text{even}$

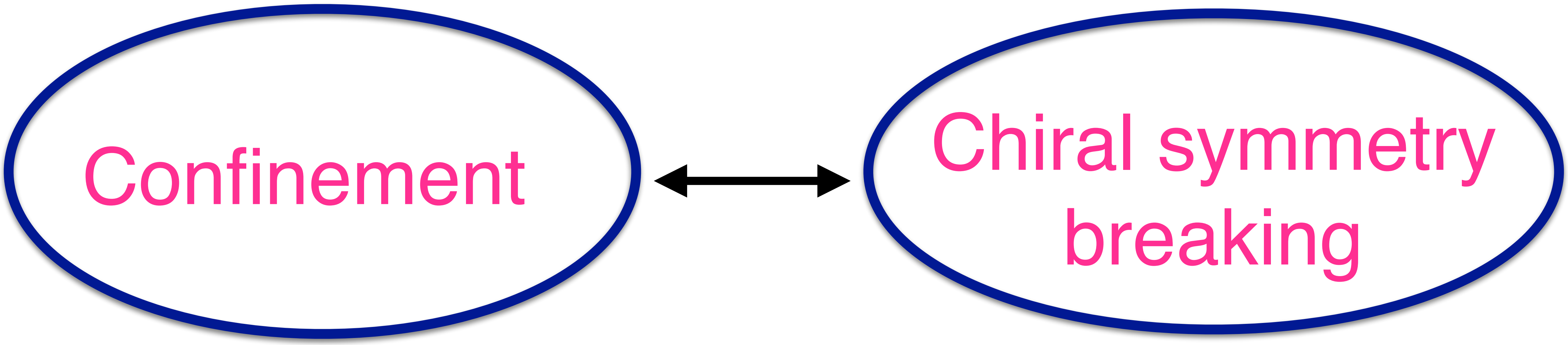
All gauge invariant composites are bosons,
so there is no fermion to match the anomaly.

Therefore, chiral symmetry must be broken
if the theory is in confinement phase.

Other values of N_c requires more complicated discussions.

't Hooft anomaly in QCD

't Hooft anomaly matching gives an important relation between the two most important concepts in QCD:



Other theories

How about other theories?

- Pure Yang-Mills theories
- Pure $\mathcal{N}=1$ Super-Yang-Mills theories
- …(any gauge theory)

They don't have continuous symmetries.
Pure Yang-Mills don't even have fermions.

Is there no useful anomaly at all?

Subtler symmetry & anomaly

It turns out:

- They have 1-form symmetry (called center symmetry)
- The theories have a mixed anomaly between the 1-form symmetry and a discrete symmetry

Discrete symmetry:

- Time-reversal for pure Yang-Mills
- Axial symmetry for Super-Yang-Mills

Subtler symmetry & anomaly

The existence of the subtle symmetry and the anomaly implies that:

The discrete symmetries (time-reversal or discrete axial symmetry) must be broken after confinement. (More exotic possibilities neglected.)

[Gaiotto-Kapstin-Komargodski-Seiberg,2017]
[Komargodski-Sulejmanpašić-Unsal, 2017]
[Shimizu-Yonekura, 2017]

More surprisingly, the subtle anomaly can even constrain finite temperature phase transition.

I will review more details of this topic in later lectures.

Contents

Summary of the overview

- The concepts of symmetry and anomaly are organized and generalized in recent years.
- They are very useful for the studies of strong dynamics. There are many more applications.
- String theory has extremely subtle and sophisticated topological structures related to anomaly which need to be investigated further.