Some new extensions of Hecke endomorphism algebras I

Speaker: Leonard Scott (University of Virginia)

Abstract: This is the first of two talks by myself and Brian Parshall. It is based on our joint work with Jie Du, in progress. I will give some of the history and framework leading to the main conjecture we had made, asserting the existence of a kind of generalized q-Schur algebra, suitable for studying cross-characteristic representation theory of finite groups of Lie type. The conjecture is now a theorem, with some of its proof to be sketched in Parshall's talk. I will mention some applications as time permits.

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# Finite dimensional algebras and highest weight categories[1])

By *E. Cline* at Worcester, *B. Parshall* at Urbana and Charlottesville and *L. Scott* at Charlottesville

Fix a field $k$. We recall the definition of the class $\mathcal{Q}=\mathcal{Q}(k)$ of (right) quasi-hereditary algebras, given in [18]. $\mathcal{Q}$ is the class of finite dimensional algebras $A$ over $k$ defined recursively by assuming that $A$ has a nonzero ideal $J$ such that

(i) $J$ is projective as a right $A$-module;

(ii) $\operatorname{Hom}_A(J_A, A/J)=0$ and $J \cdot \operatorname{rad}(A) \cdot J=0$;

(iii) if $J \neq A$, $A/J$ belongs to $\mathcal{Q}$.

From a later CPS paper (J. Algebra 1990).

We keep the notations and conventions of the previous section. Thus, $k$ is a commutative Noetherian ring, and we only consider $k$-algebras which are $k$-finite. We begin with the following definition.

(3.1) DEFINITION. An ideal $J$ in a $k$-projective algebra $A$ is called a *heredity ideal* if $A/J$ is $k$-projective and the following conditions hold:

(i) $J$ is projective as a right ideal;

(ii) $J^2 = J$;

(iii) the $k$-algebra $E = \operatorname{End}_A(J_A)$ is $k$-semisimple.

Thus, if $A$ is a quasi-hereditary algebra, there exists a strictly increasing sequence of ideals $0=J_0\subset J_1\subset J_2\subset\cdots\subset J_t=A$ such that each $A/J_i$ is quasi-hereditary and $J_{i+1}/J_i$ is a projective right ideal in $A/J_i$ satisfying the quasi-hereditary condition above for $J$. We call such a sequence $\{J_i\}$ of ideals a "defining system of ideals for the algebra $A$".

Among the properties of a quasi-hereditary algebra $A$ which we will use, we record the following (cf. [16], Theorem 4. 3, [18]): (a) $A$ has finite global dimension; (b) $A$ is quasi-hereditary if and only if the opposite algebra $A^{\mathrm{op}}$ is quasi-hereditary; and (c) the inclusion $i_*: D^b(\text{mod-}A/J)\to D^b(\text{mod-}A)$ is a full embedding of triangulated categories, and gives rise to a recollement

$$D^b(\text{mod-}A/J)\leftrightarrows D^b(\text{mod-}A)\leftrightarrows \mathscr{D}''$$

of $D^b(\text{mod-}A)$, as discussed in § 1 (with $\mathscr{D}''$ as the derived category of a semisimple algebra).

The following key step is the first half of a strong connection between quasi-hereditary algebras and highest weight categories.

**Lemma 3. 4.** *Let $A$ be a quasi-hereditary algebra over $k$. Then the category $\mathscr{C} = \text{mod-}A$ of right $A$-modules is a highest weight category.*

An important representation-theoretic feature of highest weight categories is that the "character" of an irreducible module can be written in terns of the characters of standard modules, though the coefficients may not be easy to determine. Put another way, the associated decompostion matrix is invertible. In fact, it is unitriangular.

More later....

Schur algebras arise as endomorphism algebras of a tensor product of n copies of a vector space with itself. They are useful classically in characteristic 0 and positive characteristic representation theory of the general linear group GLn (F) in characteristic agreeing with a field F. These algebras are quasi-hereditary

q-Schur algebras play a smilar role in cross characteristic representation theory for GLn(F), with F a finite field.

Generalized Schur algebras may be defined by truncating hyperalgebras of semisimple algebraic groups in any type. For type A, this leads to the Schur algebras. Once more they are useful in the defining characteristic case.

Generalized q-Schur algebras may be defined by truncating quantum enveloping algebras. They give q-Schur algebras in type A.. But the same process has, so far, not appeared to be useful in other types.

In addition, the direct use of q-permutation module endomorphism algebras does not lead, in types other than A, to the same good homological properties. This suggested to the authors, some twenty years ago, to look for new analogs of q-Schur algebras in other types.

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# Stratifying Endomorphism Algebras Associated to Hecke Algebras*

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## Abstract

Let $G$ be a finite group of Lie type and let $k$ be a field of characteristic *distinct* from the defining characteristic of $G$. In studying the *non-describing* representation theory of $G$, the endomorphism algebra $S(G,k) = \mathrm{End}_{kG}(\bigoplus_J \mathrm{ind}_{P_J}^G k)$ plays an increasingly important role. In type $A$, by work of Dipper and James, $S(G,k)$ identifies with a $q$-Schur algebra and so serves as a link between the representation theories of the finite general linear groups and certain quantum groups. This paper

presents the first systematic study of the structure and homological algebra of these algebras for $G$ of arbitrary type. Because $S(G, k)$ has a reinterpretation as a Hecke endomorphism algebra, it may be analyzed using the theory of Hecke algebras. Its structure turns out to involve new applications of Kazhdan–Lusztig cell theory. In the course of this work, we prove two stratification conjectures about Coxeter group representations made by E. Cline, B. Parshall, and L. Scott (*Mem. Amer. Math. Soc.* **591**, 1996) and we formulate a new conjecture about the structure of $S(G, k)$. We verify this conjecture here in all rank 2 examples.

1.2. *Integral Stratifying Systems*. Let $\mathscr{Z}$ be a fixed (Noetherian) commutative ring. Let $\tilde{A}$ be a $\mathscr{Z}$-algebra which is finite and projective over $\mathscr{Z}$.

An idea $\tilde{J}$ of $\tilde{A}$ is called a *stratifying ideal* provided that $\tilde{J}$ is a $\mathscr{Z}$-direct summand of $\tilde{A}$ and, for any $\tilde{M}, \tilde{N} \in \mathrm{Ob}({}_{\tilde{A}/\tilde{J}}\mathscr{C})$, inflation defines an isomorphism $\mathrm{Ext}^{\bullet}_{\tilde{A}/\tilde{J}}(\tilde{M}, \tilde{N}) \cong \mathrm{Ext}^{\bullet}_{\tilde{A}}(\tilde{M}, \tilde{N})$. A sequence $0 = \tilde{J}_0 \subsetneq \tilde{J}_1 \subsetneq \cdots \subsetneq \tilde{J}_n = \tilde{A}$ of stratifying ideals in $\tilde{A}$ is called a *stratification of length* $n$. If each $\tilde{J}_i/\tilde{J}_{i-1}$ is a projective (left) $\tilde{A}/\tilde{J}_{i-1}$-module, the stratification is called *standard*.

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# Local and global methods in representations of Hecke algebras

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**Definition 2.1.** Assume that the $k$-algebra $A$ is projective over $k$. An ideal $J$ in $A$ is called a *heredity ideal* provided the following conditions hold.

(0) $A/J$ is projective over $k$.

(1) $J$ is a projective as a left $A$-module.

(2) $J^2 = J$.

(3) The $k$-algebra $E := \mathrm{End}_A({}_AJ)$ is $k$-semisimple.

The heredity ideal $J$ is of separable (resp., semisplit, split) type provided that $E$ is separable (resp., semisplit, split) over $k$. Recall that a $k$-algebra $E$ is separable if the $(E, E)$-bimodule map $E \otimes_k E \to E$ is split. One says that $E$ is semisplit if it is a finite direct product of algebras, each of which is separable and has center $k$ (i.e., each factor is an Azumaya algebra over $k$). If each factor is the endomorphism algebra of a finite projective $k$-module, then $E$ is called split[1)].

As in Section 2, let $k$ be a Noetherian commutative ring, and let $A$ be a $k$-algebra, always assumed to be a finite $k$-module which projective over $k$. We make the following definition, analogous to the notion of a heredity ideal.

**Definition 3.1.** An ideal $J$ in $A$ is called a *standard stratifying ideal* if the following conditions hold:

(0) $A/J$ is projective over $k$;
(1) $J$ is $A$-projective as a left $A$-module.
(2) $J^2 = J$.

Observe that, in particular, a heredity ideal is a standard stratifying ideal. With the above notion, we can make the following definition.

The definition below of a stratifying system for ${}_{\tilde{A}}\mathscr{C}$ generalizes that given in [CPS4, Subsect. 6.4] for fields.

(1.2.4) DEFINITION. Let $\Lambda$ be a finite quasi-poset. Suppose for $\lambda \in \Lambda$ there is given $\tilde{\Delta}(\lambda) \in \mathrm{Ob}({}_{\tilde{A}}\mathscr{C})$ which is projective over $\mathscr{Z}$. Then $\{\tilde{\Delta}(\lambda)\}_\lambda$ is a *stratifying system* for ${}_{\tilde{A}}\mathscr{C}$ provided that, for each $\lambda \in \Lambda$, there is given $\tilde{P}(\lambda) \in \mathrm{Ob}(\mathrm{proj}({}_{\tilde{A}}\mathscr{C}))$ and a surjective morphism $\tilde{P}(\lambda) \twoheadrightarrow \tilde{\Delta}(\lambda)$ satisfying the following three conditions:

(1) For $\lambda, \mu \in \Lambda$, if $\mathrm{Hom}_{\tilde{A}}(\tilde{P}(\lambda), \tilde{\Delta}(\mu)) \neq 0$, then $\lambda \leq \mu$.

(2) For $\lambda \in \Lambda$, there is a given decreasing filtration $\tilde{G}^\lambda$ of $\tilde{P}(\lambda)$ (of length $t(\lambda)$) with $\mathrm{Gr}_i\tilde{G}^\lambda \cong \tilde{\Delta}(\nu_{\lambda,i})$ $(0 \leq i < t(\lambda))$ where $\lambda \leq \nu_{\lambda,i} \in \Lambda$ and $\nu_{\lambda,0} = \lambda$.

(3) Any irreducible $\tilde{A}$-module is a homomorphic image of some $\tilde{\Delta}(\lambda)$.

Suppose condition (2) is replaced by the stronger condition:

(2′) For $\lambda \in \Lambda$, there is given a decreasing filtration $\tilde{G}^\lambda$ of $\tilde{P}(\lambda)$ (of length $t(\lambda)$) satisfying $\mathrm{Gr}_i\tilde{G}^\lambda \cong \tilde{\Delta}(\nu_{\lambda,i})$ where $\bar{\lambda} < \bar{\nu}_{\lambda,i}$ for $0 < i < t(\lambda)$ and $\nu_{\lambda,0} = \lambda$.

In this case, we say the stratifying system is *strict*.

When $\{\tilde{\Delta}(\lambda)\}_\lambda$ is a stratifying system for ${}_{\tilde{A}}\mathscr{C}$, we call the $\tilde{\Delta}(\lambda)$ the *standard objects* for ${}_{\tilde{A}}\mathscr{C}$. We now give conditions for checking that a collection of modules is a stratifying system. (In particular, the notion behaves well under base change.)

(1.2.5) LEMMA. *Let $\Lambda$ be a finite quasi-poset, and for each $\lambda \in \Lambda$, let $\tilde{\Delta}(\lambda) \in \mathrm{Ob}({}_{\tilde{A}}\mathscr{C})$. For $\lambda \in \Lambda$, assume that $\tilde{\Delta}(\lambda) \in \mathrm{Ob}(\mathrm{proj}(\mathscr{C}_{\mathscr{Z}}))$ and that there is given $(\tilde{P}(\lambda), \tilde{G}^\lambda) \in \mathrm{Ob}({}_{\tilde{A}}\mathscr{C}_{\mathrm{filt}})$ where $\tilde{P}(\lambda) \in \mathrm{Ob}(\mathrm{proj}({}_{\tilde{A}}\mathscr{C}))$ and $\tilde{G}^\lambda$ satisfies* (1.2.4(2)). *Then the following four statements are equivalent*:

(1) $\{\tilde{\Delta}(\lambda)\}_{\lambda \in \Lambda}$ *is a stratifying system for* ${}_{\tilde{A}}\mathscr{C}$.

(2) $\{\tilde{\Delta}(\lambda)_k\}_{\lambda \in \Lambda}$ *is a stratifying system for* ${}_{\tilde{A}_k}\mathscr{C}$, *for any $k = \mathscr{Z}/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of $\mathscr{Z}$.*

(3) $\{\tilde{\Delta}(\lambda)_k\}_{\lambda \in \Lambda}$ *is a stratifying system for* ${}_{\tilde{A}_k}\mathscr{C}$, *for each field $k$ which is a $\mathscr{Z}$-algebra.*

(4) $\{\tilde{\Delta}(\lambda)_{\mathscr{Z}'}\}_{\lambda \in \Lambda}$ *is a stratifying system for* ${}_{\tilde{A}_{\mathscr{Z}'}}\mathscr{C}$, *for each commutative $\mathscr{Z}$ algebra $\mathscr{Z}'$.*

The argument for [CPS4, (6.4.5)] and (1.2.7) give the following result.

(1.2.8) THEOREM. *Assume that* ${}_{\tilde{A}}\mathscr{C}$ *has a stratifying system* $\{\tilde{\Delta}(\lambda)\}_{\lambda \in \Lambda}$. *The algebra* $\tilde{A}$ *has a stratification* $0 = \tilde{J}_0 \subsetneq \tilde{J}_1 \subsetneq \cdots \subsetneq \tilde{J}_n = \tilde{A}$ *of length* $n = \#\overline{\Lambda}$. *If the stratifying system is strict, then the stratification can be chosen to be standard.*

The following hypothesis (generalizing [CPS4, (6.4.7)]) gives a criterion for stratifying ${}_{\tilde{A}}\mathscr{C}$, when $\tilde{A}$ is an endomorphism algebra.

(1.2.9) HYPOTHESIS. *Let* $\tilde{R}$ *be a finite and projective algebra over* $\mathscr{Z}$. *Let* $\tilde{T}$ *be a finitely generated right* $\tilde{R}$*-module, projective over* $\mathscr{Z}$, *and put* $\tilde{A} =$

$\mathrm{End}_{\tilde{R}}(\tilde{T})$. *Assume* $\tilde{T} = \bigoplus_{\lambda \in \Lambda} \tilde{T}_\lambda^{\oplus m_\lambda}$ *is a fixed direct sum decomposition, where* $\Lambda$ *is a finite quasi-poset.* (*No assumption that the* $\tilde{T}_\lambda$ *are indecomposable or even non-isomorphic is imposed.*) *For* $\lambda \in \Lambda$, *assume given an* $\tilde{R}$*-submodule* $\tilde{S}_\lambda \hookrightarrow \tilde{T}_\lambda$ *and an increasing filtration* $\tilde{F}_\lambda$ *of* $\tilde{T}_\lambda$ (*of length* $t(\lambda)$) *such that the following three conditions hold*:

(1) *For* $\lambda \in \Lambda$, *there is a fixed sequence* $\nu_{\lambda,0}, \nu_{\lambda,1}, \dots, \nu_{\lambda,t(\lambda)-1}$ *in* $\Lambda$ *such that* $\nu_{\lambda,0} = \lambda$ *and, for* $i > 0$, $\nu_{\lambda,i} \geq \lambda$. *For* $0 \leq i < t(\lambda)$, *there is given a fixed isomorphism* $\mathrm{Gr}^i \tilde{F}_\lambda \cong \tilde{S}_{\nu_{\lambda,i}}$.

(2) *For* $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_{\tilde{R}}(\tilde{S}_\mu, \tilde{T}_\lambda) \neq 0 \Rightarrow \lambda \leq \mu$.

(3) *For all* $\lambda \in \Lambda$, *we have* $\mathrm{Ext}^1_{\tilde{R}}(\tilde{T}_\lambda/\tilde{F}^i_\lambda, \tilde{T}) = 0$ *for all* $i$.

In establishing the following result, we use the notation (1.2.3).

(1.2.10) THEOREM. *Assume that Hypothesis* (1.2.9) *holds. For* $\lambda \in \Lambda$, *put* $\tilde{\Delta}(\lambda) = \mathrm{Hom}_{\tilde{R}}(\tilde{S}_\lambda, \tilde{T})$ *and assume that each* $\tilde{\Delta}(\lambda)$ *is* $\mathscr{Z}$*-projective. Then* $\{\tilde{\Delta}(\lambda)\}_{\lambda \in \Lambda}$ *is a stratifying system for* ${}_{\tilde{A}}\mathscr{C}$ *with respect to* $\Lambda$. *If the inequalities in* (1.2.9(1)) *can be replaced by strict inequalities* $\overline{\nu}_{\lambda,i} > \overline{\lambda}$ *for all* $i > 0$, *the system is strict.*

(2.5.2) CONJECTURE. *Suppose that $\tilde{H}$ is a Hecke algebra of Lie type over $\mathscr{Z}$ (cf. Subsection 2.1). There exists $\tilde{X} \in \mathrm{Ob}(\mathscr{C}_{\tilde{H}})$ such that:*

(1) *$\tilde{X}$ has a (increasing) filtration with sections of the form $\tilde{S}_\omega$, $\omega \in \Omega$.*

(2) *Let $\tilde{T}^+ = \tilde{T} \oplus \tilde{X}$, and let $\mathscr{Z}'$ be any commutative $\mathscr{Z}$-algebra. Put $\tilde{A}^+_{\mathscr{Z}'} = \mathrm{End}_{\tilde{H}'}(\tilde{T}^+_{\mathscr{Z}'})$ and, for $\omega \in \Omega$, $\tilde{\Delta}^+(\omega)_{\mathscr{Z}'} = \mathrm{Hom}_{\tilde{H}'}(\tilde{S}_\omega, \tilde{T}^+)_{\mathscr{Z}'}$. Then*

$\{\tilde{\Delta}^{+}(\omega)_{\mathscr{Z}'}\}_{\omega}$ *is a strict stratifying system relative to the quasi-poset* $(\Omega, \leq_{LR}^{\mathrm{op}})$ *for the module category* ${}_{\tilde{A}^{+}_{\mathscr{Z}'}}\mathscr{C}$.

(2.4.4) THEOREM. *Assume that* $\#S > 1$, *let* $\mathscr{Z}'$ *be a commutative* $\mathscr{Z}$*-algebra, and put* $\tilde{H}' = \tilde{H}_{\mathscr{Z}'}$. *Let* $\tilde{A}' = \mathrm{End}_{\tilde{H}'}(\tilde{T}_{\mathscr{Z}'})$. *Then*

(a) *The data consisting of* $\tilde{T} = \bigoplus_{\lambda \in \Lambda} \tilde{T}_\lambda$, $\tilde{S}_\lambda^{\mathscr{R}}$ *and* $\tilde{F}_{\lambda, \mathscr{R}}$ *satisfy Hypothesis* (1.2.9).

(b) *For* $\lambda \in \Lambda$, *put* $\tilde{\Delta}(\lambda)^{\mathscr{R}} = \mathrm{Hom}_{\tilde{H}}(\tilde{S}_\lambda^{\mathscr{R}}, \tilde{T})$. *Then each* $\tilde{\Delta}(\lambda)_{\mathscr{Z}'}^{\mathscr{R}}$ *is* $\mathscr{Z}'$*-free, and* $\{\tilde{\Delta}(\lambda)_{\mathscr{Z}'}^{\mathscr{R}}\}_\lambda$ *is a stratifying system for* ${}_{\tilde{A}'}\mathscr{C}$ *with respect to the quasi-poset structure defined on the power set* $\Lambda$ *of* $S$ *in* (2.4.3). *Also, there is an isomorphism*

$$\mathrm{End}_{\tilde{H}}(\tilde{T})_{\mathscr{Z}'} \cong \mathrm{End}_{\tilde{H}'}\big(\tilde{T}_{\mathscr{Z}'}\big). \tag{2.4.4.1}$$

*Finally, if* $\xi = S$ *and* $\zeta = \emptyset$, *then* $\tilde{\Delta}(\xi)_{\mathscr{Z}'}^{\mathscr{R}}$ *is* $\tilde{A}'$*-projective and has* $\mathscr{Z}'$*-rank* $2^{\#S}$, *while* $1 = \mathrm{rank}\, \tilde{\Delta}(\zeta)_{\mathscr{Z}'}^{\mathscr{R}} < \mathrm{rank}\, \tilde{\Delta}(\mu)_{\mathscr{Z}'}^{\mathscr{R}}$, $\mu \neq \zeta$.

(2.4.9) COROLLARY. *Consider a family* $\mathscr{G} = \{G(q)\}$ *of finite groups of Lie type, and let* $k$ *be a field of characteristic* $p$ *relatively prime to* $q$. *Assume that* $G(q)$ *has rank* $> 1$. *Let* $B(q)$ *be a Borel subgroup of* $G(q)$. *Then the algebras*

$$S(G(q), k) = \mathrm{End}_{kG(q)}\left( \bigoplus_{P(q) \supseteq B(q)} \mathrm{ind}_{P(q)}^{G(q)} k \right) \tag{2.4.9.1}$$

*have a stratification of length* $\geq 3$.

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# Stratifying Endomorphism Algebras

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## 6.4. Stratifying systems

In this section, we present a very general method for stratifying a module category ${}_A\mathcal{C}$ which leads to a stratification of the algebra $A$ in the sense of

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[26] Stratifications which are standard and thus have a richer structure may naturally arise though a modification of the construction, replacing $T$ by $T' = T \oplus X$ for a suitable $W$-module $X$. Following remarks of Jie Du, we expect (with Du) that $A' = \mathrm{End}_{\mathcal{C}_R}(T')$ could have a natural standard stratification with the same number of strata as two-sided cells, and with $T$ and $X$ filtered by dual right cell modules. (Note that $A \cong eA'e$ for an idempotent $e \in A'$, so that ${}_A\mathcal{C}$ is "approximated" (as a quotient category) by the standardly stratified module category ${}_{A'}\mathcal{C}$, if $X$ is sufficiently small or natural.) We expect similar statements to hold for integral liftings $\tilde{T}$ of $T$ in the sense of Chapter 4. According to (4.7.2(b)) this predicts the vanishing of certain $\mathrm{Ext}^1$-groups associated to $\tilde{T}$ and its filtration by dual right cell modules.

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# CELLS AND $q$-SCHUR ALGEBRAS

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$$(1) \qquad \widetilde{S}_q(n,r) = \mathrm{End}_{\widetilde{H}}\Big(\bigoplus_{\lambda\in\Lambda^+(n,r)} \mathrm{ind}_{\widetilde{H}_\lambda}^{\widetilde{H}} \mathrm{IND}_\lambda\Big).$$

(This page is all for type A.)

Here $\Lambda^+(n,r)$ is the set of partitions of $r$ into at most $n$ parts, $\widetilde{H}_\lambda$ is the parabolic subalgebra corresponding to $\lambda \in \Lambda^+(n,r)$, and $\mathrm{IND}_\lambda$ is the index character on $\widetilde{H}_\lambda$. (See Section 2 for further explanation of notation.) The algebras $\widetilde{S}_q(n,r)$, which behave well with respect to base change, go back to the work of Dipper–James [DJ2]. They derive their importance, in part, from their close relationship with the non-describing representation theory of finite general linear groups $GL(n,q)$; cf. [D].

There is a preorder $\leq_{LR}$ defined on $W$ whose cells (equivalence classes for the relation $x \leq_{LR} y$ & $y \leq_{LR} x$) are the two-sided Kazhdan–Lusztig cells of $W$. Denoting this set of two-sided cells by $\Xi$, $\leq_{LR}$ defines a poset structure on this set. Let $\leq_{LR}^{\mathrm{op}}$ denote the *opposite* poset structure.

**Main Theorem.** *The algebra $\widetilde{A} = \widetilde{S}_q(n,r)$ is $\mathcal{Z}$-quasi-hereditary. In particular, for any field $k$ and ring homomorphism $\mathcal{Z} \to k$, the algebra $\widetilde{A}_k = \widetilde{A} \otimes_{\mathcal{Z}} k$ is quasi-hereditary. There exists a subposet $(\Xi(n,r), \leq_{LR}^{\mathrm{op}})$ of $(\Xi, \leq_{LR}^{\mathrm{op}})$ which serves, for all $k$, as the weight poset for the associated highest weight category $\widetilde{A}_k$–*mod. *There exist $\widetilde{A}$-modules $\widetilde{\Delta}(\xi)$, $\xi \in \Xi(n,r)$, such that, for any $k$ as above, the $\widetilde{A}_k$-modules $\widetilde{\Delta}(\xi)_k$ obtained from $\widetilde{\Delta}(\xi)$ by base-change, are the standard objects for the highest weight category $\widetilde{A}_k$–*mod.

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# EXTENDING HECKE ENDOMORPHISM ALGEBRAS

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*We dedicate this paper to the memory of Robert Steinberg.*

**The (Iwahori–)Hecke algebra in the title is a $q$-deformation $\mathcal{H}$ of the group algebra of a finite Weyl group $W$. The algebra $\mathcal{H}$ has a natural enlargement to an endomorphism algebra $\mathcal{A} = \mathrm{End}_{\mathcal{H}}(\mathcal{T})$ where $\mathcal{T}$ is a $q$-permutation module. In type $A_n$ (i.e., $W \cong \mathfrak{S}_{n+1}$), the algebra $\mathcal{A}$ is a $q$-Schur algebra which is quasi-hereditary and plays an important role in the modular representation of the finite groups of Lie type. In other types, $\mathcal{A}$ is not always quasi-hereditary, but the authors conjectured 20 year ago that $\mathcal{T}$ can be enlarged to an $\mathcal{H}$-module $\mathcal{T}^+$ so that $\mathcal{A}^+ = \mathrm{End}_{\mathcal{H}}(\mathcal{T}^+)$ is at least standardly stratified, a weaker condition than being quasi-hereditary, but with "strata" corresponding to Kazhdan–Lusztig two-sided cells.**

**The main result of this paper is a "local" version of this conjecture in the equal parameter case, viewing $\mathcal{H}$ as defined over $\mathbb{Z}[t, t^{-1}]$, with the localization at a prime ideal generated by a cyclotomic polynomial $\Phi_{2e}(t)$, $e \neq 2$. The proof uses the theory of rational Cherednik algebras (also known as RDAHAs) over similar localizations of $\mathbb{C}[t, t^{-1}]$. In future papers, the authors hope to prove global versions of the conjecture, maintaining these localizations.**

Each two-sided cell may be identified with the set of left cells it contains, and the resulting collection $\overline{\Omega}$ of sets of left cells is a partition of $\Omega$. There are various natural preorders on $\Omega$, but we will be mainly interested in those whose associated equivalence relation has precisely the set $\overline{\Omega}$ as its associated partition. We call such a preorder *strictly compatible* with $\overline{\Omega}$.

**1C.** ***A conjecture.*** Now we are ready to state the following conjecture, which is a variation (see the Appendix) on [op. cit., Conjecture 2.5.2]. We informally think of the algebra $\mathcal{A}^+$ in the conjecture as an extension of $\mathcal{A}$ as a Hecke endomorphism algebra (justifying the title of the paper).

**Conjecture 1.2.** *There exists a preorder $\leq$ on the set $\Omega$ of left cells in $W$, strictly compatible with its partition $\overline{\Omega}$ into two-sided cells, and a right $\mathcal{H}$-module $\mathcal{X}$ such that the following statements hold*:

(1) *$\mathcal{X}$ has an finite filtration with sections of the form $S_\omega$, $\omega \in \Omega$.*

(2) *Let $\mathcal{T}^+ := \mathcal{T} \oplus \mathcal{X}$ and put*

$$\begin{cases} \mathcal{A}^+ := \mathrm{End}_{\mathcal{H}}(\mathcal{T}^+), \\ \Delta^+(\omega) := \mathrm{Hom}_{\mathcal{H}}(S_\omega, \mathcal{T}^+), \quad \textit{for any } \omega \in \Omega. \end{cases}$$

*Then, for any commutative, Noetherian $\mathcal{Z}$-algebra $R$, the set $\{\Delta^+(\omega)_R\}_{\omega\in\Omega}$ is a strict stratifying system for $\mathcal{A}^+_R$-mod relative to the quasi-poset $(\Omega, \leq)$.*

## 6. Identification of $\tilde{\mathcal{A}}^+ = \mathrm{End}_{\tilde{\mathcal{H}}}(\tilde{\mathcal{T}}^+)$

The constructions in Section 5B of the modules $\tilde{X}_\omega$ in the previous section work just as well using the modules $\tilde{S}_E := \tilde{S}(E)^*$ for $E \in \mathrm{Irr}(\mathbb{Q}W)$ defined in (3A.5) to replace the dual left cell modules $\tilde{S}_\omega$. This results in right $\mathcal{H}$-modules $\tilde{X}_E$. As in the case of $\tilde{X}_\omega$, we have the following property, with the same proof. In the statement of the following proposition, $\tilde{X}_E$ can be defined using either of the two constructions.

**Proposition 6.1.** *If $e \neq 2$, then $\mathrm{Ext}^1_{\tilde{\mathcal{H}}}(\tilde{S}_{E'}, \tilde{X}_E) = 0$ for all $E, E' \in \mathrm{Irr}(\mathbb{Q}W)$.*

In the result below, we allow $c_s \neq 1$. In case $c_s = 1$, assumption (2) is satisfied for $R = \mathscr{Q}$ if and only if $e \neq 2$.

**Corollary 4.5.** *Suppose $R$ is a commutative domain with fraction field $F$, and assume that $R$ is also a $\mathcal{Z}$-algebra. Let $\lambda \subseteq S$. Assume that*

(1) *$\mathcal{H}_F$ is semisimple;*

(2) *$q^{c_s} + 1$ is invertible in $R$, for each $s \in \lambda$.*

*Then, for any dual left cell module $S_{\omega,R}$ over $R$,*

$$\mathrm{Ext}^1_{\mathcal{H}_R}(S_{\omega,R}, x_\lambda \mathcal{H}_R) = 0.$$

The next slides are taken from a paper in progress by Du, Parshall, and Scott.

The first slide uses the notion of a height function compatible with a given preorder, which will be discussed further in the next talk.

For $\omega, \omega' \in \Omega$, define a preorder

$$\omega \preceq \omega' \iff \text{either } [\mathfrak{h}(\omega) < \mathfrak{h}(\omega')] \text{ or } [\mathfrak{h}(\omega) = \mathfrak{h}(\omega') \text{ and } \omega \sim_{LR} \omega'] \tag{5.0.27}$$

(compare the order $\leq_f$ given on [DPS15, p.236]). Recall that $\mathfrak{h} = \mathfrak{h}_r$ has been constructed so that it is compatible with $\leq_{LR}^{\mathrm{op}}$ on $\Omega$. In particular, it is constant on two-sided cells.

The following is the main result of this paper.

**Theorem 5.8.** *Let* $T^+ = T \oplus \mathcal{X}$ *and* $A^+ = \mathrm{End}_{\mathcal{H}}(T^+)$. *For* $\omega \in \Omega$, *put* $\Delta(\omega) = \mathrm{Hom}_{\mathcal{H}}(S_\omega, T^+)$ *and* $P(\omega) = \mathrm{Hom}_{\mathcal{H}}(T^+_\omega, T^+)$. *Then* $\{\Delta(\omega), P(\omega)\}_{\omega\in\Omega}$ *is a stratifying system for the category* $A^+$*-mod with respect to the quasi-poset* $(\Omega, \preceq)$.

**Remark 5.9.** If $W$ is the Weyl group of a finite group of Lie type, the above theorem proves the conjecture stated in [DPS15, Conj. 1.2]. The second part of this conjecture states that the theorem holds for some preorder $\preceq$ with the same equivalence classes (two-sided cells) as $\leq_{LR}^{\mathrm{op}}$. The actual preorder $\preceq$ in the theorem above satisfies this condition on equivalence classes, and is quite close to $\leq_{LR}^{\mathrm{op}}$. This second part of [DPS15, Conj. 1.2] is phrased in the language of base ring extensions of $\mathcal{Z}$, but it is sufficient to check it over $\mathcal{Z}$, by [DPS98a, Lem. 1.2.5(4)]. The first (remaining) part of the conjecture deals with the filtration of $\mathcal{T}^\dagger$ by dual left cell modules, and follows

## 8. Application: the quasi-heredity of $\mathcal{A}^{\dagger\natural}$

invert "bad' primes in {2,3,5}

Keep the notation of §5. We prove a result which strengthens Theorem 5.8. We maintain the notation of §5, so, in particular, $A^+ = \mathrm{End}_{\mathcal{H}}(T^+)$ is a certain endomorphism algebra over $\mathcal{Z}$ has a stratifying system $\{\Delta(\varpi), P(\varpi)\}_{\omega\in\Omega}$. Now base change to $\mathcal{Z}^\natural$ and set $A^{+\dagger} = \mathcal{Z}^\natural \otimes A^{+\natural}$.

**Theorem 8.1.** *The algebra $A^{+\natural}$ is split quasi-hereditary.*

Corollaries/applications:

--a unitriangular decomposition matrix, in the "good prime" case, for the new quasi-hereditary algebra as well as the Hecke algebra, and an approximation for all primes.

--a similar unitriangular matrix for the original “q-Schur algebra” style endomorphism algebra for the direct sum of all permutation modules associated with G(q) actions on coset spaces G(q)/P(q) associated to parabolic subgroups containing a fixed Borel B(q).

More speculative:

--a parameterization of most, and, eventually, all, the irreducible modular constituents of the G(q)/B(q) permutation module (not just those in the "head"), with a generalization to all primes.