Some new Hecke endomorphism algebras

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This talk is an informal discussion of joint work with Jie Du (UNSW) and Leonard Scott (UVA), and it continues a talk on the same joint work by L. Scott. What follows below are the slides for the talk. The reader is warned that there a probably lots of typos and obscurities, etc.

1. MOTIVATION

Let G be a reductive group (G' simply connected) over a alg. closed field F of positive characteristic p, and let σ be an endomorphism s.t. G_σ is finite. Then G_σ is a finite group of Lie type. One often writes $G(q)$ for G_σ, $q = p^d$. Now let k be alg. closed field, $\text{char}(k) = r > 0$. The rep. theory of kG_σ has two cases

• $p = r$: **defining characteristic** case. Attacked by relating to rational representations of G.

• $p \neq r$: **cross characteristic** case. Subject of this talk (sort of). So from now on assume $p \neq r$.

In case $G = GL_n$, there are close relationships $\mathrm{Rep}kGL_n(q) \leftrightarrow \mathrm{Rep}\ q\text{-Schur alg.} \leftrightarrow \text{Quantum gps for } \mathfrak{gl}_n$ etc. (Dipper-James theory and many others). A key player is the Hecke algebra

$$H := \mathrm{End}_{G(q)}(\mathrm{ind}_{B(q)}^{G(q)} k)$$

which has a nice presentation, canonical basis, ... which all play a role.

2. Generic (Iwahori-Hecke) algebra $\mathcal{H}$ and cell modules

We follow Lusztig's CRM book. Let W, S be a Weyl group with simple reflections S. Let $\ell : W \to \mathbb{Z}$ be the length function and $L : W \to \mathbb{Z}^+$ a **L weight function:** $\ell(xy) = \ell(x) + \ell(y) \implies L(xy) = L(x) + L(y)$. (E.g., $L = \ell$.) Let $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$, and set $t_s = t^{L(s)}$. Then $\mathcal{H}$ is the free $\mathcal{Z}$-algebra with basis T_w, $w \in W$, and relations

$$T_sT_w = \begin{cases} T_{sw} \quad \text{if} \quad \ell(sw) = 1 + \ell(w); \\ T_{sw} + (t_s - t_s^{-1})T_w, \quad \text{if}\, \ell(sw) = -1 + \ell(w). \end{cases}$$

In particular, if $w = s_1s_2\cdots s_r$ is reduced, then

$$T_w = T_{s_1}T_{s_2}\cdots T_{s_r}.$$

The algebra $\mathcal{H}$ has a ring involution $h \mapsto \overline{h}$:

$$\begin{cases} T_w \mapsto T_{w^{-1}}^{-1} \\ t \mapsto t^{-1}. \end{cases}$$

Theorem 1: (Kazhdan-Lusztig) For $w \in W$, there is a unique $c_w \in \mathcal{H}_{\leq 0} = \bigoplus_w \mathcal{Z}_{\leq 0} T_w$ such that
(a) $\overline{c}_w = c_w$;
(b) $c_w \equiv T_w \bmod \mathcal{H}_{\leq 0}$.
Then $\{c_w\}_{w \in W}$ form a $\mathcal{Z}$-basis of $\mathcal{H}$ (the "canonical basis"). The $\mathcal{Z}$-span of the c_w forms an **order** in $\mathcal{H}$.

Now define $w' \longleftarrow_L w$ if $c_{w'}$ "appears" with nonzero coef. in $c_s c_w$ for some $s \in S$, when $c_s c_w$ is written as a linear combination of the $c_{w'}$.

This relation generates a pre-order (reflexive and transitive) $\leq_L$ on W. (In the above example, $w' \leq_L w$.) A corresponding equivalence classes ω are called **left cells** of W. Let Ω be the set of left cells in W. If $y \in \omega$,

$$S(\omega) := \bigoplus_{w;\, w \leq_L y} \mathcal{Z} c_w / \bigoplus_{w;\, w <_L y} \mathcal{Z} c_w$$

is a left $\mathcal{H}$-module, called a **left cell module**, while its dual

$$S_\omega := \mathrm{Hom}_{\mathcal{Z}}(S(\omega), \mathcal{Z})$$

is a **DUAL left cell module**. So $S(\omega) \in \mathcal{H}-\mathrm{mod}$ and $S_\omega \in \mathrm{mod} - \mathcal{H}$.

Similarly, there are right cell modules and two-sided cell modules.

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3. EXACT CATEGORIES

We follow Quillen as arranged by B. Keller. Let $\mathscr{A}$ be an additive category. A pair (i, d) of composable morphisms $i : X \to Y$ and $d : Y \to Z$ in $\mathscr{A}$ is an **exact pair** if $i : X \to Y$ is the kernel of $d : Y \to Z$ and d is the cokernel of i. Thus, we have a "short exact sequence"

$$0 \to X \xrightarrow{i} Y \xrightarrow{d} Z \to 0.$$

in $\mathscr{A}$.

Let $\mathscr{E}$ be a class of exact pairs. If $(i, d) \in \mathscr{E}$, then i (resp., d) is called an inflation (resp., deflation).

The pair $(\mathscr{A}, \mathscr{E})$ is an **EXACT CATEGORY** if the following axioms hold:

0. $1_0 \in \mathrm{Hom}(0, 0)$ is a deflation, where 0 is the zero object in $\mathscr{A}$.

1. The composition of two deflations is a deflation.

2. A diagram

$$\begin{array}{ccc} & & Z' \\ & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[d]{} & Z \end{array}$$

in $\mathscr{A}$ in which d is a deflation, can be completed to a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{d} & Z \end{array}$$

in which d' is a deflation.

2°. The dual of axiom 2 holds

Let $(\mathscr{A}, \mathscr{E})$ be an exact category. For $X, Z \in \mathscr{A}$, let $\mathscr{E}(Z, X)$ be the set of sequences $X \to Y \to Z$ in $\mathscr{E}$. Define the usual equivalence relation $\sim$ on $\mathscr{E}(Z, X)$ by putting

$$(X \to Y \to Z) \sim (X \to Y' \to Z)$$

provided there is a morphism $Y \to Y'$ giving a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \| & & \downarrow & & \| \\ X & \longrightarrow & Y' & \longrightarrow & Z \end{array}$$

The morphism $Y \to Y'$ is necessarily an isomorphism. Let $\mathrm{Ext}^1_{\mathscr{E}}(Z, X) = \mathscr{E}(Z, X)/\sim$. Some familiar properties of Ext are still valid: A short exact sequence in $\mathscr{E}$ leads to 4-term "long" exact sequence (covariant and contravariant), sequence, and sometimes even a 6-term exact sequence.

General references for exact categories are:

[1] T. Bühler, Exact categories, *Expo. Math.* **28** (2010) 1–69.

[2] B. Keller, Chain complexes and stable categories, *Manuscripta Math.* **67** (1990), 379–417.

[3] B. Keller, Appendix to *Trans. Amer. Math.* **351** (1999), 647-682. (Paper by Dräxler, Reiten, Smalo, Solberg.)

4. FILTRATIONS OF MODULES

We make considerable use of natural filtrations on our modules. Here is some notation:

(1) $\mathscr{K}$: Noetherian domain with fraction field K,
(2) H: finite, torsion-free $\mathscr{K}$-algebra with H_K semisimple.
(3) $\mathscr{A}$: additive category of $\mathscr{K}$-finite, torsion-free left (or maybe right) H-modules.
(4) $\Lambda \leftrightarrow \mathrm{Irr}(\text{mod-}H_K)$, $\lambda \longrightarrow E_\lambda$
(5) A fixed function $\mathfrak{h} : \Lambda \longrightarrow \mathbb{Z}^{>0}$. We call $\mathfrak{h}$ a **height function.**

Given $N \in \text{mod}–H_K$, $\mathfrak{h}$ defines a natural increasing finite filtration

$$0 = N^0 \subseteq \cdots \subseteq N^i \subseteq N^{i+1} \subseteq \cdots \subseteq N^t = N,$$

in which N^i is the sum of all submodules of N isomorphic to some E_λ with $\mathfrak{h}(\lambda) \leq i$.

Main Point: Now any $M \in \mathscr{A}$ has an induced filtration:

$$0 = M^0 \subseteq \cdots \subseteq M^i \subseteq M^{i+1} \subseteq \cdots \subseteq M$$

defined by putting

$$M^i := M \cap (M_K)^i, \forall i \geq 0.$$

Observations: (1) M^i, M/M^i, $M^i/M^{i-1} \in \mathscr{A}$.

(2) $(M^i/M^{i-1})_K \cong \bigoplus_{\mathfrak{h}(\lambda)=i} E_\lambda^{\oplus m_\lambda}$, various integers $m_\lambda \geq 0$

(3) A morphism $X \xrightarrow{f} Y$ in $\mathscr{A}$ induces maps

$$\begin{cases} f_i : X^i \to Y^i \\ \overline{f_i} : X^i/X^{i-1} \to Y^i/Y^{i-1}. \end{cases}$$

In addition, if $g : Y \to Z$ is another morphism, then $(gf)_i = g_i f_i$ and $\overline{g_i}\,\overline{f_i} = \overline{g_i f_i}$ for each i.

(4) If

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is an exact sequence in mod–H, the following statements hold for any integer i:

(a) The sequence $0 \to X^i \to Y^i \to Z^i$ is exact in mod–H.

(b) The sequence $0 \to X^h \to Y^h \to Z^h \to 0$ is a short exact sequence in mod–H, $\forall h \leq i$, $\iff$ $0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$ is exact for each $j \leq i$.

Remark: Later when $H = \mathcal{H}$, we always assume that the height function $\mathfrak{h}$ is constant on irreducible modules in the same two-sided cell.

5. EXAMPLES OF EXACT CATEGORIES

We will exhibit some exact categories and then mention some results which are relevant to representation theory. We keep the previous notation, so that $\mathscr{A}$ is an additive category consisting of $\mathscr{K}$-finite and torsion free right H-modules, H_K is semisimple, height function $\mathfrak{h}$, etc.

Example 1: Define $\mathscr{E}$ as follows. A short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ belongs to $\mathscr{E}$ (by definition) if and only if the various sequences (or "layers")

$$0 \to X^i/X^{i-1} \to Y^i/Y^{i-1} \to Z^i/Z^{i-1} \to 0$$

($i \in \mathbb{N}$) are all exact in mod-H.

By observations above, each sequence $0 \to X^i \to Y^i \to Z^i \to 0$ is also exact for any integer i.

Theorem 2: (DPS) The pair $(\mathscr{A}, \mathscr{E})$ is an exact category.

Proof: This can be deduced from the following lemma:

Lemma: (DPS 2017) Let $\mathscr{C}$ be an abelian category. Let $(\mathscr{A}, \mathscr{E})$ be an exact category and let $F : \mathscr{A} \to \mathscr{C}$ be a left $\mathscr{E}$-exact, additive functor. Define $\mathscr{E}$ to be the class of those $0 \to X \to Y \to Z \to 0$ in $\mathscr{E}$ s.t. $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact in $\mathscr{C}$. Then $(\mathscr{A}, \mathscr{E})$ is an exact category.

Proof: Check the axioms. $\square$

Example 2: (DPS 2017) For each integer i, let $\mathscr{S}_i$ be a full, additive subcategory of $\mathscr{A}$ such that if $S \in \mathscr{S}_i$, then S_K is a direct sum of irreducible right H_K-modules having the same height i. Let $\mathscr{S}$ be the set-theoretic union of the $\mathscr{S}_i$. Let $\mathscr{A}(\mathscr{S})$ be the full subcategory of $\mathscr{A}$ above having objects M satisfying $M^j/M^{j-1} \in \mathscr{S}_j$ for all j.

Define $\mathscr{E}(\mathscr{S})$ to be the class of those conflations $X \to Y \to Z$ in $\mathscr{E}$ such that $X, Y, Z \in \mathscr{A}(\mathscr{S})$ and with the additional property that, for each integer i,

$$0 \to X^i/X^{i-1} \to Y^i/Y^{i-1} \to Z^i/Z^{i-1} \to 0$$

is a **SPLIT** short exact sequence in mod–H.

Theorem 3: (DPS 2017) The pair $(\mathscr{A}(\mathscr{S}), \mathscr{E}(\mathscr{S}))$ is an exact category.

Proof: Check the axioms. $\square$

Example 3. Now let $\mathscr{K} = \mathcal{Z} := \mathbb{Z}[t, t^{-1}]$, and let $\mathcal{H}$ be the generic Hecke algebra. For any integer h, let $\mathscr{L}_h$ be the full subcategory of $\mathscr{A}$ (left modules) consisting of modules N which are a finite direct sum of modules which are isomorphic to left cell modules $S(\omega)$ of height $h < 0$ (by convention). Let $\mathscr{L}$ be the set-theoretic union of the $\mathscr{L}_h$, $h \in \mathbb{Z}$. Next, let $\mathscr{A}(\mathscr{L})$ be the full additive subcategory of $\mathcal{H}$-mod with objects M (necessarily in $\mathscr{A}_l$) whose $\mathfrak{h}$-filtration

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_{n-1} \supseteq M_n = 0$$

has the property that each $M_{h-1}/M_h \in \mathscr{L}_{-h}$. We can form the LEFT module version $(\mathscr{A}(\mathscr{L}), \mathscr{E}(\mathscr{L}))$ of $(\mathscr{A}(\mathscr{S}), \mathscr{E}(\mathscr{S}))$ in Example 2. E.g., $\mathscr{E}(\mathscr{L})$ consists of those sequences $(A \to B \to C)$ in $\mathscr{A}(\mathscr{L})$ such that each

$$0 \to A_{h-1}/A_h \to B_{h-1}/B_h \to C_{h-1}/C_h \to 0$$

is a split short exact sequence in $\mathscr{L}_{-h}$. Then (as in Example 2 which worked with right modules)

$$(\mathscr{A}(\mathscr{L}), \mathscr{E}(\mathscr{L}))$$

is an exact category.

If M is a left $\mathcal{H}$-module and $J \subseteq S$, define

$$M^{\mathcal{H}_J} := \{v \in M \mid c_s v = (t_s + t_s^{-1})v, \forall s \in J\}.$$

(Recall that $t_s = t^{L(s)}$ for all $s \in S$.) We call $M^{\mathcal{H}_J}$ the **fixed point subspace** $\mathcal{H}_J$ for its action on M. (Of course, they are not really fixed!)

Lemma: Let $I \subseteq K$ be left ideals in $\mathcal{H}$. Assume that both I and K are spanned by Kazhdan-Lusztig basis elements c_x that is contains. Then, for any $J \subseteq S$, the natural map

$$K^{\mathcal{H}_J} \longrightarrow (K/I)^{\mathcal{H}_J}$$

is surjective.

Remark: Typical (left) ideals spanned by canonical basis elements are the so-called q-permutation modules $\mathcal{H}x_J$ for $J \subset S$, with $x_J := \sum_{w \in W_J} t^{L(w)} T_w$.

Let $\mathscr{E}^\flat(\mathscr{L})$ denote the full subcategory of $\mathscr{E}(\mathscr{L})$ with objects $0 \to M \to N \to P \to 0 \in \mathscr{E}(\mathscr{L})$ such that, for any integer h and any subset J of S, the sequence

$(*)\quad 0 \to (M_h)^{\mathcal{H}_J} \to (N_h)^{\mathcal{H}_J} \to (P_h)^{\mathcal{H}_J} \to 0$

of $\mathcal{H}_J$-fixed points is exact. Note that, for any h, each $0 \to M_h/M_{h-1} \to N_h/N_{h-1} \to P_h/P_{h-1} \to 0$ is split in $\mathcal{H}$-mod, since $(M \to N \to P) \in \mathscr{E}(\mathscr{L})$, but this does not mean that that the short exact sequence $0 \to M_h \to N_h \to P_h \to 0$ is split. In particular, the exactness of above sequence (*) is a non-trivial property.

Let $\mathscr{A}^\flat(\mathscr{L})$ be the full subcategory of $\mathscr{A}(\mathscr{L})$ with objects M having the property that, for any integer h and any $J \subseteq S$, the map

$$(M_{h-1})^{\mathcal{H}_J} \longrightarrow (M_{h-1}/M_h)^{\mathcal{H}_J}$$

is surjective.

Theorem 4: $(\mathscr{A}^\flat(\mathscr{L}), \mathscr{E}^\flat(\mathscr{L}))$ is an exact category.

The following illustrates some of the techniques in proving certain results:

Proposition: Let $M, P \in \mathscr{A}^\flat(\mathscr{L})$, and assume given $(M \to N \to P) \in \mathscr{E}(\mathscr{L})$. Then $N \in \mathscr{A}^\flat(\mathscr{L})$ if and only if $(M \to N \to P) \in \mathscr{E}^\flat(\mathscr{L})$. In particular, the full subcategory $\mathscr{A}^\flat(\mathscr{L})$ of $\mathscr{A}(\mathscr{L})$ is closed under extensions in $(\mathscr{A}(\mathscr{L}), \mathscr{E}(\mathscr{L}))$.

Sketch of proof: Consider the case

$$0 \to M \to N \to P \to 0 \in \mathscr{E}^\flat(\mathscr{L})$$

and we want to prove that $N \in \mathscr{A}^\flat(\mathscr{L})$.

For an integer h and a subset $J \subseteq S$, form the commutative diagram

$$\begin{array}{ccccc}
(M_{h-1}/M_h)^{\mathcal{H}_J} & \longrightarrow & (N_{h-1}/N_h)^{\mathcal{H}_J} & \longrightarrow & (P_{h-1}/P_h)^{\mathcal{H}_J} \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
(M_{h-1})^{\mathcal{H}_J} & \longrightarrow & (N_{h-1})^{\mathcal{H}_J} & \longrightarrow & (P_{h-1})^{\mathcal{H}_J} \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
(M_h)^{\mathcal{H}_J} & \longrightarrow & (N_h)^{\mathcal{H}_J} & \longrightarrow & (P_h)^{\mathcal{H}_J}
\end{array}$$

Since $M, P \in \mathscr{A}^\flat(\mathscr{L})$, columns 1 and 3 are short exact sequences. Since $(M \to N \to P) \in \mathscr{E}^\flat(\mathscr{L})$, the three rows are all short exact sequences as well. (For row 1, it is already, before taking fixed points, a split short exact sequence, because $(M \to N \to P) \in \mathscr{E}(\mathscr{L})$). Thus, it remains exact upon taking $\mathcal{H}_J$-fixed points.) Thus, column 2 is exact as well, using the 3×3 lemma, so that $N \in \mathscr{A}^\flat(\mathscr{L})$, as required.

Another important fact involves the q-permutation modules:

Theorem 5: For any $J \subseteq S$, $\mathcal{H}x_J \in \mathscr{A}^\flat(\mathscr{L})$.

Example 4: Apply the duality functor $(-)^* = \mathrm{Hom}_{\mathcal{Z}}(-, \mathcal{Z})$ to the exact category in Theorem 4 above to get an exact category

$$(\mathscr{A}_\flat(\mathscr{L}^*), \mathscr{E}_\flat(\mathscr{L}^*)).$$

Using the fact that $x_J\mathcal{H} \cong (\mathcal{H}x_J)^*$, we get

Corollary: For any left cell $\omega \in \Omega$, there is a vanishing

$$\mathrm{Ext}^1_{\mathscr{E}_\flat(\mathscr{L}^*)}(S_\omega, x_J\mathcal{H}) = 0.$$

An Abstraction

Maintain the above notation. Writing $\mathscr{S}$ for $\mathscr{L}^*$, the exact category $(\mathscr{A}_\flat(\mathscr{S}), \mathscr{E}_\flat(\mathscr{S}))$ of Example 4 is an exact category $(\mathscr{A}_\flat, \mathscr{E}_\flat)$. It satisfies the following 5 properties:

(1) $\mathscr{A}_\flat$ is object-closed under isomorphisms in $\mathscr{A}(\mathscr{S})$.
(2) All objects of $\mathscr{S}$ are contained in $\mathscr{A}_\flat$.
(3) Let $M \in \mathscr{A}_\flat$ and let h be an integer. Then M^h, M^{h-1} and M^h/M^{h-1} belong to $\mathscr{A}_\flat$, and $(M^{h-1} \to M^h \to M^h/M^{h-1})$ belongs to $\mathscr{E}_\flat$.
(4) For $M, N, P \in \mathscr{A}_\flat$, if $(M \to N \to P) \in \mathscr{E}_\flat$, then $(M^h \to N^h \to P^h) \in \mathscr{E}_\flat$ for all integers h.

There are a number of results about Ext^1 in such an exact category that one wants to prove. The following is a difficult example.

Lemma: Let h be an integer. Let $S \in \mathscr{S}_h$ and $M \in \mathscr{A}_\flat$, and suppose the map

$$\mathrm{Hom}_{\mathscr{A}_\flat}(S, M^h/M^{h-1}) \to \mathrm{Ext}^1_{\mathscr{E}_\flat}(S, M^{h-1})$$

defined by pull-back is surjective. Then $\mathrm{Ext}^1_{\mathscr{E}_\flat}(S, M) = 0$.

In particular, the lemma holds for our exact category $(\mathscr{A}_\flat(\mathscr{S}), \mathscr{E}_\flat(\mathscr{S}))$. Also, we have:

Theorem 5 (DPS 2018) Suppose $(\mathscr{A}_\flat, \mathscr{E}_\flat)$ is a full exact subcategory of $(\mathscr{A}(\mathscr{S}), \mathscr{E}(\mathscr{S})$ satisfying conditions (1)–(4). Assume each $\mathscr{S}_d$ is "finitely generated" by dual left cell modules $S_\omega := S(\omega)^*$. Then, given $M \in \mathscr{A}_\flat$, there exists an object $X = X_M$ in $\mathscr{A}_\flat$ and an inflation $M \xrightarrow{i} X$ in $(\mathscr{A}_\flat, \mathscr{E}_\flat)$ such that $\mathrm{Ext}^1_{\mathscr{E}_\flat}(S, X) = 0$ for all $S \in \mathscr{S}$. Also, if h is chosen minimal with $M^{h-1} \neq 0$, can assumed that the inflation induces an isomorphism $M^{h-1} \cong X^{h-1}$.

Now consider the exact category

$$(\mathscr{A}_\flat(\mathscr{S}), \mathscr{E}_\flat(\mathscr{S}) = (\mathscr{A}_\flat(\mathscr{L}^*), \mathscr{E}_\flat(\mathscr{L}^*)) = (\mathscr{A}^\flat(\mathscr{L}), \mathscr{S}^\flat(\mathscr{L}))^*$$

constructed above. If $M = S_\omega \in \mathscr{L}^*$ for a left cell $\omega \in \Omega$, then write T_ω for X_M defined in the statement of Theorem 5 above.

Let Ω' be the set of all left cells that do **not** contain the longest element $w_{J,0}$ of a parabolic subgroup W_J

of W. Put

$$T := \bigoplus_{J \subseteq S} x_J \mathcal{H} \in \mathscr{A}_\flat(\mathscr{L}^*) \quad \text{and} \quad X := \bigoplus_{\omega \in \Omega'} T_\omega \in \mathscr{A}_\flat(\mathscr{L}^*). \tag{0.0.1}$$

If $\omega \notin \Omega'$, then ω contains the longest word $w_{0,J}$ of a parabolic subgroup W_J of W. Thus, if $\omega \in \Omega \backslash \Omega'$ we set $T_\omega^+ := x_J \mathcal{H}$ for some $J \subseteq S$. Put

$$T^+ := \bigoplus_{\omega \in \Omega} T_\omega^+ = T \oplus X. \tag{0.0.2}$$

The multiplicities of the summands T_ω can be increased (giving Morita equiv. algebras).

It turns out that $A^+ := \mathrm{End}_{\mathcal{H}}(T^+)$ solves the conjecture that Leonard talked about. But we can also simplify matters by using stratifying systems in the exact category setting:

6. STRATIFYING SYSTEMS IN EXACT CATEGORY SETTING

Let A be an algebra which,. is finite (= finitely generated) and projective as a $\mathscr{K}$-module. Assume $\leq$ is a pre-order on Λ (reflexive and transitive). The equivalence classes of Λ defines a poset $\overline{\Lambda}$, and a pre-order map

$$\Lambda \longrightarrow \overline{\Lambda}, \quad x \mapsto \overline{x}.$$

For $\lambda \in \Lambda$, assume that there is given an A-module $\Delta(\lambda)$ which is finite and projective over $\mathscr{K}$. Also, there is given a finite and projective A-module $P(\lambda)$, together with an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$. The following conditions are assumed to hold:

(SS1) For $\lambda, \mu \in \Lambda$,

$$\mathrm{Hom}_A(P(\lambda), \Delta(\mu)) \neq 0 \implies \lambda \leq \mu.$$

(SS2) Every irreducible A-module is a homomorphic image of some $\Delta(\lambda)$.

(SS3) For $\lambda \in \Lambda$, the A-module $P(\lambda)$ has a finite filtration by A-submodules with top section $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ with $\bar{\mu} > \bar{\lambda}$.

When these conditions hold, the data

$$\{\Delta(\lambda), P(\lambda)\}_{\lambda \in \Lambda} \tag{0.0.3}$$

form (by definition) a *stratifying system* for A–mod. (Clearly, this setup works well w.r.t. base change $\mathscr{K} \to \mathscr{K}'$, provided $\mathscr{K}'$ is a Noetherian commutative ring.

Stratifying systems can be constructed in an endomorphism ring setting. Originally, the construction required a difficult Ext^1-vanishing condition. This problem can be partly solved by introducing "designer" exact categories which have smaller Ext^1-groups. The new version was somewhat awkward to use in the Quillen axiom system. The updated version below removes this issue, by replacing all Ext^1-vanishings with existence assertions for needed short exact sequences.

Let R be a finite and projective $\mathscr{K}$-algebra. We provide ingredients in mod–R, including an R-module T with suitable filtrations, which enable the construction of a stratifying system for A–mod, where $A := \mathrm{End}_R(T)$. Clearly, T is naturally a left A-module.

We will construct T as an object in a full subcategory $\mathscr{A}$ of mod–R, with the following assumptions. Let $\mathscr{A}$ be a full additive subcategory of mod–R, which is part of an exact category $(\mathscr{A}, \mathscr{E})$. We assume the exact sequences $(X \to Y \to Z) \in \mathscr{E}$ are among the short exact sequences $0 \to X \to Y \to Z \to 0$ in mod–R. Thus, $(\mathscr{A}, \mathscr{E})$ is a full exact subcategory of mod–R. Assume there is given a collection of objects $S_\lambda, T_\lambda \in \mathscr{A}$ indexed by Λ. Assume that T is a finite and projective $\mathscr{K}$-module. For each $\lambda \in \Lambda$, S_λ is a subobject of T_λ, with the inclusion

$S_\lambda \hookrightarrow T_\lambda$ an inflation. For each $\lambda \in \Lambda$, fix a positive integer m_λ. Define

$$T := \bigoplus_{\lambda \in \Lambda} T_\lambda^{\oplus m_\lambda}.$$

Note that T is an object in $\mathscr{A}$, so that $A = \mathrm{End}_R(T)$.

In particular, $(\mathscr{A}, \mathscr{E})$ is an exact category and $T \in \mathscr{A}$. Let $\diamond$ be the contravariant functor $\mathscr{A} \to \text{mod–}T$, given by $M^\diamond := \mathrm{Hom}_{\mathscr{A}}(M, T)$. Consider the

Stratification Hypothesis (Exact category edition):

(1) For $\lambda \in \Lambda$, there is an increasing filtration of T_λ:

$$0 = T_\lambda^{-1} \subseteq T_\lambda^0 \subseteq \cdots \subseteq T_\lambda^{l(\lambda)} = T_\lambda$$

in which each inclusion $T_\lambda^{i-1} \subseteq T_\lambda^i$ is an inflation. In addition, $T_\lambda^0 \cong S_\lambda$, and, for $i > 0$, the "section" $T_\lambda^i / T_\lambda^{i-1}$ is a direct sum of various S_μ, $\mu \in \Lambda$ and $\mu > \lambda$ (repetitions allowed).

(2) For $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_{\mathscr{A}}(S_\mu, T_\lambda) \neq 0 \implies \lambda \leq \mu$.

(3) For all $\lambda \in \Lambda$ and integer $i \geq 0$, the natural sequence

$$0 \to (T_\lambda^i / T_\lambda^{i-1})^\diamond \longrightarrow T_\lambda^{i,\diamond} \longrightarrow T_\lambda^{i-1,\diamond} \to 0$$

of A-modules is exact.

Theorem 6: (DPS) Assume that the Stratification Hypothesis holds. For $\lambda \in \Lambda$, put

$$\begin{cases} \Delta(\lambda) := \mathrm{Hom}_R(S_\lambda, T) = \mathrm{Hom}_{\mathscr{A}}(S_\lambda, T) \in A\text{–mod}; \\ P(\lambda) := \mathrm{Hom}_R(T_\lambda, T) \in A\text{–mod}. \end{cases} \tag{0.0.4}$$

Assume that each $\Delta(\lambda)$ is projective over $\mathscr{K}$. Then $\{\Delta(\lambda), P(\lambda)\}_{\lambda\in\Lambda}$ is a stratifying system for A–mod.

Remark: If $\mathscr{K} = \mathcal{Z} := \mathbb{Z}[t, t^[- 1]]$, then $\Delta(\lambda)$ is projective by Auslander-Goldman, and then it is free by Swan.

8. FINAL COMMENTS

Put $\mathcal{Z}^\natural := \mathcal{S}^{-1}\mathcal{Z}$, the localization of $\mathcal{Z}$ at the multiplicative set $\mathcal{S}$ generated by the bad primes of W. Given a $\mathcal{Z}$-algebra B, let $B^\natural = \mathcal{Z}^\natural \otimes_{\mathcal{Z}} B$.

Theorem 7: In the notation above, $A^{+\natural}$ is quasi-hereditary.

Remarks: (a) (DPS) The original DPS conjecture considered all rank 2 examples and determined that Theorem 7 is not always true if $A^{+\natural}$ is replaced by A^+.

(b) Is it necessary to invert all bad primes for Theorem 7 to be true?