Modular Hamiltonian

$$K_A \equiv -\log \rho_A \leftrightarrow \rho_A = e^{-K_A}$$

$$\rightsquigarrow \Delta_{\psi|\phi} = \rho_{\phi,A} \otimes \rho_{\psi,\bar{A}}^{-1}$$

$$= e^{-K_{\phi,A}} \otimes e^{+K_{\psi,\bar{A}}}$$

$$= e^{-\hat{K}_{\psi|\phi,A}}$$

$$\hat{K}_{\psi|\phi,A} \equiv \underline{K_{\phi,A} \otimes \mathbb{1}_{\bar{A}}} - \mathbb{1}_{A} \otimes K_{\psi,\bar{A}}$$

Full (relative modular Hamiltonian) $= K_{\phi,A} - K_{\psi,\bar{A}}$

Relative entropy using K_A

$$S(\rho \| \sigma) = \mathrm{tr}[\rho(\log \rho - \log \sigma)]$$

$$= \underbrace{\mathrm{tr}[\rho \log \rho]}_{= -S(\rho)} - \underbrace{\mathrm{tr}[\sigma \log \sigma]}_{= +S(\sigma)}$$

$$+ \mathrm{tr}[\sigma \underbrace{\log \sigma}_{= -K_\sigma}] - \mathrm{tr}[\rho \underbrace{\log \sigma}_{= -K_\sigma}]$$

$$= -\Delta S + \Delta \langle K_\sigma \rangle$$

$$S(\rho\|\sigma) = \underline{-\Delta S} + \underline{\Delta\langle K_\sigma\rangle}$$

$$\Delta S \equiv S(\rho) - S(\sigma)$$

$$\Delta\langle K_\sigma\rangle \equiv \mathrm{tr}[\rho K_\sigma] - \mathrm{tr}[\sigma K_\sigma]$$

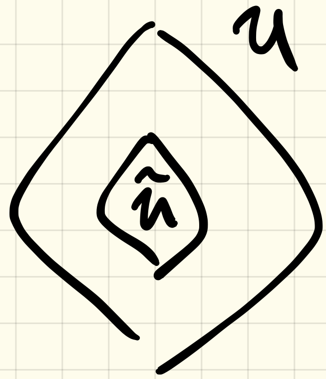
- Positivity : $S(\rho\|\sigma) \geq 0$

$$\Longleftrightarrow \boxed{\Delta S \leq \Delta\langle K_\sigma\rangle}$$

Monotonicity

For $\Lambda : \mathcal{U} \to \hat{\mathcal{U}} = \Lambda(\mathcal{U})$

inclusion



$$S(\Lambda(\rho)\,\|\,\Lambda(\sigma)) \leq \underbrace{S(\rho\,\|\,\sigma)}_{= -\langle\psi|\log\Delta_{\psi|\phi}|\psi\rangle}$$

proved even in Type III

$$\Longleftrightarrow \quad \boxed{\Delta_{\psi|\phi,\hat{\mathcal{U}}} \geq \Delta_{\psi|\phi,\mathcal{U}}}$$

$\Leftrightarrow$ $\hat{K}_{\psi|\phi;\hat{u}} \leq \hat{K}_{\psi|\phi:u}$ ↖ Hermitian

for $\hat{u} \subset u$

$A \geq B$ $\Leftrightarrow$ $A-B \geq 0$ is a non-negative operator

↑ ↗ operators

$\Leftrightarrow$ For $\forall |\psi\rangle$

$\langle\psi| A-B |\psi\rangle \geq 0$

Path-integral formulation

$$\Delta_{\psi|\phi,A} = \rho_{\phi,A} \otimes \rho_{\psi,\bar{A}}^{-1}$$

We will derive the path-int. rep. of the reduced density mat. ρ_A

In QFT

1. $\langle \phi_A | \rho_A | \phi_A' \rangle$

in the path-int. form

2. To compute EE

we use replica trick

Replica trick

Start with Rényi entropy

$$S_{n,A} \equiv \frac{1}{1-n} \log \mathrm{tr}_A(\rho_A^n)$$

$$\rightsquigarrow \quad S_A = \lim_{n\to 1} S_{n,A}$$

Matrix element $\langle \phi_A | \rho_A | \phi_A' \rangle$

We consider $\rho = |\Psi\rangle\langle\Psi|$ ← G.S.
(the ground state is pure)

- For any state $|\psi\rangle$ w/ overlap with $|\Psi\rangle$, $\quad (H|\Psi\rangle = 0)$

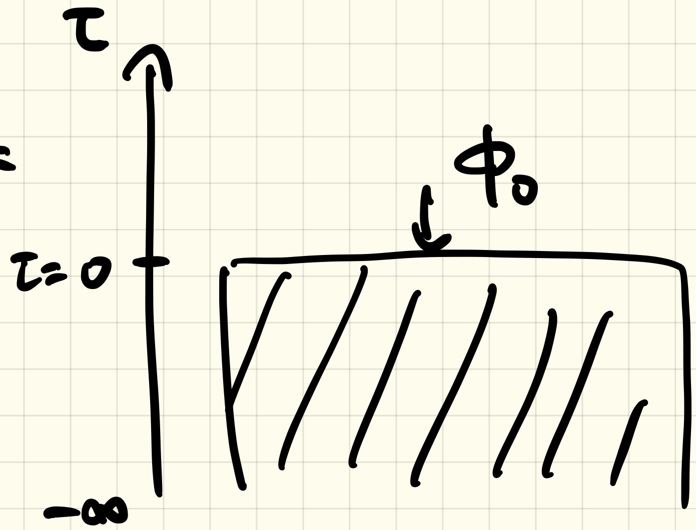
$$|\Psi\rangle = e^{-\tau H}|\psi\rangle \quad \tau \to \infty$$

In path-int.

$$\Psi(\phi_0) = \langle \phi_0 | \Psi \rangle = \int_{\tau=-\infty}^{\tau=0,\ \phi=\phi_0} D\phi\, e^{-I[\phi]}$$

($\Psi(x)$ in QM)

$=$



$|\Psi\rangle$ is not normalized

$$Z = \langle \Psi | \Psi \rangle = \int_{\tau=-\infty}^{\tau=\infty} D\phi \, e^{-I[\phi]}$$

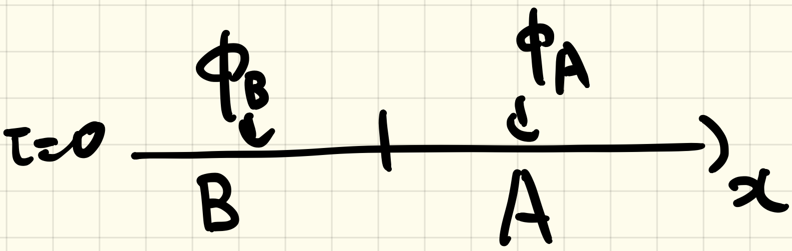
partition function

$I[\Phi]$ Euclidean action

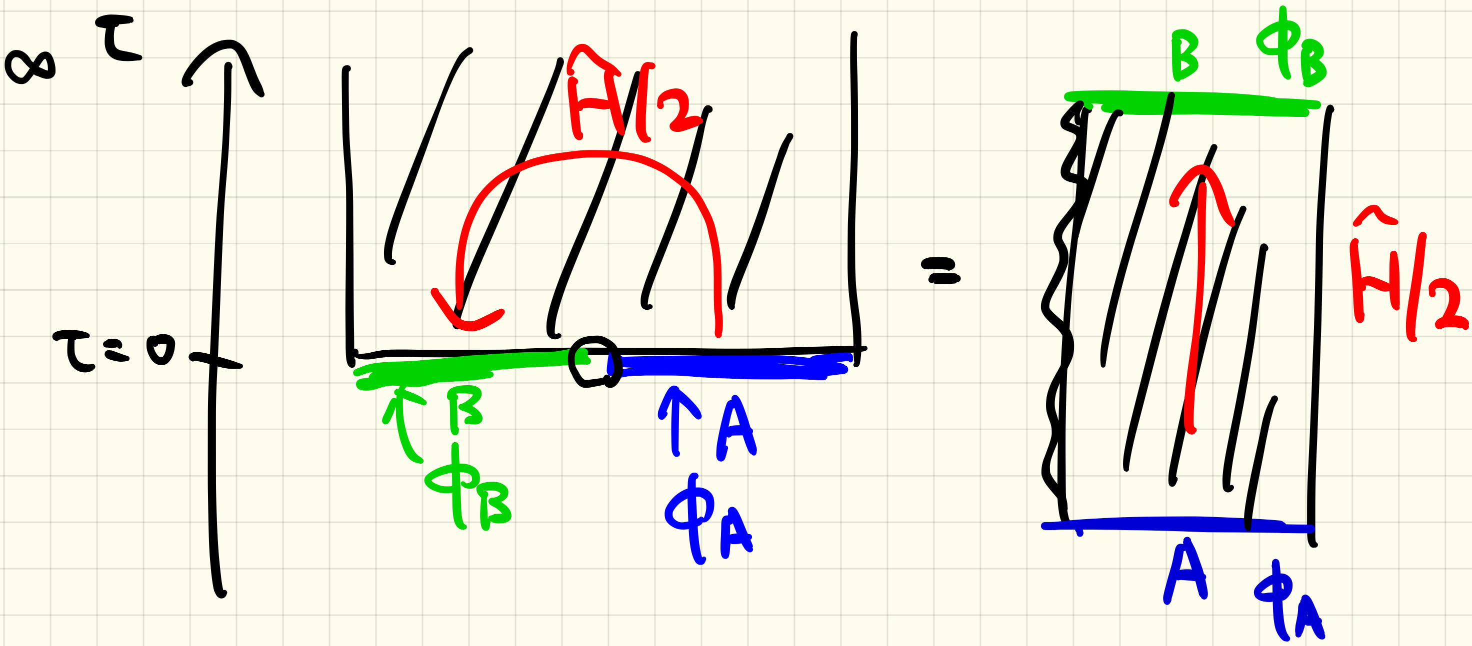
Let's take

$$\phi_0(x) = \begin{cases} \phi_A(x) & x \in A \\ \phi_B(x) & x \in B \end{cases}$$

at $\tau = 0$



$$\langle \Psi | \phi_A \phi_B \rangle$$
$$= \langle \phi_B | e^{-\hat{H}/2} | \phi_A \rangle$$



I will show

$$\tilde{H} = K_A \quad \text{(modular Hamiltonian)}$$

$$+ \text{(const term)}$$

To this end

compute

$$\langle \mathcal{O}_A \rangle = \mathrm{tr}_A[\rho_A \mathcal{O}_A] \qquad \mathcal{O}_A(x) \quad x \in A$$

$$= \boxed{\frac{\langle \tilde{\Psi} | \mathcal{O}_A | \Psi \rangle}{Z}}$$

$$\langle \Psi | \mathcal{O}_A | \Psi \rangle$$

$$= \int D\phi_A \, D\phi_B \, D\phi_A' \, D\phi_B'$$

$$\times \underbrace{\langle \Psi | \phi_A \phi_B \rangle}_{\langle \phi_B | e^{-\hat{H}/2} | \phi_A \rangle} \boxed{\langle \phi_A \phi_B | \mathcal{O}_A | \phi_A' \phi_B' \rangle}$$

$$\times \underbrace{\langle \phi_A' \phi_B' | \Psi \rangle}_{\langle \phi_A' | e^{-\hat{H}/2} | \phi_B' \rangle}$$

$$\boxed{\langle \phi_A \phi_B | \mathcal{O}_A | \phi_A' \phi_B' \rangle} = \underbrace{\langle \phi_B | \phi_B' \rangle}_{= \delta(\phi_B - \phi_B')} \cdot \langle \phi_A | \mathcal{O}_A | \phi_A' \rangle$$

$$= \int D\phi_A \, \langle \phi_A | \mathcal{O}_A \, e^{-\hat{H}} | \phi_A \rangle$$

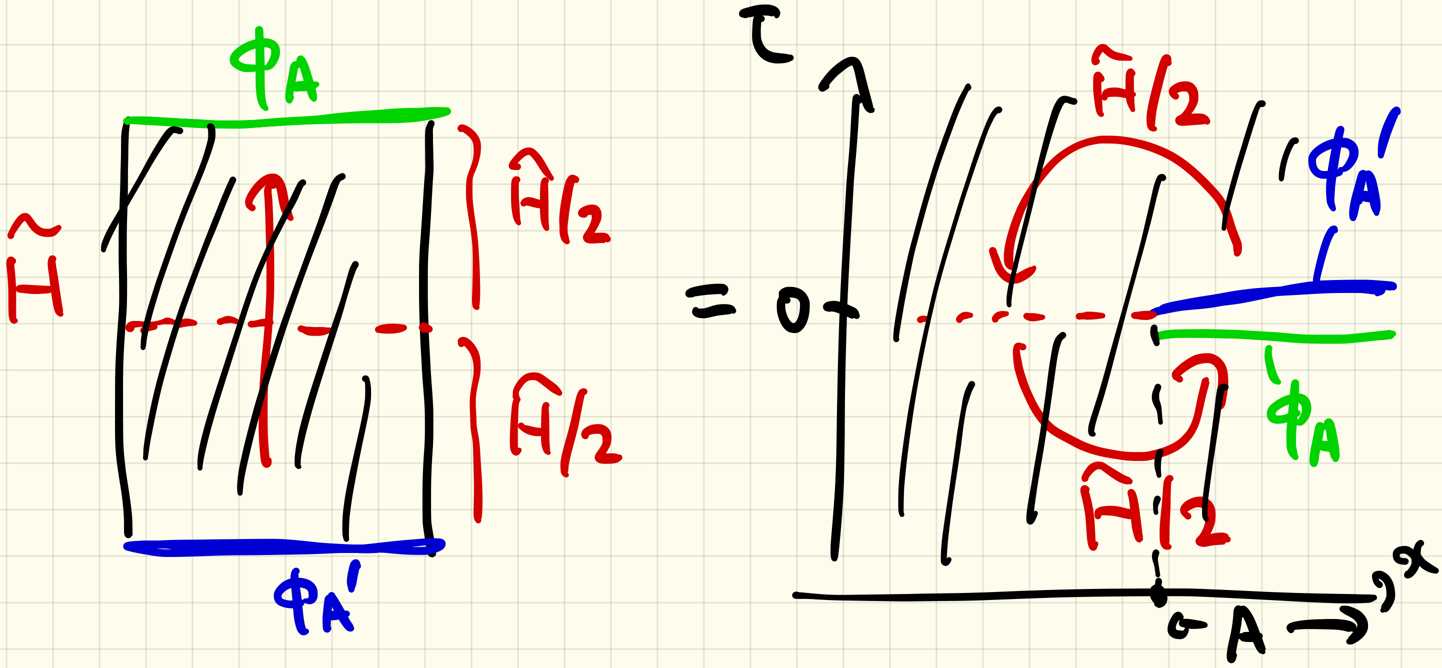
$$= \mathrm{tr}_A \left[e^{-\hat{H}} \mathcal{O}_A \right]$$

Finally. comparing two expressions

$$\boxed{\tilde{H} = K_A - \log Z}$$

$$\langle \phi_A | \rho_A | \phi_A' \rangle$$

$$= \frac{\langle \phi_A | e^{-\tilde{H}} | \phi_A' \rangle}{Z}$$



Calculation of $tr_A[\rho_A^n]$

$$tr_A[\rho_A^n] = tr_A[e^{-nK_A}]$$

$$= \frac{1}{Z^n} \underbrace{tr_A[e^{-n\tilde{H}}]}_{\rotatebox{90}{$\equiv$}\ Z_n}$$

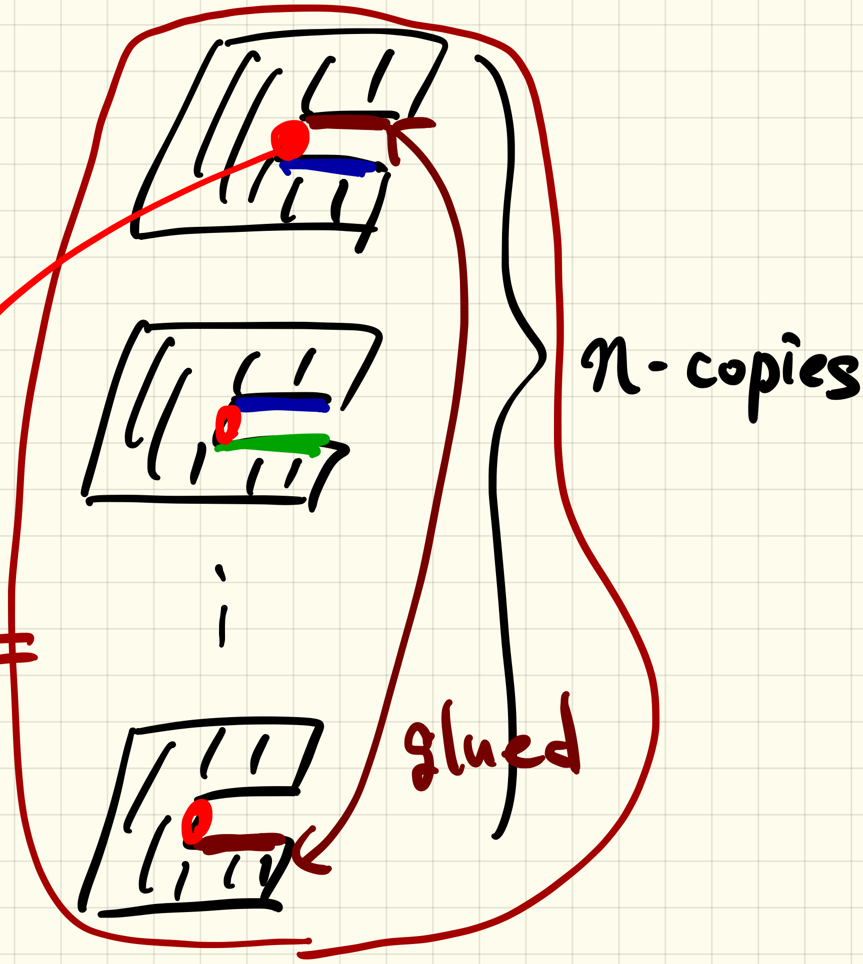
"thermal"

Z_n : partion func.

for the "Hamiltonian" $\tilde{H}$

at $\beta = n$

$$\mathrm{tr}_A[\rho_A^n] = \frac{1}{Z^n}$$



$Z_n =$

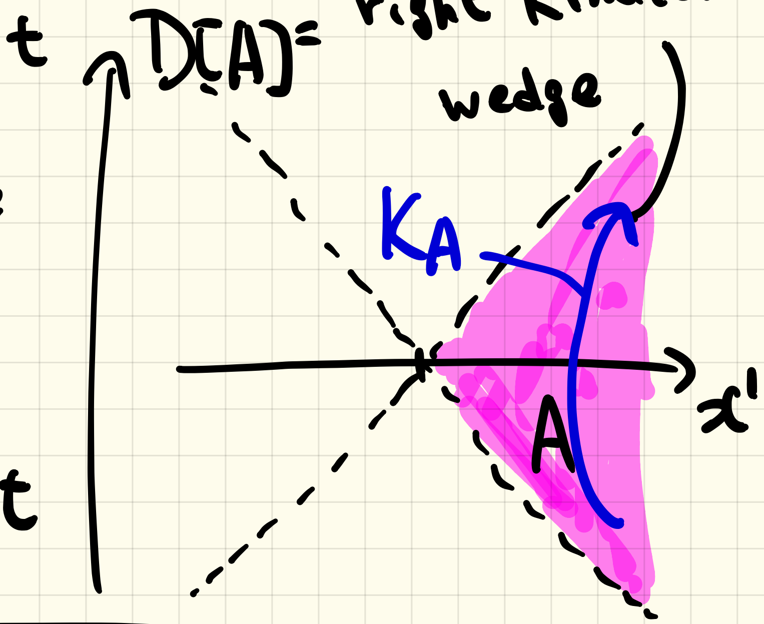
p.f. on the n-fold cover of $\mathbb{R}^d$

w/ singularity at $\Sigma \equiv \partial A$

Bisognano - Wichman theorem

- When
  A is a half-space
  in flat space.

K_A is the Lorentz boost

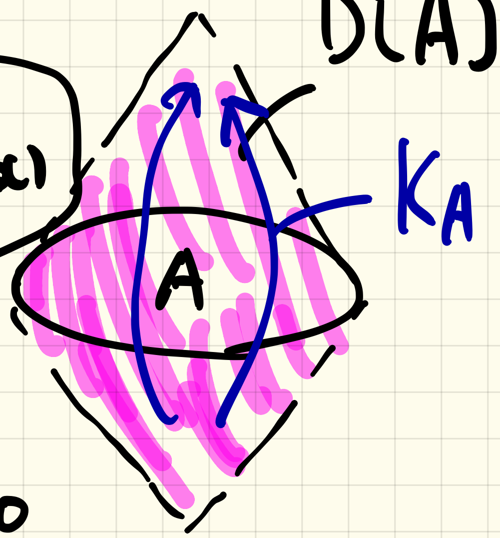


$$\boxed{K_A = 2\pi \int_{t=0,\, x^1 \geq 0} d^{d-1}x \; x^1 \, T_{00}(x)}$$

$\mathbb{R}^{1,d-1}$

- when A is a ball region of rad. R in CFT_d on $\mathbb{R}^{1,d-1}$

$$\boxed{K_A = 2\pi \int_{t=0,\, r\leq R} d^{d-1}x \, \frac{R^2 - r^2}{2R} \, T_{00}(x)}$$



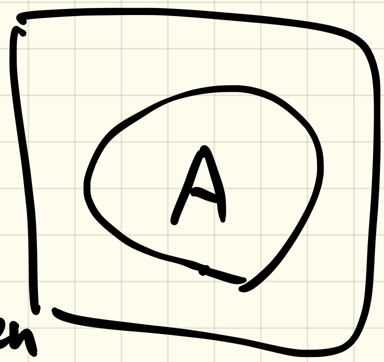
Casini-Huerta-Myers 10

EE in QFT_d has UV divergences

$$S_A = \frac{c_{d-2}}{\epsilon^{d-2}} + \frac{c_{d-4}}{\epsilon^{d-4}} + \cdots$$

$\epsilon \ll 1$

UV cutoff



$$\begin{cases} c_0 \log \frac{l}{\epsilon} & d : \text{even} \\ f & d : \text{odd} \end{cases}$$

In even d, due to the conformal anomaly

there are log divergences

In CFT, C_0 = (central charges)

× (geometric structure)

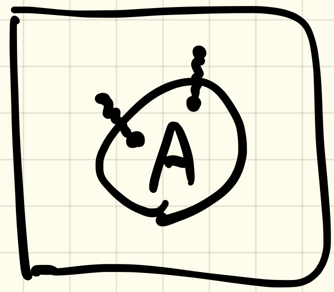
In odd d

CFT γ

universal

Area law

$t=0$



$$C_{d-2} \propto \mathrm{Area}(A) = \mathrm{Vol}(\partial A)$$

holds for local QFT

QFT vacuum is highly entangled

At finite temperature

EE has contributions

from quantum entanglement

and classical thermal

fluctuation

$T \to \infty$

$$S \to S_{\text{Thermal}}$$

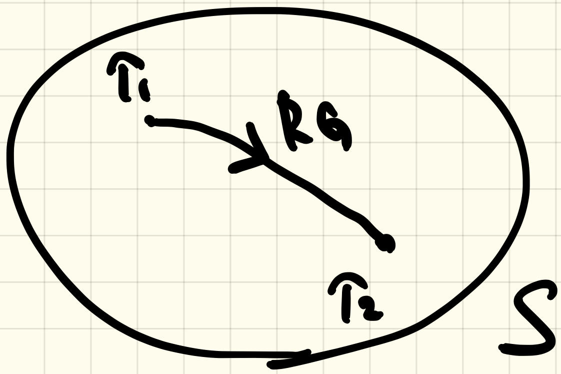
Applications of entanglement measures

- Constraints on RG flows (C-theorem)

- Bounds on energy and entropy

C - thm

$$T_1 \xrightarrow{RG} T_2$$



space of QFTs

A partial order

$$T_1 < T_2$$

Q. When can T_1 flow to T_2 ?

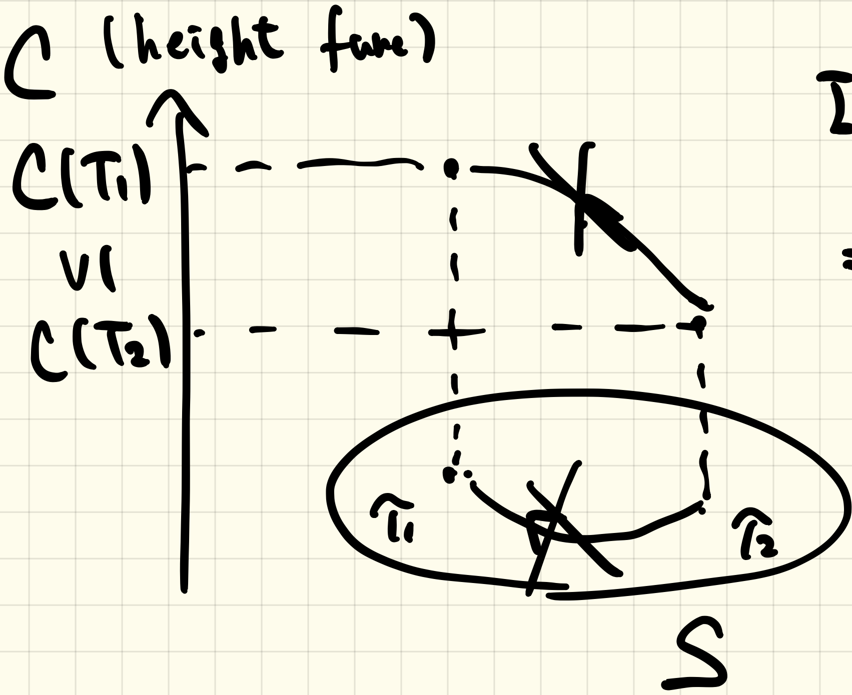
To quantify the "order" bet QFTs

we want to find a C-function

QIT	QFT
LOCC	RG
entangled states	QFTs
entanglement measure	C-func
separable states	trivial thy ?

C-func

$$\tau_1 < \tau_2 \implies C(\tau_1) \geq C(\tau_2)$$



If $C(\tau_1) > C(\tau_2)$

$\implies \tau_2 \not< \tau_1$

Examples of c-thm

- $d=2$ Zamoldchikov's c-thm

  $c_{UV} \geq c_{IR}$ (c : central charge)

- $d=4$ a-thm (Cardy

  $a_{UV} \geq a_{IR}$ Komargodski - Schwimmer)

- In even d CFT

Euler density

$$\langle T_\mu{}^\mu \rangle = \frac{(-1)^{\frac{d}{2}}}{2} A\, E_d + \sum_i B_i\, I_i$$

Weyl invariants

Central charges

- Conjecture

  In even d

$$A_{UV} \geq A_{IR}$$

- In odd d, no conformal anomaly

  but $\exists$ conjecture

  the F-thm (Klebanov et.al)

$$F_{UV} \geq F_{IR} \quad , \quad F \equiv (-1)^{\frac{d-1}{2}} \log Z[S^d]$$

- In general d

(conjecture)

the generalized F-thm

$$\tilde{F}_{UV} \geq \tilde{F}_{IR}, \quad \tilde{F} \equiv \sin\left(\frac{\pi d}{2}\right) \log Z[S^d]$$

$$\tilde{F} = \begin{cases} F & \text{in odd } d \\ \frac{\pi}{2} A & \text{in even } d \end{cases}$$

Entanglement measures as a C-func

Relative entropy of entanglement

$$E^{X}_{rel}(\tau) = \inf_{\tau' \in X} S(\tau \| \tau')$$

$$= S(\tau \| \tau_0)$$

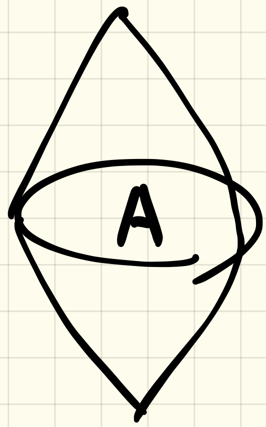
trivial thy

$$= \Delta (K_0) - \Delta S$$

When A is spherical

$$K_0 - \int \# T_{00}$$

$$\to 0 \quad \text{for } \hat{T}_0$$



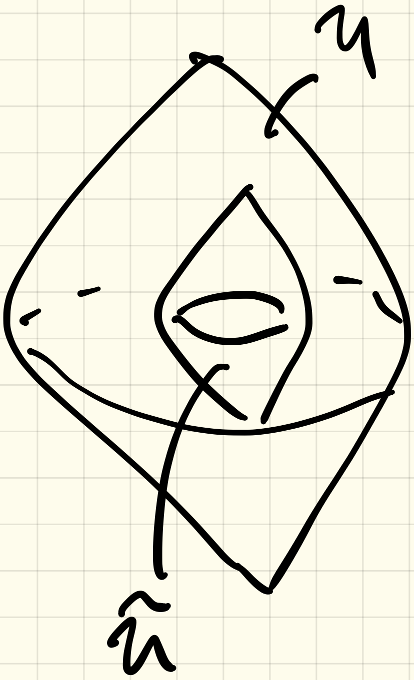
$$\Delta \hat{S} = \hat{S}_{\hat{T}}(R)$$

$$\Rightarrow \boxed{\bar{E}^{x}_{\mathrm{rel}}(\hat{T}) = -\hat{S}_{\hat{T}}(R)}$$

Monotonicity of E^{x}_{rel}

$\hat{\mathcal{U}} \subset \mathcal{U}$

$\Rightarrow \quad \hat{S}_{\hat{\mathcal{U}}}(\hat{\tau} \| \hat{\tau}_0) \leq \hat{S}_{\mathcal{U}}(\hat{\tau} \| \hat{\tau}_0)$



$\rightsquigarrow \quad \hat{S}_{\hat{\tau}}(R') \leq \hat{S}_{\hat{\tau}}(R)$

for $R' \leq R$

$$S_T(R) = \alpha \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + (\text{univ})$$

$$\rightarrow \quad \boxed{\alpha > 0}$$

For CFT w/ sperical region

$$\boxed{S_{CFT}(R) = \log Z[S^d]}$$

(Casini - Huerta - Myers)

up to UV divergences

$$\tilde{F}\Big|_{CFT} = \sin\left(\frac{\pi d}{2}\right) \tilde{S}_{CFT}(R)$$

(up to UV div)

We want to show the monotonicity of $\tilde{F}$ using inequalities of EE

$$\tilde{S}_{CFT}(R) = \frac{C_{d-2}}{\epsilon^{d-2}} + \cdots + \# \tilde{F}$$

Define

$$C(\mathcal{T}, R) \equiv \left[R\partial_R - (d-2)\right] \times \left(S_{\mathcal{T}}(R) - S_{\mathcal{T}_{UV}}(R)\right)$$

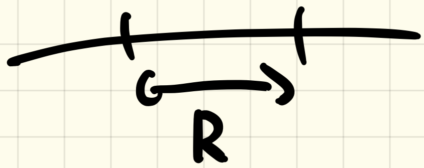
[Casini et. al]

$$\left(C(\mathcal{T}, 0) = 0\right)$$

$$\Rightarrow \quad \partial_R C(\mathcal{T}, R) \leq 0$$

$$\Rightarrow \quad C(\mathcal{T}, R) \leq 0$$

- In $d=2$



$$S_{CFT}(R) = \frac{c}{3}\log\frac{R}{\epsilon} + \cdots$$

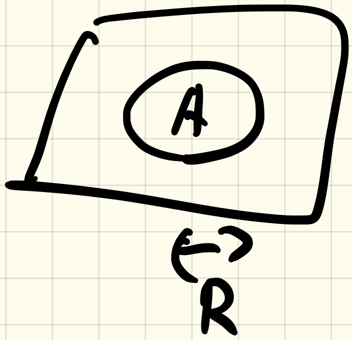
$$C(T_{IR},R) = R\,\partial_R\left(S_{IR}(R) - S_{UV}(R)\right)$$

$$= \frac{c_{IR}-c_{UV}}{3} \leq 0$$

$$\Rightarrow \boxed{c_{IR} \leq c_{UV}}$$

• In $d=3$

$$\hat{S}_{CFT}(R) = \alpha \frac{2\pi R}{\epsilon} - F$$



$$\Rightarrow \quad C(T_{IR}, R) = (R\partial_R - 1) \times \left((\alpha_{IR} - \alpha_{UV}) \frac{2\pi R}{\epsilon} - (F_{IR} - F_{UV}) \right)$$

$$= F_{IR} - F_{UV}$$

$$\leq 0 \qquad \rightsquigarrow \boxed{F_{IR} \leq F_{UV}}$$

- In $d=4$

$$S_{CFT}(R) = \alpha \frac{R^2}{\epsilon^2} - a \log \frac{R}{\epsilon} + \cdots$$

$$\rightarrow \quad C(T_{IR}, R) = (R\partial_R - 2)\left(\# R^2 - (a_{IR} - a_{UV}) \times \log \frac{R}{\epsilon} \right)$$

$$= 2(a_{IR} - a_{UV}) \log \frac{R}{\epsilon} + \text{finite}$$

$$\leq 0 \quad \Rightarrow \quad \boxed{a_{IR} \leq a_{UV}}$$

- In $d \geq 5$

$C(T_{IR}, R)$ has power-law UV divergences

$$S_{CFT}(R) = \alpha \frac{R^{d-2}}{\epsilon^{d-2}} + \beta \frac{R^{d-4}}{\epsilon^{d-2}} + \cdots$$

$$\Rightarrow \quad \boxed{\beta_{IR} \geq \beta_{UV}}$$

- Bounds on energy and entropy

$$S(\rho \| \sigma) \geq 0 \quad \left(\begin{array}{l} \text{Refinement} \\ \text{of Bekenstein} \\ \text{bound} \end{array} \right)$$

$$\Leftrightarrow \quad \Delta S \leq \Delta \langle K_\sigma \rangle$$

Take σ vacuum in CFT

A sphere

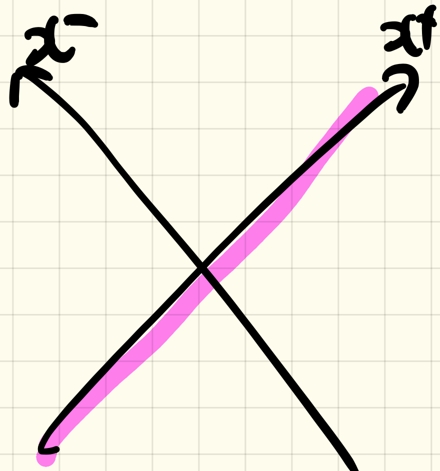
energy for ρ

$$\Delta \langle K_\sigma \rangle \sim \int_{r \in R} \# T_{00} \propto R E$$

- Averaged Null Energy condition (ANEC)

$$\int_{-\infty}^{\infty} dx^+ \langle T_{++} \rangle_\psi \geq 0$$

on $\mathbb{R}^{1,d-1}$



$\rightsquigarrow$ Hofman - Maldacena bound (4d)

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$$

For half space A

$K_A = \int dx\, x^1 T_{00}$

Take B s.t

$D[B] \subset D[A]$

$\Rightarrow \quad \hat{K}_B \leq \hat{K}_A \equiv K_A - K_{\bar{A}}$

[Faulkner et. al 16]