Quantum Information Measures in QFT

Tatsuma Nishioka

Quantum Information Measures in QFT

0. Introduction

$\boxed{1.1}$ Quantum Information Theory (QIT)

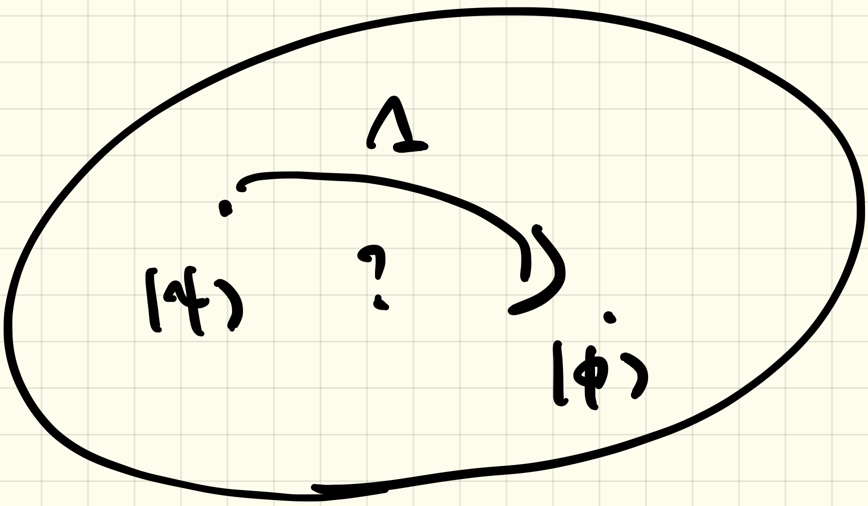
state conversion

$$|\psi\rangle \xrightarrow[1]{} |\phi\rangle$$

- What operations Λ are allowed?

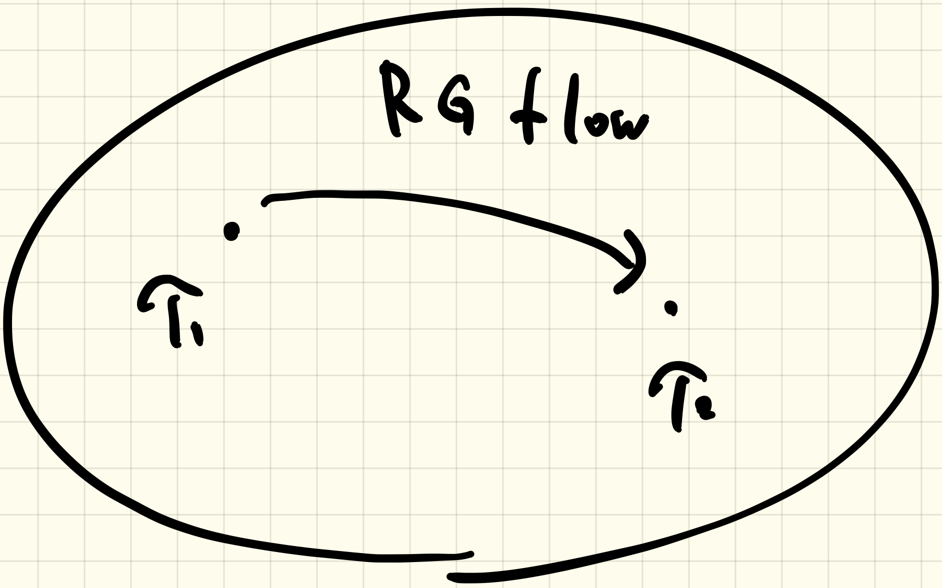
- When $|\psi\rangle$ is transformed to $|\phi\rangle$?

$\rightsquigarrow$ Ordering of quantum states



Space of quantum states

$\boxed{1.1}$ QFT



Space of QFTs

Under RG flow

$$\mathcal{T}_1 \longrightarrow \mathcal{T}_2$$

- When $\mathcal{T}_1$ flows to $\mathcal{T}_2$?

$\rightsquigarrow$ Ordering QFTs

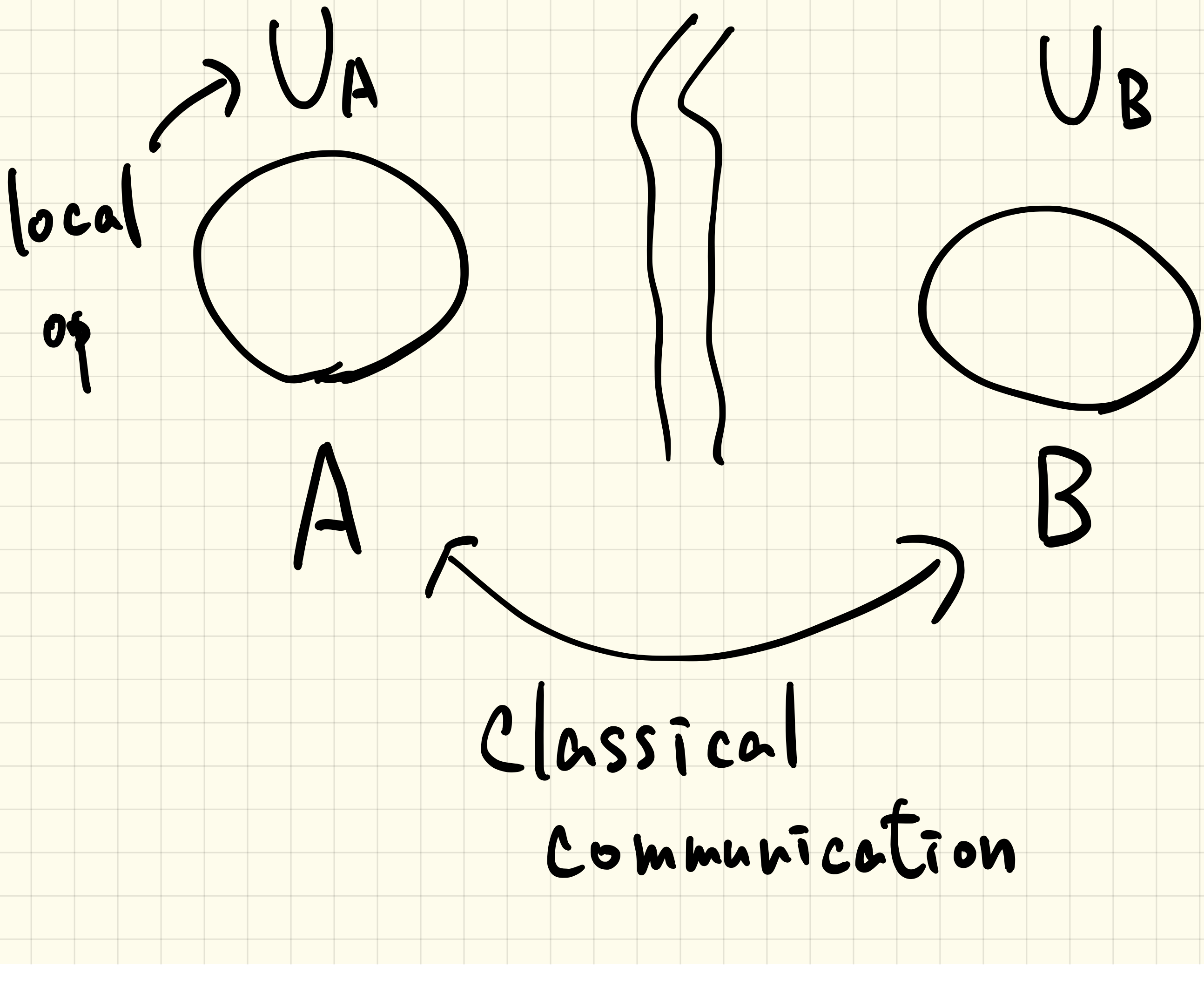
(C - theorems)

1. QIT

$|\psi\rangle$: quantum states

Λ : LO CC

local op classical communication


local op
U_A
U_B
A
B
Classical communication

Classification of quantum states

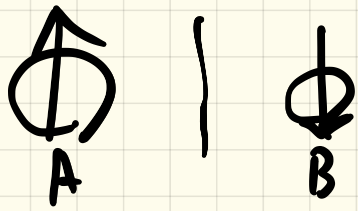
- pure bipartite system

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\overset{\Downarrow}{|\psi\rangle} \longleftrightarrow \rho_\psi = |\psi\rangle\langle\psi|$$

(mixed state $\longleftrightarrow \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$)

- Separable state



$$|\psi\rangle = \underset{\substack{\uparrow \\ \mathcal{H}_A}}{|\psi_A\rangle} \otimes \underset{\substack{\uparrow \\ \mathcal{H}_B}}{|\psi_B\rangle}$$

- Entangled state

  = non-separable

Bell pair (EPR)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle|1_B\rangle + |1_A\rangle|0_B\rangle\right)$$

$$\neq |\psi_A\rangle \otimes |\psi_B\rangle$$

for any ψ_A, ψ_B

Schmidt decomposition

Canonical form
of a bipartite quantum state

In general

$$|\psi\rangle = \sum_{i,j} \underset{\substack{\uparrow \\ \text{cpx}}}{c_{ij}} \underset{\substack{\uparrow \\ \mathcal{H}_A}}{|i_A\rangle} \underset{\substack{\uparrow \\ \mathcal{H}_B}}{|j_B\rangle}$$

By changing the orthonormal basis

$$|i_A\rangle \to U_{ik}\,|k_A\rangle$$
$$|j_B\rangle \to V_{jl}\,|l_B\rangle$$

U, V unitaries

C_{ij} : $d_A \times d_B$ rectangular cpx matrix

$$C_{ij} = \overbrace{\begin{pmatrix} & & \\ & & \end{pmatrix}}^{d_A} \updownarrow d_B$$

Singular value decomposition

$$C_{ij} = \left(U \cdot \Lambda \cdot V \right)_{ij}$$

$$\begin{array}{cc} \Lambda = \underbrace{\left(\begin{array}{ccc|c} \sqrt{\lambda_1} & & 0 & \\ & \ddots & & 0 \\ 0 & & \sqrt{\lambda_{d_B}} & \end{array} \right)}_{d_A} & \updownarrow d_B \end{array}$$

$(d_A \geq d_B)$

- $\lambda_i \geq 0$

- $\langle \psi | \psi \rangle = 1$

$$\Leftrightarrow \sum_{i=1}^{\min(d_A, d_B)} \lambda_i = 1$$

$$|\Psi\rangle = \sum_{i=1}^{\min(d_A, d_B)} \sqrt{\lambda_i} \, |i_A\rangle |i_B\rangle$$

- $\lambda_1^{\psi} \geq \lambda_2^{\psi} \geq \cdots \geq \lambda_{\min(d_A, d_B)}^{\psi} \geq 0$

$$\lambda^{\psi} = (\lambda_1^{\psi}, \lambda_2^{\psi}, \cdots, \lambda_{\min(d_A, d_B)}^{\psi})$$

$\uparrow$

Schmidt vector

- We want to introduce "order" between quantum states

Partial order $\prec$

For $x = (x_1, x_2, \cdots) \in \mathbb{R}^n$
$y = (y_1, y_2, \cdots)$

$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$

$x \prec y$ iff $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$ for any k

x is majorized by y

Example

$$\left(\frac{1}{n}, \frac{1}{n}, \cdots \frac{1}{n}\right) < \left(\frac{1}{n-1}, \frac{1}{n-1}, \cdots, \frac{1}{n-1}, 0\right)$$

$$< \left(\frac{1}{n-2}, \frac{1}{n-2}, \cdots, \frac{1}{n-2}, 0, 0\right)$$

$$< \cdots$$

$$< (1, 0, \cdots, 0)$$

There are vectors which cannot be compared

$$x \not\prec y \ , \ y \not\prec x$$

e.g.)

$$x = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

$$y = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$$

$k=1$ $\quad x_1 = \frac{1}{2} > \frac{2}{5} = y_1$

$k=2$

$x_1 + x_2 = \frac{3}{4}$

$\wedge$

$y_1 + y_2 = \frac{4}{5}$

QIT problem

When can a state $|\psi\rangle$ be transformed into a state $|\phi\rangle$ by LOCC?

$$|\psi\rangle \overset{?}{\rightarrow} |\phi\rangle$$

LOCC $\subset$ SEP (Separable op)

$$\Lambda(\rho) = \sum_m (A_m \otimes B_m)\, \rho\, (A_m \otimes B_m)^\dagger$$

$$\boxed{\sum_m A_m^\dagger A_m \otimes B_m^\dagger B_m = \mathbb{1}_A \otimes \mathbb{1}_B}$$

LOCC

A performs op. A_i

A sends the results i to B

Then B performs some op

$B_j^{(i)}$ | $A_i \otimes B_j^{(i)}$

$m = (i, j)$

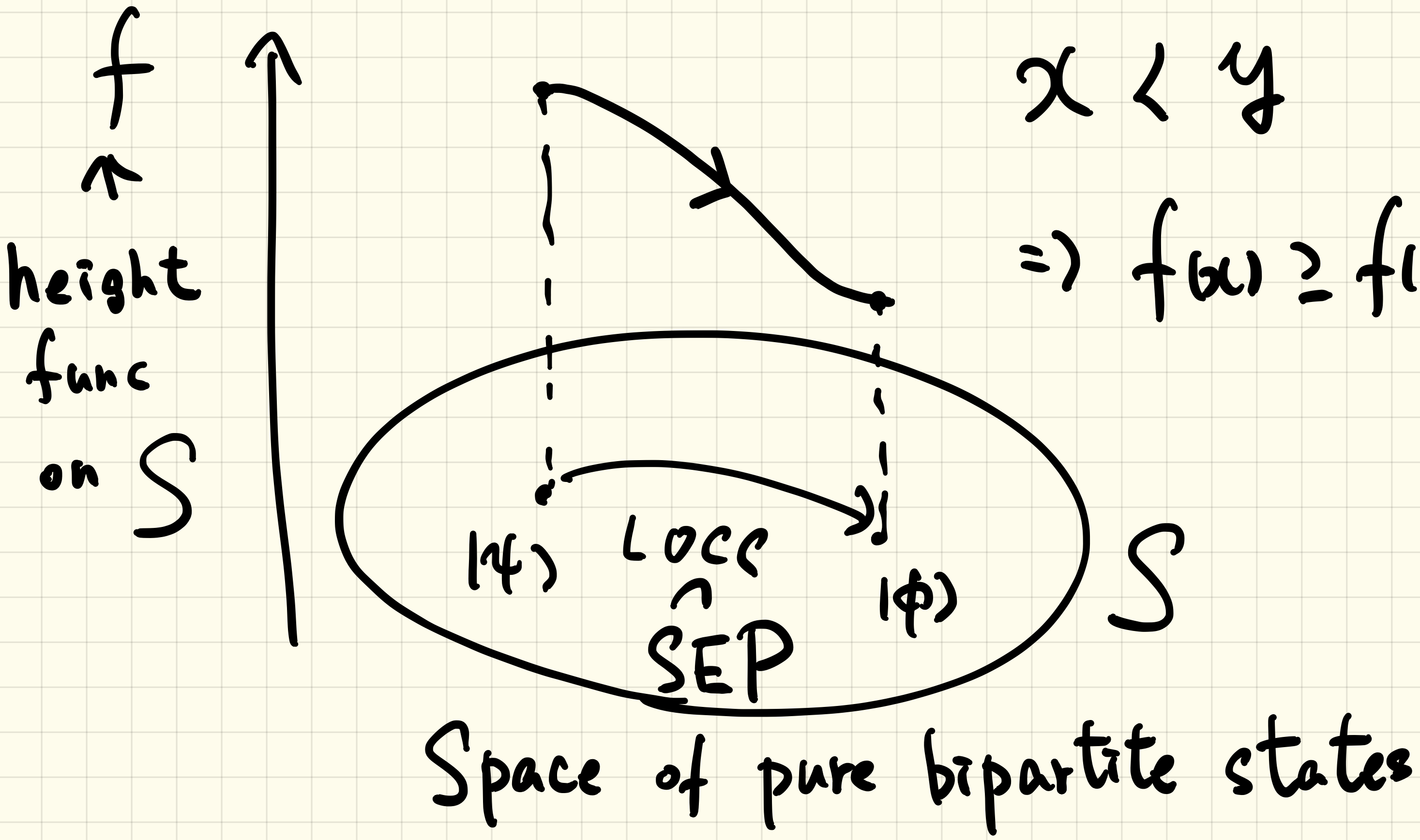
Theorem (Michael Nielsen's majorization)

For bipartite pure states $|\psi\rangle, |\phi\rangle$

$|\psi\rangle$ is transformed to $|\phi\rangle$ under SEP iff $\lambda^{\psi} \prec \lambda^{\phi}$

$$|\psi\rangle \xrightarrow[\text{SEP}]{} |\phi\rangle \quad \text{iff} \quad \lambda^{\psi} \prec \lambda^{\phi}$$

· How to quantify the order?



$x < y$

$\Rightarrow f(x) \ge f(y)$

Schur-concave function

$$x \prec y \Rightarrow f(x) \geq f(y)$$

<u>NB</u>. $f(x) \geq f(y)$ does not necessarily mean $x \prec y$

but it means $y \nprec x$

Quantum information measures (entanglement)

A1. Monotonicity

$$|\psi\rangle \xrightarrow[\text{LOCC}]{} |\phi\rangle \Rightarrow f(\rho_\psi) \geq f(\rho_\phi)$$

A2. Vanishing for sep. states

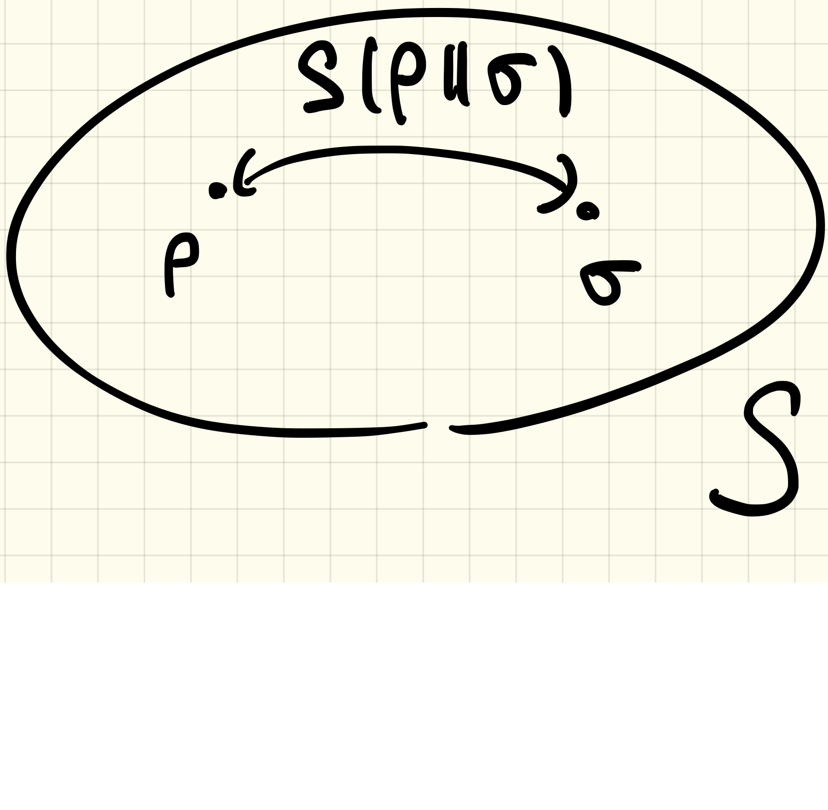
$|\psi\rangle$: separable. $\Leftrightarrow f(\rho_\psi) = 0$

A3. Convexity

$$f\left(\sum_i p_i \rho_i\right) \le \sum_i p_i f(\rho_i)$$

Relative entropy

$\simeq$ "distance" bet quantum states



$$S(\rho \| \sigma)$$

TP : $\mathrm{Tr}(\rho) = 1 \Rightarrow \mathrm{Tr}(\Lambda(\rho)) = 1$

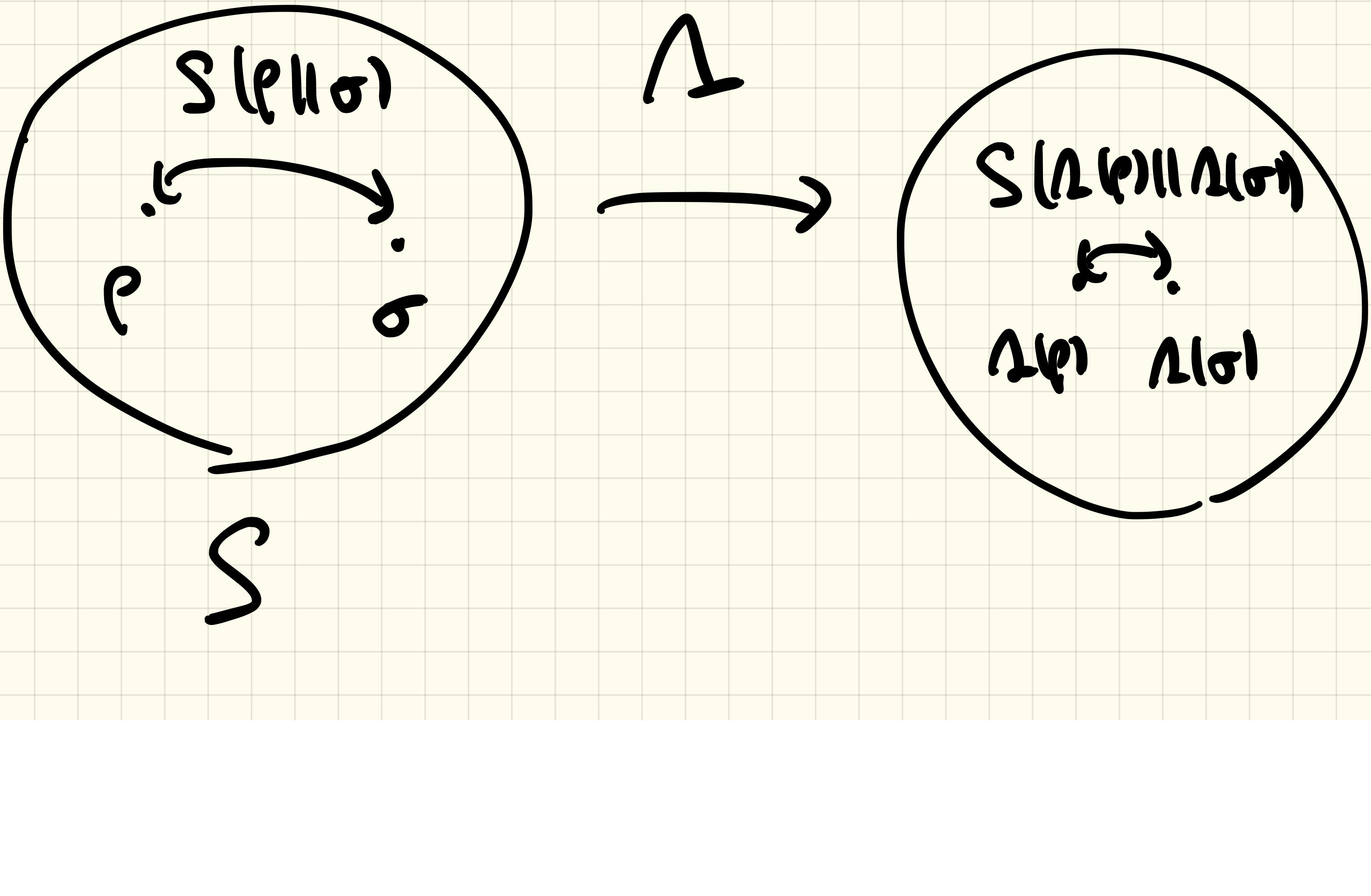
$$= \mathrm{tr}[\rho(\log \rho - \log \sigma)]$$

- Monotonicity

Λ : SEP $\subset$ CPTP

$$S(\Lambda(\rho) \| \Lambda(\sigma)) \leq S(\rho \| \sigma)$$

completely positive trace preserving


S(ρ||σ)
ρ
σ
S
Λ
S(Λ(ρ)||Λ(σ))
Λ(ρ)
Λ(σ)

- Positivity

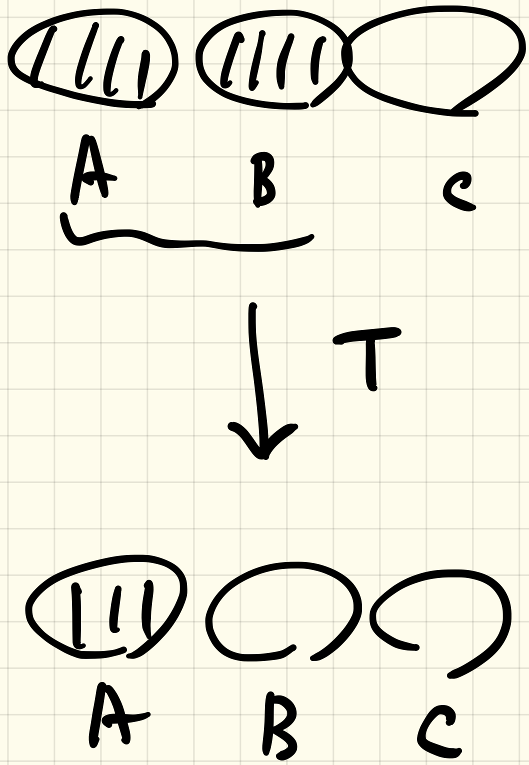
$$S(\rho \| \sigma) \geq 0$$

- $S(\rho \| \sigma) = 0$

$$\Leftrightarrow \quad \rho = \sigma$$

Example of CPTP

· Partial trace



suppose $\rho_{A\cup B}$

$$T = \mathbb{1}_A \otimes \mathrm{Tr}_B$$

$$T : \rho_{A\cup B} \to \rho_A$$

$$\Rightarrow \hat{S}(\rho_A \| \sigma_A) \leq \hat{S}(\rho_{A\cup B} \| \sigma_{A\cup B})$$

Entanglement measures

- Entanglement entropy

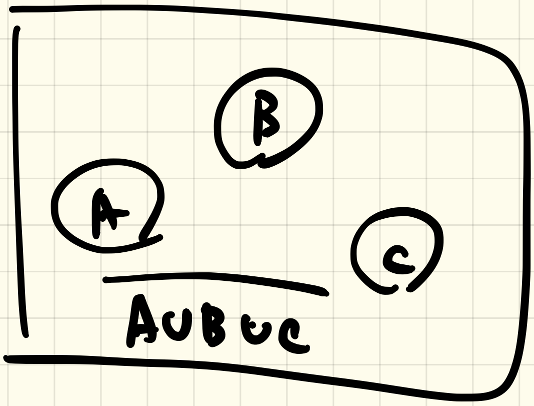
$$\rho_A = \mathrm{Tr}_{\bar{A}}(\rho)$$

$$S_A(\rho) \equiv -\mathrm{tr}_A[\rho_A \log \rho_A]$$

$$= \log d_A - S(\rho_A \,\|\, \underbrace{\mathbb{1}_A / d_A}_{\text{maximally entangled state}})$$

- From the monotonicity of rel. ent.
  (MONO)

Strong subadditivity
SSA



$$S_{A\cup C} + S_{B\cup C} \geq S_A + S_B$$

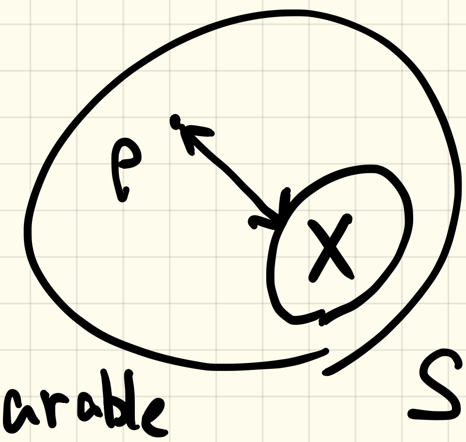
$$\boxed{\text{SSA} \Longleftrightarrow \text{MONO}}$$

E.g. 2) Relative ent. of entanglement

$$E_R^X(\rho) \equiv \inf_{\sigma \in X} S(\rho \| \sigma)$$

In QFT.

[Hollands - Sanders 17]



X : separable state

QIT	Resource thy
separable states	free states
CPTP	free operations
entangled states	resource
entanglement measures	resource measure

Ref)

- M. Nielsen
  "An introduction to majorization and its application to quantum mechanics"
- Plenio - Virmani quant-ph/0504163
- Chitambar - Gour 1806.06107

2. Quantum entanglement in QFT

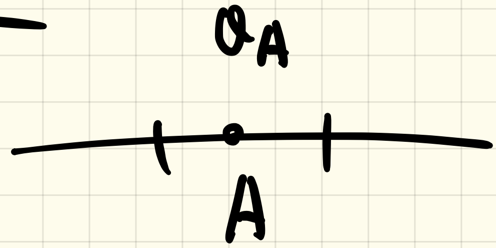
- Algebraic formulation
- Path-integral formulation

⟩ useful for proving inequalities

⟩ practical computation

Algebraic formulation

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$$



- Reduced density matrix

$$\rho_A = \mathrm{Tr}_{\bar{A}}[\rho]$$

$$\mathcal{O}_A = \mathcal{O}_A \otimes \mathbb{1}_{\bar{A}}$$

$$\langle \mathcal{O}_A \rangle = \mathrm{Tr}[\rho(\mathcal{O}_A \otimes \mathbb{1}_{\bar{A}})]$$

$$= \mathrm{Tr}_A[\rho_A \mathcal{O}_A] \quad \text{for any } \mathcal{O}_A$$

$\langle O_A \rangle$ invariant under transformation

$$O_A \rightarrow U O_A U^\dagger \qquad U : \mathcal{H}_A \rightarrow \mathcal{H}_A$$

$$\rho_A \rightarrow U \rho_A U^\dagger$$

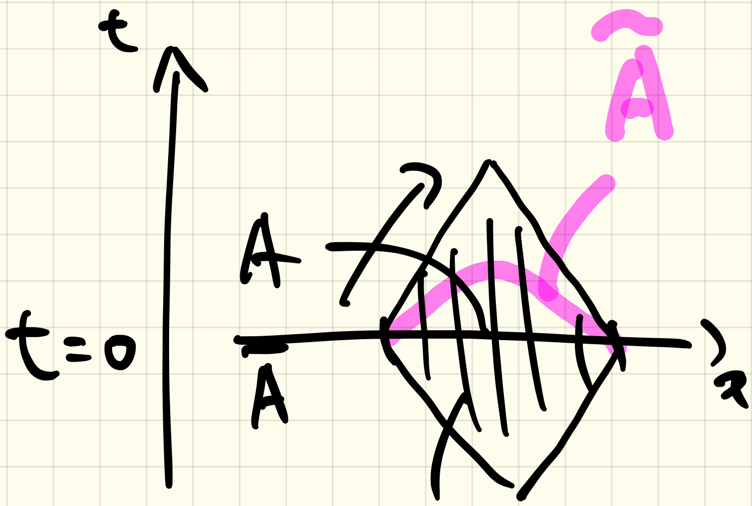
$S_A(\rho)$ is also inv.

U : time evolution on A

If $D[\hat{A}] = D[A]$

then $\exists U$ s.t.

$$\rho_{\hat{A}} = U \rho_A U^\dagger$$

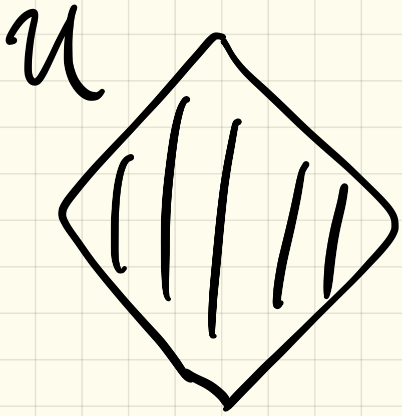


$D[A]$

domain of dependence

AQFT

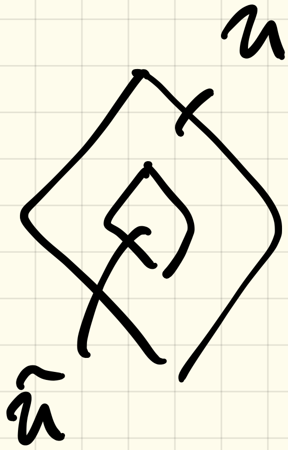
- Operator algebra $\mathcal{A}_{\mathcal{U}}$ associated with a spacetime region $\mathcal{U}$



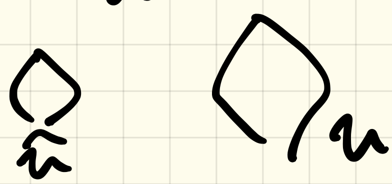
Postulates

1. $\hat{\mathcal{U}} \subset \mathcal{U}$

   $\Rightarrow \mathcal{A}_{\hat{\mathcal{U}}} \subset \mathcal{A}_{\mathcal{U}}$

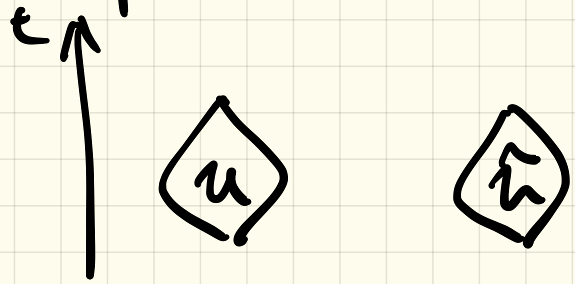


$\rightsquigarrow$ Partial order



2. If $\mathcal{U}, \hat{\mathcal{U}}$ are spacelike

$$[\mathcal{A}_{\mathcal{U}}, \mathcal{A}_{\hat{\mathcal{U}}}] = 0$$



3. Covariance [Haag's textbook]

Classification of $\mathcal{A}$

- Type I (QM)
- Type II
- Type III (QFT)

  III_1

Relative entropy in QFT

For a spacetime region

$\mathcal{U} = D[A]$

$\rho_\psi = |\psi\rangle\langle\psi|$

$\rho_\phi = |\phi\rangle\langle\phi|$

the rel. ent. between $|\psi\rangle, |\phi\rangle \in \mathcal{H}$

$$S_A(\rho_\psi \| \rho_\phi) = -\langle\psi| \log \Delta_{\psi|\phi} |\psi\rangle$$

$$\Delta_{\psi|\phi} = \rho_{\phi,A} \otimes \rho_{\psi,\bar{A}}^{-1}$$ [Araki 76]

(relative modular operator)

$$S_A(\rho_\psi \| \rho_\phi) = S(\rho_{\psi,A} \| \rho_{\phi,A})$$

<u>Proof</u>

$$
\begin{aligned}
S_A(\rho_\psi \| \rho_\phi) &= -\mathrm{tr}\left[\rho_\psi \log \Delta_{\psi|\phi}\right] \\
&= -\mathrm{tr}\left[\rho_\psi \left((\log \rho_{\phi,A} \otimes \mathbb{1}_{\bar{A}} \right.\right. \\
&\qquad\qquad\qquad \left.\left. - \mathbb{1}_A \otimes \log \rho_{\psi,\bar{A}} \right)\right] \\
&= -\mathrm{tr}_A\left[\rho_{\psi,A} \log \rho_{\phi,A}\right] \\
&\quad + \mathrm{tr}_{\bar{A}}\left[\rho_{\psi,\bar{A}} \log \rho_{\psi,\bar{A}}\right]
\end{aligned}
$$

Schmidt decomp. $(\bar{A} \to A)$

$$= -\mathrm{tr}_A[\rho_{\psi,A} \log \rho_{\phi,A}]$$
$$+ \mathrm{tr}_A[\rho_{\psi,A} \log \rho_{\psi,A}]$$
$$= \mathrm{tr}_A[\rho_{\psi,A}(\log \rho_{\psi,A} - \log \rho_{\phi,A})]$$
$$= S(\rho_{\psi,A} \| \rho_{\phi,A}) \quad //$$

This is the case for Type I

The new definition of rel. ent
is valid for Type III (w/ Tomita-Takesaki theory)

Ref)

- Witten 1803.04993