Some topological properties of gauge theories

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Plan:

- 1-form center symmetry in gauge theory
- Anomalies of center symmetry and their applications
- Variants of center symmetry: global structure of symmetry groups (SO(3) $\neq$ SU(2))

1 Higher form symmetry

QFT on d-manifold M.

Continuous 0-form symmetry:
Conserved current $J = J_\mu dx^\mu$.
Hodge dual

$$*J = \frac{1}{(n-1)!}\epsilon_{\nu_1\cdots\nu_d}J^{\nu_1}dx^{\nu_2}\cdots dx^{\nu_d}: \ d-1\text{-form} \tag{1.1}$$

Charge: defined codimension one (dimension $d-1$) submanifold $\Sigma_{d-1} \subset M$

$$Q(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} *J: \ \text{total charge on } \Sigma_{d-1} \tag{1.2}$$

Symmetry operator

$$U(\Sigma_{d-1}, \alpha) = \exp(i\alpha Q(\Sigma_{d-1})) \tag{1.3}$$

Charge conservation

$$\partial_\mu J^\mu \propto d(*J) = 0: \ \text{closed} \tag{1.4}$$

$$\xrightarrow{\text{Stokes thm}} Q(\Sigma_{d-1}) \text{ depends only on topology of } \Sigma_{d-1} \tag{1.5}$$

Discrete 0-form symmetry:
Example: $G = \mathbb{Z}_N$.
No current J.
$U(\Sigma_{d-1}, \alpha)$ for each element $\alpha \in G$.
Invariance under deforming Σ_{d-1}.

Continuous p-form symmetry:
Conserved current $J = \frac{1}{(p+1)!}J_{\mu_1\cdots\mu_{p+1}}dx^{\mu_1}\cdots dx^{\mu_{p+1}}$: $p+1$-form
Hodge dual

$$*J = \frac{1}{(p+1)!(n-p-1)!}\epsilon_{\nu_1\cdots\nu_d}J^{\nu_1\cdots\nu_{p+1}}dx^{\nu_{p+2}}\cdots dx^{\nu_d}: \ d-p-1\text{-form} \tag{1.6}$$

Charge: defined on codimension $p+1$ (dimension $d-p-1$) submanifold Σ_{d-p-1}

$$Q(\Sigma_{d-p-1}) = \int_{\Sigma_{d-p-1}} *J \tag{1.7}$$

$$U(\Sigma_{d-p-1}, \alpha) = \exp(i\alpha Q(\Sigma_{d-p-1})) \tag{1.8}$$

Charge conservation

$$\partial_{\mu_1} J^{\mu_1 \cdots \mu_{p+1}} \propto d(*J) = 0 : \text{ closed} \tag{1.9}$$

$$\rightarrow \; Q(\Sigma_{d-p-1}) \text{ depends only on topology of } \Sigma_{d-p-1} \tag{1.10}$$

Example:
String theory Dp-branes have p-form charge (more precisely: K-theory)

Discrete p-form symmetry:
No J, but $U(\Sigma_{d-p-1}, \alpha)$ exists.
Example:
Non-BPS but stable Dp-branes have discrete p-form charge
(more precisely: K-theory)

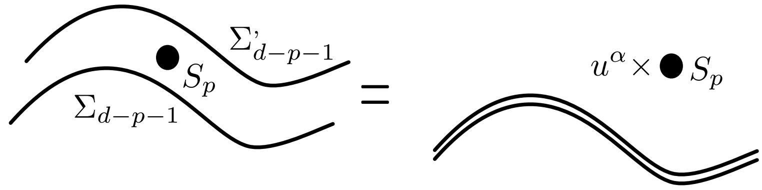
Operator charge
$O(S_p)$: operator on p-dim. submanifold S_p.
Example: Wilson loop $W = \mathrm{tr}\, \mathrm{P} \exp(i \int A)$ with $p=1$ etc.

Σ_{d-p-1}, Σ'_{d-p-1}: such that $\Sigma'_{d-p-1} - \Sigma_{d-p-1} = \partial T_{d-p}$, intersection $T_{d-p} \cdot S_p = 1$.

$$U(\Sigma'_{d-p-1}, \alpha) O(S_p) U^{-1}(\Sigma_{d-p-1}, \alpha) = u^\alpha \cdot O(S_p) \tag{1.11}$$

Here

$$u^\alpha (= e^{i\alpha q}) : \text{representation of } U(\alpha) \tag{1.12}$$



Coupling to background fields

Continuous p-form symmetry:
Background fields: $p+1$-form B_{p+1}. Coupling

$$\int_M B_{p+1} \wedge *J + \cdots . \tag{1.13}$$

Discrete p-form symmetry: More abstract approach is needed.

One way to think:

$$\text{insertion of } U(\Sigma_{d-p-1}, \alpha) \longleftrightarrow \text{Coupling to } B_{p+1}$$

Roughly

$$(\Sigma_{d-p-1}, \alpha) \xrightarrow{\text{Poincare dual}} B_{p+1} \sim \alpha\delta(\Sigma_{d-p-1}),\ \delta : \text{delta function localized on } \Sigma_{d-p-1}$$

More abstractly,

$$(\Sigma_{d-p-1}, \alpha) \in H_{d-p-1}(M, G) \quad (G \text{ abelian}) \tag{1.14}$$

$$B_{p+1} \in H^{p+1}(M, G)\text{: cohomology with } G \text{ coefficient} \tag{1.15}$$

Assume: no torsion in (co)homology (for technical simplicity)
$H^{p+1}(M, G)$: closed differential forms $dB_{p+1} = 0$ up to exact ones (de Rham theorem.)

$G = \mathbb{Z}_N$: $\int B_{p+1} =$ integer mod N.

Dimensional reduction
S^1 reduction $M = L \times S^1 \to L$,

$$p\text{-form} \to (p-1)\text{-form} \oplus p\text{-form}$$

Cohomology:

$$H^{p+1}(L \times S^1, G) = H^{p+1}(L, G) \oplus H^p(L, G) \tag{1.16}$$

Spontaneous breaking

Proposal:
(1)Consider operators $O(S_p)$ with nonzero charge.
(2)See $\langle O(S_p) \rangle \to 0$ or not in the limit $S_p \to \infty$.

Example: Wilson loop area law: $\langle W \rangle \sim \exp(-\text{const.Area}) \to 0$ in confinement phase.

Remark:
Operator renormalization

$$O(S_p) \to O(S_p)_{\text{ren}} = O(S_p) \exp \int_{S_p} (c_0 + c_1(\text{Riemann, extrinsic, curvature}) + \cdots) \tag{1.17}$$

$\langle O(S_p) \rangle \to 0$ means faster than any renormalization terms.

2 1-form center symmetry

2.1 Example: U(1) gauge theory

Remark 1:
Any gauge group G contains Cartan subgroup

$$\mathrm{U}(1)^r \subset G, \qquad r\text{: rank of } G \tag{2.1}$$

Some properties of G captured by U(1)s.

Remark 2:
I discuss $d = 4$. It is easy to generalize the following to d-dim.

Free Maxwell:

$$\mathcal{L} = \frac{1}{2g^2} F \wedge *F. \tag{2.2}$$

$F = dA$.

Two 1-form symmetry:

$$\text{electric 1-form sym. current } *J_E = \frac{1}{g^2} * F \tag{2.3}$$

$$\text{magnetic 1-form sym. current } *J_M = \frac{1}{2\pi} F \tag{2.4}$$

Free Maxwell equations = conservation equations $d(*J_E) = d(*J_M) = 0$.

Electric 1-form symmetry operator as Gukov-Witten operator
Σ: 2-dim. surface.

Insertion of

$$U_E(\Sigma, \alpha) := \exp(i\alpha \int_\Sigma *J_E) \tag{2.5}$$

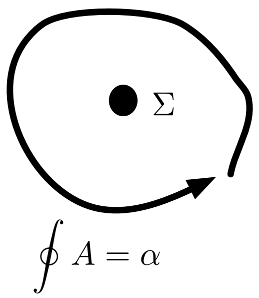
in path integral:

$$iS = i \int_M \frac{1}{2g^2} F \wedge *F + i\alpha \int_\Sigma \frac{1}{g^2} * F \tag{2.6}$$

$$= i \int_M \frac{1}{2g^2} F' \wedge *F' + (\text{pure background.}) \tag{2.7}$$

$$F' = \alpha\delta(\Sigma) + F \quad (\delta(\Sigma): \text{ delta function localized on } \Sigma) \tag{2.8}$$

$$\rightarrow \oint_{\text{around } \Sigma} A' = \alpha, \quad (dA' = F') \tag{2.9}$$



Insertion of $U_E(\Sigma, \alpha) \longleftrightarrow$ holonomy α around Σ.

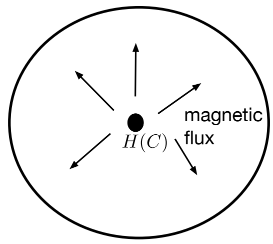
Right hand side: Gukov-Witten operator

Operator charge
Wilson loop operator

$$W(C) = \exp(i \oint_C A) \tag{2.10}$$

Crossing $W(C)$ and $U_E(\Sigma, \alpha) \to e^{i\alpha}$ by definition of Gukov-Witten:
W has charge 1 under electric 1-form symmetry.

’t Hooft operator $H(C)$: charge 1 under magnetic 1-form symmetry.



$$\int_{\text{around } C} *J_M = \int_{\text{around } C} \frac{1}{2\pi} F = 1 \tag{2.11}$$

In d-dim., it is a $d-3$-form symmetry.
$d = 3$: monopole operator & topological symmetry.

Explicit breaking
The symmetries are explicitly broken by charged particles

$$d(*J_E) = *j_E, \leftarrow \text{ electric current} \tag{2.12}$$
$$d(*J_M) = *j_M, \leftarrow \text{ magnetic current} \tag{2.13}$$

Not conserved.

Assume: all particles have electric charges e multiple of N: $e \in N\mathbb{Z}$.

$$\text{Gukov-Witten with } \alpha = 2\pi n/N: \quad U_E(\Sigma, \frac{2\pi n}{N}) : \oint_{\text{around } \Sigma} A = \frac{2\pi n}{N} \tag{2.14}$$

$n \in \mathbb{Z}_N$. Locally invisible to particles since

$$\exp(ie \oint A) = 1. \tag{2.15}$$

Explicit breaking

$$\mathrm{U}(1) \to \mathbb{Z}_N \tag{2.16}$$

The similar statement true for magnetic 1-form.

Spontaneous breaking
C: loop with size $\to \infty$

	Coulomb	Higgs	Confine
Wilson $W(C)$	$\neq 0$	$\neq 0$	$\to 0$
't Hooft $H(C)$	$\neq 0$	$\to 0$	$\neq 0$
electric 1-form	broken	broken	unbroken
magnetic 1-form	broken	unbroken	broken

- $W(C) \neq 0$ in Higgs phase is obvious. ($A_\mu \to 0$)
- $H(C) \to 0$ in Higgs phase: Meissner effect (superconductor)
  Magnetic fluxes are confined.
- Confinement = electric-magnetic dual of Higgs.

1-form symmetries distinguish phases. One of the most important motivations!
Confinement = electric 1-form unbroken

2.2 1-form Center symmetry in Yang-Mills

SU(2) theory with matter in adjoint rep. (e.g. SYM)

$U(1) \subset SU(2)$: Cartan subgroup. Electric-magnetic charge (e, m) under U(1):

	W-boson	monopole
(e, m)	$(2, 0)$	$(0, 1)$

$U(1) \subset SU(2)$ has electric $\mathbb{Z}_2$ 1-form sym.

SU(N)

Def: 1-form $\mathbb{Z}_N$ center symmetry

$$U_E(\Sigma, n) : \text{Gukov-Witten op. with } P\exp(i\oint A) = e^{\frac{2\pi i n}{N}} I_N \in [\text{center of SU}(N)] \quad (2.17)$$

$n \in \mathbb{Z}_N$. Sym. if the center acts trivially on all fields.

Remark: More precise definition of singularity along Σ:

$$M \setminus \Sigma : \text{spacetime with } \Sigma \text{ eliminated.} \quad (2.18)$$

$$D_\Sigma : \text{ infinitesimal tube surrounding } \Sigma \text{ (tubular neighborhood).} \quad (2.19)$$

Transition function between them

$$g(\theta) = \exp\left(\frac{2\pi i n}{N} \operatorname{diag}(1, \cdots, 1, -N+1)\theta\right) \quad (2.20)$$

$g(\theta + 2\pi) = e^{\frac{2\pi i n}{N}} g(\theta)$. Transition function of SU(N)/$\mathbb{Z}_N$ bundle.

$$P\exp(i\oint_\Sigma A) = \begin{cases} e^{\frac{2\pi i n}{N}} & \text{on } M \setminus \Sigma \\ 1 & D_\Sigma \end{cases} \quad (2.21)$$

A_μ extended smoothly to D_Σ, but as a SU(N)/$\mathbb{Z}_N$ bundle.

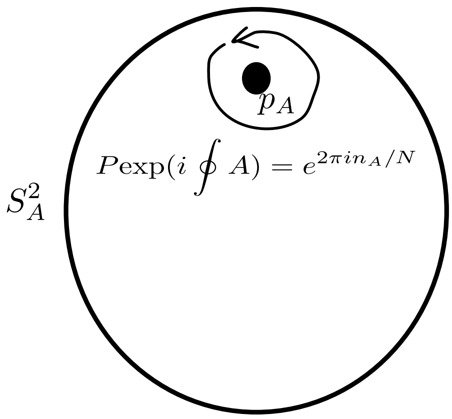
SU(N) bundle $\rightarrow$ SU(N)/$\mathbb{Z}_N$ bundle

2.3 Topology

Demonstration:
$M = S_A^2 \times S_B^2$ ($M = T^4$ works as well)
$\Sigma = p_A \times S_B^2$, p_A=north pole of S_A^2
Insert $U(\Sigma, n_A)$.

$$\text{Ex:} \quad A_A = \frac{(1+\cos\theta_A)}{2} d\phi_A \cdot \frac{n_A}{N} \,\mathrm{diag}(1, \cdots, 1, -N+1) \text{ on } S_A^2 \setminus p_A \tag{2.22}$$

- Non-singular except at $p_A (\theta_A = 0)$.
- $P\exp(i \oint_{\text{around } p_A} A) = e^{\frac{2\pi i n_A}{N}}$.



magnetic flux on S_A^2.

$$F_A = -f_A \cdot \frac{1}{N} \mathrm{diag}(1, \cdots, 1, -N+1) \tag{2.23}$$

$$f_A = \frac{n_A}{2} \sin\theta_A d\theta_A d\phi_A \qquad : \text{monopole flux on } S_A^2 \tag{2.24}$$

$$\int_{S_A^2} \frac{f_A}{2\pi} = n_A \tag{2.25}$$

Take another $\Sigma' = S_A^2 \times p_B$.

$$A_B = (A \leftrightarrow B) \tag{2.26}$$

Instanton number of $A = A_A + A_B$:

$$\int_M \frac{1}{8\pi^2} \mathrm{tr}(F^2) = \int_{S_A^2 \times S_B^2} \mathrm{tr}(\frac{F_A}{2\pi} \frac{F_B}{2\pi}) = \frac{N-1}{N} n_A n_B \equiv -\frac{n_A n_B}{N} \mod 1. \tag{2.27}$$

Fractional instanton number!

More generally,

$$B_2 \in H^2(M, \mathbb{Z}_N) : \text{2-form bkg field for 1-form center sym} \tag{2.28}$$
$$(\leftrightarrow \text{inclusion of } U(\Sigma, n)) \tag{2.29}$$

In the example,

$$B_2 = \frac{1}{2\pi}(f_A + f_B) \qquad \text{mod } N \tag{2.30}$$
$$\int_{S_A^2} B_2 = n_A, \quad \int_{S_B^2} B_2 = n_B \tag{2.31}$$

Generally

$$N_{\text{instanton}} \equiv -\frac{1}{2N}\int_M (B_2)^2 \quad \text{mod } 1. \tag{2.32}$$

3 't Hooft anomaly involving center symmetry

3.1 SU(N) with adjoint fermion

Concrete example:

$$A_\mu : \mathrm{SU}(N) \text{ gauge fields} \tag{3.1}$$

$$\lambda : \text{Weyl fermion in adjoint rep.} \tag{3.2}$$

$\mathcal{N} = 1$ SYM, but SUSY irrelevant.
(Multiple λ's OK)

Discrete axial sym:

$$\mathbb{Z}_{2N} : \ \lambda \to e^{i\alpha}\lambda = e^{\frac{2\pi i}{2N}}\lambda \tag{3.3}$$

Path integral measure

$$[\mathscr{D}\lambda] \to \exp(2iN\alpha \cdot N_{\text{instanton}})[\mathscr{D}\lambda] = \exp(2\pi i N_{\text{instanton}})[\mathscr{D}\lambda] \tag{3.4}$$

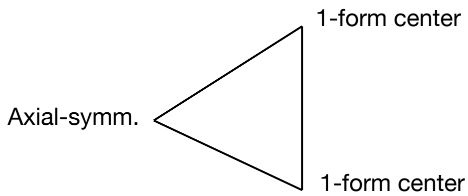
For $N_{\text{instanton}} \in \mathbb{Z}$, anomaly free.

Nonzero bkg B_2,

$$\exp(2\pi i N_{\text{instanton}}) = \exp(-\frac{2\pi i}{2N}\int_M (B_2)^2) : \ \text{anomaly} \tag{3.5}$$

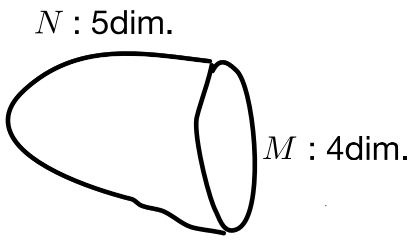
't Hooft anomaly.

Schematically...



3.2 $d + 1 = 5$ topological phase

Anomaly of d-dim. theory = $d + 1$-dim. topological phase.



Let

$$C_1 \in H^1(M, \mathbb{Z}_{2N}) \quad : \text{background for axial } \mathbb{Z}_{2N} \tag{3.6}$$

$$B_2 \in H^2(M, \mathbb{Z}_N) \quad : \text{background for 1-form center } \mathbb{Z}_N \tag{3.7}$$

5d topological phase: Guess:

$$Z[B_2, C_1] = \exp\left(-\frac{2\pi i}{2N}\int_N C_1(B_2)^2\right). \tag{3.8}$$

Rougly, by "gauge transformation" $\delta C_1 = d\alpha$,

$$\delta \log Z[B_2, C_1] = -\frac{2\pi i}{2N}\int_N d\alpha(B_2)^2 \tag{3.9}$$

$$= -\frac{2\pi i}{2N}\int_M \alpha(B_2)^2 \tag{3.10}$$

Cancelled against the anomaly of 4dim theory.

Remark:
Above description not precise.

3.3 Application

Zero temperature
't Hooft anomaly implies either

- One of the symmetries spontaneously broken.
- Extra DOF which match the anomaly.

Neglect the second possibility.

Confinement: 1-form center (electric) sym. unbroken.
$\to$ Axial-sym. must be broken.
(In the usual QCD, $\mathrm{SU}(N_f)^3_{L,R}$ 't Hooft anomaly $\to$ chiral symmetry breaking.)

Many SUSY techniques show it: $\langle\lambda\lambda\rangle \neq 0$.
The technique here is very general, no SUSY necessary.

Finite temperature
S^1 compactification with $\beta = 1/T$

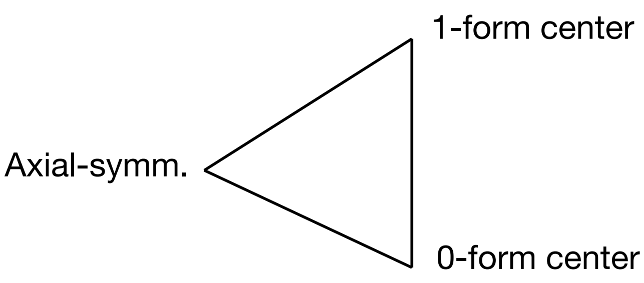
Dimensional reduction

$$\text{1-form center} \xrightarrow{S^1} \text{0-form center} \oplus \text{1-form center} \tag{3.11}$$

$$B_2 \to B_1 \oplus B_2 \tag{3.12}$$

$$-\frac{2\pi i}{2N}\int_{S^1\times M_3}(B_2)^2 \to -\frac{2\pi i}{N}\int_{M_3} B_1 B_2 \tag{3.13}$$

't Hooft Anomaly still survives.

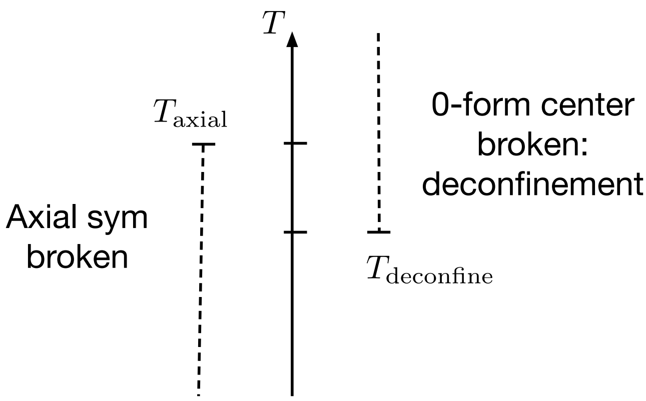


Implications: either

- Axial broken
- 0-form center broken (def: deconfinement)
- 1-form center broken
- Extra DOF matching the anomaly

The third case: “Higgs or Coulomb phase of 3d YM after dimensional reduction”.
Neglect it.
Neglect also the Fourth case.

The first or second case realized at any temperature:



$$T_{\text{deconfine}} \le T_{\text{axial}}$$

Equality only for 1st order transition

- Strong constraints on strongly coupled system
- SUSY useless. (Broken by finite temperature)

4 Variants of center symmetry

Quarks in fundamental rep. of SU(N):
No center symmetry at all. Center acts nontrivially on quarks

4.1 Gauge-flavor center

SU(N_c) with N_f flavors of quarks.

$$q: \quad \text{bi-fundamental of } \mathrm{SU}(N_c)\times\mathrm{SU}(N_f) \tag{4.1}$$

Suppose

$$\text{greatest common divisor}(N_c, N_f) := K \neq 1 \tag{4.2}$$

Subgroup

$$\mathbb{Z}_K \subset \mathrm{SU}(N_c)\times\mathrm{SU}(N_f) \tag{4.3}$$

generated by

$$g = (e^{2\pi i/K}, e^{2\pi i/K}) \in \mathrm{SU}(N_c)\times\mathrm{SU}(N_f) \tag{4.4}$$

g acts trivially on all fields.
We can take the group to be

$$[\mathrm{SU}(N_c)\times\mathrm{SU}(N_f)]/\mathbb{Z}_K \tag{4.5}$$

Fractional instanton possible.
Demonstration: $N_c = N_f = K := N$ and

$$A_{\text{color}} = A_{\text{flavor}} = A \tag{4.6}$$

q: bifundamental $\to$ adjoint of A
Take A: previous fractional instanton

$$\int_M \frac{1}{8\pi^2}\,\mathrm{tr}(F^2) \equiv -\frac{n_A n_B}{N} \quad \mathrm{mod}\ 1. \tag{4.7}$$

4.2 General lesson

Symmetry group H (color, flavor, Lorentz, mixture of them)
Suppose a center subgroup

$$C \subset H \tag{4.8}$$

acts trivially on all fields.
The group

$$H/C \tag{4.9}$$

realize more refined topology. (Fractional instanton etc.)
New anomaly from refined topology.

4.3 Spin-gauge center

Assume the following:

- G: internal symmetry
- Exists an element $\epsilon \in G$ in the center such that
- $\epsilon = -1$ for all fermions
- $\epsilon = +1$ for all bosons

Possible to combine Lorentz group Spin(d) and G as

$$[\mathrm{Spin}(d) \times G]/\mathbb{Z}_2 \tag{4.10}$$

$\mathbb{Z}_2$ generated by $((-1)^F, \epsilon)$.

Mathematical terminology:
If $G = \mathrm{U}(1)$,

$$\mathrm{Spin}^c(d) := [\mathrm{Spin}(d) \times \mathrm{U}(1)]/\mathbb{Z}_2 \tag{4.11}$$

Possible if boson: U(1) charge even, fermion: U(1) charge odd

Bundle of this group: called Spin^c structure.
Manifolds with Spin^c structure: called Spin^c manifolds.
Example: $\mathbb{CP}^2$ not Spin but Spin^c.

Fractional instanton, new anomaly, etc. (Explore the literature)