Worst Unstable Points of a Hilbert Scheme

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"state polytope of a point"

Everything is over $k = \bar{k}$ of char. 0.

T: alg. torus, V: T-repn

$$\Rightarrow V = \bigoplus_{\chi \in X(T)} V_\chi \quad \text{where } V_\chi := \{ v \in V \mid t.v = \chi(t) v \}$$

i.e, $\forall v \in V \; \forall \chi \in X(T) \; \exists v_\chi \in V_\chi$ s.t. $v = \sum_{\chi \in X(T)} v_\chi$

$$\Xi_v := \{ \chi \in X(T) \mid v_\chi \neq 0 \}. \quad (\text{state of } v)$$

$$\Delta_v := \operatorname{Conv} \Xi_v \subseteq X(T) \otimes_{\mathbb{Z}} \mathbb{R}$$

(state polytope of v)

From now on, $V = \bigwedge^{Q(d)} k[x_0, \cdots, x_r]_d$

for some Hilbert polynomial Q of a graded ideal

$I \subseteq k[x_0, \cdots, x_r]$ and $d \geq$ (gotzmann # corresponding to Q)

$G := SL_{r+1}(k)$

$T :=$ (group of diagonal matrices in $SL_{r+1}(k)$)

Theorem (Ian Morrison, David Swinarski (2011))

A Hilbert point $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ is semi-stable under the canonical $SL_{r+1}(k)$ action and linearization induced by the Plücker coordinate

$$\mathrm{Hilb}^P(\mathbb{P}^r_k) \longrightarrow \mathbb{P}\left(\bigwedge^{Q(d)} k[x_0, \cdots, x_r]\right)$$

if and only if

$$\Delta_{g.x} \ni \xi_{Q,d} := \frac{d\,Q(d)}{r+1}(\underbrace{1, 1, \cdots, 1}_{r+1})$$

for all $g \in SL_{r+1}(k)$.

Theorem [George R. Kempf, 1978]

If $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ is unstable under the above setting, the margitude of instability of x can be measured by the number

$$\delta_{x,d} := \max_{g \in SL_{r+1}(k)} \mathrm{dist}(\xi_{a,d}, \Delta_{g \cdot x}) \quad . \qquad (\text{Kempf index})$$

There is $g_{max} \in SL_{r+1}(k)$ which attains such a maximum.

Example) $\mathrm{Hilb}^n(\mathbb{P}^1_k) \hookrightarrow \mathbb{P}(k[x_0, x_1]_n)$

(Gotzmann # $= n$, $d := n$)

$$\mathrm{dist}(\xi_{a,1}, \Delta_x) = \begin{cases} 0 & (\mathrm{mult}_{[1:0]} Z_x,\ \mathrm{mult}_{[0:1]} Z_x \leq \frac{n}{2}) \\ \max\left\{ \mathrm{mult}_{[1:0]} Z_x - \frac{n}{2},\ \mathrm{mult}_{[0:1]} Z_x - \frac{n}{2} \right\} & (\text{otherwise}) \end{cases}$$

$$\therefore\ \delta_{x,d} = \begin{cases} 0 & (\text{if } \nexists\, p \in Z_x \text{ s.t. } \mathrm{mult}_p Z_x > \frac{n}{2}) \\ \left[\max\limits_{p \in Z_x} \mathrm{mult}_p Z_x\right] - \frac{n}{2} & (\text{otherwise}) \end{cases}$$

<u>**Definition**</u> $x \in \mathrm{Hilb}^P(\mathbb{P}^r_k)$ is asymptotically worst unstable if

$$\delta_{x,d} = \max_{y \in \mathrm{Hilb}^P(\mathbb{P}^r_k)} \delta_{y,d}$$

for all $d \gg 0$.

Observation "worst unstable scheme is a monomial scheme."

Theorem [David Bayer, Ian Morrison, 1988]

Let I be a homogeneous ideal of $S = k[x_0, \dots, x_r]$ and P be the Hilbert polynomial of S/I. Suppose that d is at least the Gotzmann #.

For each vertex v of $\Delta[\bigwedge^{Q(d)} I_d]$, there is a monomial ordering $\preceq$ such that

$$\{v\} = \Delta\left(\bigwedge^{Q(d)} (\mathrm{in}_{\preceq} I)_d\right).$$

Our goal: classify all sets of monomials

$M \subseteq k[x_0, \dots, x_r]_d$ of cardinality $P(d)$

maximalizing

$$\sum_{i=0}^{r} c_i^2$$

where

$$\prod_{m \in M} m = \prod_{i=0}^{r} x_i^{c_i}$$

for sufficiently large d.

Guess: If M maximalizes $\sum_{i=0}^{r} c_i^2$,

then M also maximalizes

$\max_{0 \le i \le r} c_i$.

Claim : Our guess is true when P is a constant polynomial.

Let ℓ be the number satisfying

$$\sum_{0 \le i < \ell} \binom{r+i-1}{r-1} < p \le \sum_{0 \le i \le \ell} \binom{r+i-1}{r-1}$$

and let

$$e := \left[\sum_{0 \le i < \ell} i \binom{r+i-1}{r-1}\right] + \left[p - \sum_{0 \le i < \ell} \binom{r+i-1}{r-1}\right] \cdot \ell .$$

We can see that

$$\max_{0 \le i \le r} c_i \le dp - e$$

and such inequality is sharp.

when $c := \max_{0 \le i \le r} c_i = dp - e$,

$$\sum_{i=0}^{r} c_i^2 \ge (dp-e)^2 + \frac{e^2}{r}$$

Also,

$$\sum_{i=0}^{r} c_i^2 \le c^2 + (dp-c)^2.$$

for general c.

$\therefore$ Our Claim is true when

$$(dp-e)^2 + \frac{e^2}{r} > c^2 + (dp-c)^2 \quad \text{———} \quad (*)$$

for all integer c s.t.

$$\frac{dp}{r+1} \le c \le dp-e-1 \text{ when } d \gg 0$$

Our claim is true when

$$\frac{dp - \sqrt{d^2p^2 - 4dpe + \frac{2(r+1)}{r}e^2}}{2} < c < \frac{dp + \sqrt{d^2p^2 - 4dpe + \frac{2(r+1)}{r}e^2}}{2}$$

$\underset{A(d)}{\shortparallel}$ $\underset{B(d)}{\shortparallel}$

for all integer c s.t. $\frac{dp}{r+1} \le c \le dp-e-1$

when $d \gg 0$.

We can check that $\frac{dp}{r+1} \ge A(d)$ when $d \gg 0$

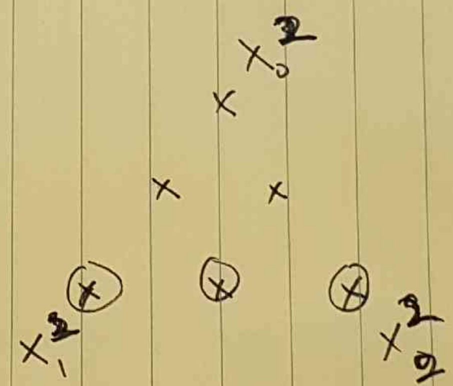
and $\lim_{d\to\infty} B(d) - (dp - e) = 0$

$\therefore$ Claimed statement holds.

Example "$\mathrm{Hilb}^3(\mathbb{P}^2_k)$"

There is a unique worst unstable orbit, which contains $[x_1^2 \wedge x_1 x_2 \wedge x_2^2] \in \mathbb{P}(\bigwedge^{Q(2)} k[x_0, x_1, x_2]_2)$

$(d = 2)$



Also, if $I \subseteq k[x_0, \dots, x_r]$ defines an asymptotically worst unstable 0-dimensional scheme Z, then

$x_i^d \in I$ for all $0 < i \leq r$, under suitable choice of coordinate.

$\Rightarrow$ The set of all closed points in Z

is $\{[1:0:\cdots:0]\}$

$\Rightarrow |Z(k)| = 1$.

__Cor.__ Asymptotically worst unstable 0-dimensional scheme has a unique closed point.

Theorem. Let $P(t) = \binom{r+t}{r} - \binom{r+t-d}{r} + p$

for some $d, p \in \mathbb{N}$.

There is a unique worst unstable orbit in $\mathrm{Hilb}^P(\mathbb{P}^r_k)$, which contains

$$\left[\bigwedge_{m \in M}^{Q(d)} m \right] \in \mathbb{P}\left(\bigwedge^{Q(d)} k[x_0, \cdots, x_r]_d \right)$$

where M is a set of monomials in $k[x_0, \cdots, x_r]_d$

satisfying
$$\begin{cases} \min\limits_{0 \le i \le r} c_i \quad \text{is minimal} \quad (d = 0) \\ \\ \max\limits_{0 \le i \le r} c_i \quad \text{is maximal} \quad (d \neq 0) \end{cases}$$

$$\left(\prod_{m \in M} m = \prod_{i=0}^{r} x_i^{c_i} \right)$$

Theorem. When $P(t) = \binom{t+r}{r} - \binom{t+r-\gamma}{r} + p$ for some $p, \gamma \in \mathbb{N}$, The Castelnuovo - Mumford regularity of the ideal sheaf corresponding to an arbitrary asymptotically worst unstable closed subscheme of $\mathbb{P}^r_k$ (whose Hilbert polynomial is P)

is $\ell_p + \gamma + 1$, where

ℓ_p is an integer satisfying

$$\begin{cases} \displaystyle\sum_{0 \le i < \ell_p} \binom{r+i-1}{r-1} < p \le \sum_{0 \le i \le \ell_p} \binom{r+i-1}{r-1} & \text{if } \gamma = 0,\ p \neq 0 \\[2ex] \displaystyle\sum_{0 \le i < \ell_p} \binom{r+i-2}{r-2} < p \le \sum_{0 \le i \le \ell_p} \binom{r+i-2}{r-2} & \text{if } \gamma \neq 0,\ p \neq 0. \end{cases}$$

(The Gotzmann # corresponding to P) $= \gamma + p$.