New perspective on moduli of commuting nilpotents

Gergely Bérczi
Centre for Quantum Geometry of Moduli Spaces, Aarhus
joint work with W. Haboush & D. Hyeon

CiGaR 2018, Jeju

Players of this talk

- **Commuting nilpotents:** regular q-tuples $(A_1,\dots,A_q)$ of commuting nilpotent $n\times n$ matrices:

$$N(q,n)^{reg}=\{(A_1,\dots,A_q): A_i^n=0, [A_i,A_j]=0, A_s^{n-1}\neq 0 \text{ for some } 1\le s\le q\}$$

$$\mathcal{N}_{q,n}=N(q,n)/\mathrm{GL}(n) \text{ orbit space under simultaneous conjugation}$$

- **Hilbert scheme of n points on $\mathbb{C}^q$**
$\mathrm{Hilb}^n(\mathbb{C}^q)=\{I\subset R=\mathbb{C}[x_1,\dots,x_q]:\dim(R/I)=n\}=$
$=\{\xi\subset\mathbb{C}^q:\dim H^0(\mathcal{O}_\xi)=n\}$
$\mathrm{Hilb}^n_0(\mathbb{C}^q)=\{I\subset\mathfrak{m}=(x_1,\dots,x_q):\dim(\mathfrak{m}/I)=n\}$ punctual Hilbert scheme
$\mathrm{Curv}^n(\mathbb{C}^q)=\{\xi\in\mathrm{Hilb}^n_0(\mathbb{C}^q):\xi\subset\mathcal{C}_0$ for some smooth curve $\mathcal{C}\subset X\}=$
$\{\xi\in\mathrm{Hilb}^n_0(\mathbb{C}^q):\mathcal{O}_\xi\simeq\mathbb{C}[z]/z^n\}$ curvilinear locus.
Note: $\mathrm{Curv}^n(\mathbb{C}^q)$ is open, its closure is a component of the punctual Hilbert scheme. For $q=2$ $\overline{\mathrm{Curv}^n(\mathbb{C}^q)}=\mathrm{Hilb}^n_0(\mathbb{C}^q)$ (Briançon)

- **Jets of curves in $\mathbb{C}^q$**

$$J_n(1,q)=\{n\text{-jets } f_{[n]}:(\mathbb{C},0)\to(\mathbb{C}^q,0)\}=\{(f',f'',\dots,f^{[n]}):f'\neq 0\}$$

Note: Composition gives

$$J_n(p,q)\times J_n(q,r)\to J_n(p,r)$$

In particular: $J_n(1,1)$ is an n-dimensional reparametrisation group acting on $J_n(1,q)$, and

$$J_n(1,q)/J_n(1,1)= \text{ moduli of } n\text{-jets of curves in } \mathbb{C}^q$$

Let's write down the action

$$J_n(1,1) \times J_n(1,q) \to J_n(1,q)$$

$$(\alpha, f) \mapsto f \circ \alpha$$

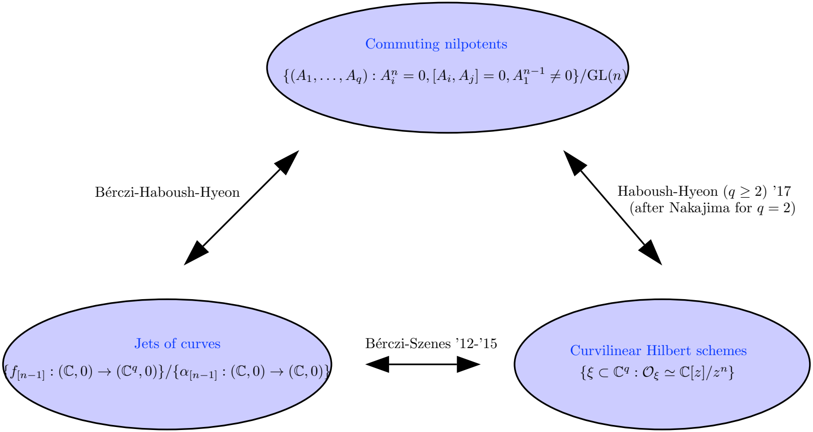
$f(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^n}{n!}f^{(n)}(0) \in J_n(1,q)$
$\alpha(z) = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_k z^n \in J_n(1,1)$
Then

$$f \circ \alpha(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots$$

$$= (f', \ldots, f^{(n)}/n!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{n-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{n-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^n \end{pmatrix} = \mathbb{C}^* \rtimes \mathbf{U}_n$$

Acting group is **non-reductive**! Reductive GIT (finite generation, Haboush's theorem) does not work...even in char 0.
Plays crucial role in hyperbolicity problems (Green-Griffiths and Kobayashi conjecture), enumerative geometry (curve counting) and singularity theory (Thom polynomials of singularities).

Moduli spaces...



$N(q,n)^{reg}/\mathrm{GL}(n) = \mathrm{Curv}^n(\mathbb{C}^q) = J_{n-1}(1,q)/J_{n-1}(1,1)$
quasi-projective

Bérczi, Szenes '12-'17

Demailly '90s
(Siu, Diverio, Rousseau, Merker, Brotbek,...)

Bérczi, Doran, Hawes, Kirwan
('10-'16)

Curvilinear component of $\mathrm{Hilb}_0^n(\mathbb{C}^q)$

$\mathrm{CHilb}^n(\mathbb{C}^q) = \overline{\mathrm{Curv}^n(\mathbb{C}^q)}$

- irreducible
- highly singular
- plays crucial role in enumerative geometry
- tautological integral formulas

Demailly-Semple monster tower (invariant jet differentials)

- smooth iterated bundle
- not a vector bundle
- plays crucial role in hyperbolicity problems

Non-reductive GIT quotient

$\mathcal{N}(q,n)^{\mathrm{nr}} = \mathrm{Proj}(\mathbb{C}[J_{n-1}(1,q)]^{J_{n-1}(1,1)})$

- smooth projective
- symplectic quotient
- coh. intersection theory (residue pairings, Betti numbers)

No morphism, need to blow up the DS tower, hard (in progress with D. Brotbek)

No morphism, need to blow up the NRGIT quotient, hard (work in progress)

- Need to blow-up the DS tower along Wronskian ideal–subspace arrangement, reminds us to wonderful compactification of C. de Concini and C. Procesi.
- Need to blow up the NRGIT compactification along subvarieties indexed by monomial ideals, i.e torus fixed-points on $\mathrm{Hilb}^n(\mathbb{C}^q)$.

- $\mathrm{Curv}^n(\mathbb{C}^q) = \{\xi \in \mathrm{Hilb}_0^n(\mathbb{C}^q) : \xi \subset \mathcal{C}_0 \subset \mathbb{C}^q\} = \{\xi : \mathcal{O}_\xi \simeq \mathbb{C}[z]/z^k\}$
- $\mathrm{CHilb}^n(\mathbb{C}^q) = \overline{\mathrm{Curv}}^n(\mathbb{C}^q)$

This is a singular irreducible projective variety of dimension $(q-1)(n-1)+q$.
Curvilinear subschemes to jets: $\xi \in \mathrm{Curv}^{n+1}(\mathbb{C}^q) \rightsquigarrow \xi \subset \mathcal{C}_0 \subset (\mathbb{C}^q, 0)$ smooth curve germ $\rightsquigarrow f_\xi : (\mathbb{C}, 0) \to (\mathbb{C}^q, 0)$ n-jet of germ parametrising $\mathcal{C}_0$. f_ξ is determined up to polynomial reparametrisation germs $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Hence

$$\mathrm{Curv}^{n+1}(\mathbb{C}^q) \to J_n(1,q)/J_n(1,1), \quad \xi \mapsto f_\xi$$

is well-defined.
Jets to curvilinear subschemes: If $g \in J_n(q,1)$ then $g \circ f_\xi = 0 \Rightarrow g \circ (f_\xi \circ \phi) = 0$

$$(\mathbb{C},0) \xrightarrow{\phi} (\mathbb{C},0) \xrightarrow{f_\xi} (\mathbb{C}^q,0) \xrightarrow{g} (\mathbb{C},0)$$

In other words: $Ann(f_\xi) \subset J_n(q,1)$ is invariant under $J_n(1,1)$. The map $f_\xi \mapsto Ann(f_\xi)$ defines an embedding

$$\rho : J_n(1,q)/J_n(1,1) \hookrightarrow \mathrm{Grass}(\mathrm{codim} = n+1, J_n(q,1))$$

Theorem [B-Szenes 2012, B 2015]

1. The dual of ρ can be written as

$$\rho : J_n(1,q)/J_n(1,1) \hookrightarrow \mathrm{Grass}(n+1, J_n(q,1)^*)$$

$$(f', \dots, f^{(n)}) \mapsto f' \wedge (f'' + (f')^2) \wedge (f''' + 2f'f'' + (f')^3) \wedge \dots \wedge \Big(\sum_{a_1+a_2+\dots+a_i=n} f^{(a_1)} f^{(a_2)} \dots f^{(a_i)}\Big).$$

2. The closure of the image is the curvilinear Hilbert scheme $\mathrm{CHilb}^{n+1}(\mathbb{C}^q)$.

Nakajima: Let $W_{q,n} = M_n(k)^{\oplus q} \oplus \hom_k(k, k^n)$ acted on by $G = \mathrm{GL}_n(k)$ via

$$g.(B, v) := g.(B_1, \ldots, B_q, v) = (g^{-1}B_1 g, \ldots, g^{-1}B_q g, g^{-1}v).$$

and let $\chi = (\det)^l$, $l > 0$. Then a G-orbit of (B, v) is closed if and only if

1. $[B_i, B_j] = 0$, $\forall i, j$.
2. If $S \subset k^n$ is a subspace satisfying $B_i(S) \subset S$ and $\mathrm{im} v \subset S$, then $S = k^n$. This is satisfied if $v(1)$ is a cyclic vector for B.

$$\mathrm{Hilb}^n(\mathbb{A}^q) \simeq \{(B_1, \ldots, B_q, v) : [B_i, B_j] = 0 + \text{ stability}\}/G$$

Proof: $\to$: $I \subset R = k[z_1, \ldots, z_q], \dim(R/I) = n \mapsto B_i \in \mathrm{End}(R/I)$ be "multiplication by" z_i mod I and $v(1) := 1$ mod I.
$\leftarrow$: $(B_1, \ldots, B_q, v) \mapsto I = \{f \in k[z_1, \ldots, z_q] : f(B_1, \ldots, B_q))(v(1)) = 0\}$
Haboush, Hyeon '17 Adapt to the nilpotent case: let $\mathcal{X}_{q,n} = N(q, n) \oplus \hom_k(k, k^n)$ acted on by $\mathrm{GL}(n)$ as before. Consider the affine GIT quotient of $\mathcal{X} \times \mathbb{A}^1$ modulo G where the action is $g.(X, v; z) = (gXg^{-1}, gv, \chi^{-1}(g)z)$. We have:

1. $\forall z \in \mathbb{A}^1 \setminus \{0\}$, $(X, v; z)$ has a closed orbit if X is regular and $v \notin \ker(X^{n-1})$.
2. (X, v) and (X, v') are identified in $\mathcal{X} /\!\!/_\chi G$ if X is regular and both are stable.
3. If (X, v) is regular and $X^{n-1}v \neq 0$, then the stabilizer of (X, v) is trivial.

This identifies $\mathcal{N}_{q,n}$ with a subscheme $\mathcal{X}_{q,n}^{reg} /\!\!/_\chi G$ of the affine GIT quotient $\mathcal{X}_{q,n} /\!\!/_\chi G$.

Observation:

- For any $f_{[n-1]} = (f_1, \ldots, f_q) \in J_{n-1}(1, q)$ with $f_i' \neq 0$ there is a unique $\alpha_f \in J_{n-1}(1,1)$ such that $(f_i \circ \alpha)(z) = z$ for $z \in \mathbb{C}$. Indeed, take $\alpha_f = (f_i)^{-1}_{[n-1]}$. Let

$$p_a(z) = (f_a \circ \alpha)(z) \text{ for } 1 \leq a \leq q \text{ the reparametrised } a^{th} \text{ coordinate}$$

- Let $\mathbb{S} = n \times n$ matrix with 1 on the subdiagonal & zero elsewhere, that is, the Jordan normal form of a regular nilpotent $n \times n$ matrix.

Then

$$J_{n-1}(1,q)/J_{n-1}(1,1) \to N(q,n)^{reg}$$

$$f_{[n-1]} \mapsto (p_1(\mathbb{S}), \ldots, \mathbb{S}, \ldots, p_q(\mathbb{S}))$$

is well-defined.

Compactifications I: the Demailly-Semple tower

Jet bundles Let X be smooth, projective variety of dimension q.

$J_nX = \{f : (\mathbb{C},0) \to (X,p), f' \neq 0\}/\sim$ where $f \sim g$ iff $f^{(j)}(0) = g^{(j)}(0)$ for $0 \leq j \leq n$

Fibration $J_nX \to X$ is simply $f \mapsto f(0)$. In local coordinates on X at p the fibre is

$$\{(f'(0), \ldots, f^{(n)}(0)/n!) : f'(0) \neq 0\} = J_n(1,q)$$

Note: J_nX is not a vector bundle over X since the transition functions are polynomial but not linear:

Demailly-Semple tower A fiberwise compactification of $J_nX/J_n(1,1)$. A directed manifold is a pair $(X.V)$ where $\dim(X) = q$ and $V \subseteq T_X$ a subbundle of rank $rk(V) = r$. We associate to (X,V) an other directed manifold $(\tilde{X}, \tilde{V})$, where $\tilde{X} = \mathbb{P}(V)$ is the projectivised bundle and $\tilde{V}$ is the subbundle of $T_{\tilde{X}}$ defined as

$$\tilde{V}_{(x_0,[v_0])} = \{\xi \in T_{\tilde{X},(x_0,[v_0])} | \pi_*(\xi) \in \mathbb{C} \cdot v_0\} \text{ where } \pi : \tilde{X} \to X$$

for any $x_0 \in X$ and $v_0 \in T_{X,x_0} \setminus \{0\}$. Lift of a regular holomorphic curve $f : (\mathbb{C},0) \to X$ (tangent to V) is $\tilde{f} : (\mathbb{C},0) \to \tilde{X}$ (tangent to $\tilde{V}$), defined as $\tilde{f}(t) = (f(t), [f'(t)])$. Iterate this and define

$$(X_0, V_0) = (X, T_X),\ (X_k, V_k) = (\tilde{X}_{k-1}, \tilde{V}_{k-1}).$$

Then $\dim X_k = q + k(q-1)$, $\mathrm{rank} V_k = q$, and the construction can be described inductively by the following exact sequence

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0$$

$$X_n \xrightarrow{\mathbb{P}^{q-1}} X_{n-1} \xrightarrow{\mathbb{P}^{q-1}} \ldots \qquad X_1 \xrightarrow{\mathbb{P}^{q-1}} X \text{ , (fiber is } J_{n-1}\mathbb{P}^{q-1} = \overline{J_n(1,q)/J_n(1,1)})$$

Compactifications II: non-reductive GIT

$$J_n(1,1) = \mathbb{C}^* \rtimes U = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{n-1} + \dots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{n-2} + \dots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^n \end{pmatrix} \text{ acts on } J_n(1,q).$$

- Let U be a graded unipotent lin. alg. group and $\hat{U} = U \rtimes \mathbb{C}^*$. Let X be an irreducible normal $\hat{U}$-variety and $L \to X$ an very ample linearisation of the $\hat{U}$ action.
- $X_{\min}^{\mathbb{C}^*} := \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{C}^*\text{-fixed point and} \\ \mathbb{C}^* \text{ acts on } L^*|_x \text{ with min weight} \end{array} \right\}$
- $X_{\min} := \{x \in X \mid \lim_{t \to 0} t \cdot x \in X_{\min}^{\mathbb{C}^*}\}$ $(t \in \mathbb{G}_m \subseteq \hat{U})$ the Bialynicki-Birula stratum.

Theorem (Non-reductive GIT theorem, B.,Doran, Hawes, Kirwan '16))

Let $L \to X$ be an irreducible normal $\hat{U}$-variety with ample linearisation L. Assume

$$(*)\ x \in X_{\min}^{\mathbb{C}^*} \Rightarrow \mathrm{Stab}_U(x) = 1$$

(or more generally $\dim \mathrm{Stab}_U(x) = \min_{y \in X} \dim \mathrm{Stab}_U(y)$) Twist the linearisation by a character $\chi : \hat{U} \to \mathbb{C}^*$ of $\hat{U}/U$ such that 0 lies in the lowest bounded chamber for the $\mathbb{C}^*$ action on X. Then

1. *the ring $A(X)^{\hat{U}}$ of $\hat{U}$-invariants is* **finitely generated**, *so that $X/\!/\hat{U} = Proj(A(X)^{\hat{U}})$, and $X/\!/\hat{U} = X^{ss,\hat{U}}/\sim$ where $x \sim y \Leftrightarrow \overline{\hat{U}x} \cap \overline{\hat{U}y} \cap X^{ss,\hat{U}} \neq \emptyset$ (same as reductive GIT).*
2. *$X^{ss,\hat{U}} = X^{s,\hat{U}}$ and $X/\!/\hat{U} = X^{s,\hat{U}}/\hat{U}$ is a* ***geometric quotient*** *of $X^{s,\hat{U}}$ by $\hat{U}$.*
3. *(**Hilbert-Mumford criterion**) $X^{s,\hat{U}} = X^{ss,\hat{U}} = \bigcap_{u \in U} uX_{\min+}^{s,\mathbb{C}^*} = X_{\min}^0 \setminus UZ_{\min}$.*

Moreover without condition (*) there is a GIT blow-up process to provide a $\hat{U}$-equivariant blow-up $\tilde{X}$ of X for which (*) holds. Theorem works for graded linear groups in general.

Tautological vector bundles on Hilbert schemes: F–rank r bundle (loc. free sheaf) on $X \rightsquigarrow F^{[n]}$–rank rn bundle over $\mathrm{Hilb}^{[n]}(X)$
Fibre over $\xi \in \mathrm{Hilb}^{[n]}(X)$ is $F \otimes \mathcal{O}_\xi = H^0(\xi, F|_\xi)$. Equivalently: $F^{[n]} = q_* p^*(F)$

where $\mathrm{Hilb}^{[n]}(X) \times X \supset$

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & \mathrm{Hilb}^{[n]}(X) \\ \downarrow{\scriptstyle p} & & \\ X & & \end{array}$$

Tautological vector bundles on commuting nilpotent matrices: We define these to be $F^{[n]}|_{\mathrm{Curv^n}(\mathbb{C}^q)}$.

Tautological integrals on Hilbert schemes $\int_{\mathrm{Hilb}^{[k]}(X)} C(F^{[k]})$ where C is a Chern polynomial

More generally: for an algebra A of dimension k let

$$Q_A = \{\xi \in \mathrm{Hilb}_0^{[k]}(X) : \mathcal{O}_\xi \simeq A\}.$$

The **geometric subset** of type $(A_1, \ldots, A_s)$ is

$$\mathrm{Hilb}^{\mathbf{A_1},\ldots,\mathbf{A_s}}(X) = \overline{\{\xi = \xi_1 \cup \ldots \cup \xi_s, \ \ \xi_i \in Q_{A_i} \subset \mathrm{Hilb}_{p_i}^{[\dim A_i]}(X) \text{ and } p_i \neq p_j\}}.$$

Tautological integrals: $\int_{\mathrm{Hilb}^{A_1,\ldots,A_s}(X)} C(F^{[k]})$ where $k = \dim A_1 + \ldots + \dim A_s$.

Why tautological integrals? Curve and hypersurface counting.

1. Counting hypersurfaces with prescribed singularities on X.
   Fact: Let $T_1, \dots, T_s$ be analytic singularity types with expected codimension $d = \sum_{i=1}^s d_i$. There is an associated geometric set $\mathrm{Hilb}^{A_1,\dots,A_s}(X)$ such that a generic hypersurface containing a $Z \in \mathrm{Hilb}^{A_1,\dots,A_s}(X)$ has precisely the singularities $T_1, \dots, T_s$. For sufficiently ample L and a generic $\mathbb{P}^d \subseteq |L|$ the number of hypersurfaces in $\mathbb{P}^d$ containing a subscheme $Z \in \mathrm{Hilb}^{A_1,\dots,A_s}(X)$ is $\int_{\mathrm{Hilb}^{A_1,\dots,A_s}(X)} c_{top}(L^{[k]})$.

Theorem (Existence of universal polynomial, Rennemo, Tzeng 2013-14)

The tautological integral $\int_{\mathrm{Hilb}^{\mathbf{Q}}(X)} R(c_i(F^{[k]}))$ is given by a universal polynomial of the Chern numbers of F and X.

2. Segre classes of tautological bundles and Lehn's conjecture. Let S be a surface. Lehn has a beautiful conjecture on

$$\int_{S^{[k]}} s_{top}(F^{[k]})$$

   Recently Voisin+Pandharipande+Szenes proved the conjecture for $F = L$ line bundle. For higher rank recent results of Marian-Oprea-Pandharipande in the K3 case. Geometry is not understood, no conjecture for top Segre numbers in higher rank.

3. Hyperbolicity conjectures: Demailly's strategy reduces the Green-Griffiths conjecture to the positivity of certain tautological integrals over the Demailly-Semple tower.

Theorem (Integrals over the curvilinear Hilbert schemes, B. 2015)

$$\int_{\mathrm{Hilb}^{\mu_k}(X)} R(c_i(F)) = \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j) Q_k(\mathbf{z}) R(c_i(z_i + \theta_j, \theta_j)) d\mathbf{z}}{\prod_{i+j \le l \le k}(z_i + z_j - z_l)(z_1 \dots z_k)^n} \prod_{i=1}^{k} s_X\left(\frac{1}{z_i}\right)$$

Features of the residue formulae:

1. The residue gives a degree n symmetric polynomial in Chern roots of F and Segre classes of X reproving Rennemo and Tzeng in this case.
2. For fixed k the formula gives a universal generating series for the integrals in the dimension of X.

Theorem (Integrals over geometric subsets, B.-Szenes '18)

Similar but more complicated residue formulas for tautological integrals over geometric subsets of Hilbert schemes. This provides nn alternative to 'virtual integration' developed in the Calabi-Yau set-up.