Arithmetic and Zariski-dense subgroups:

weak commensurability, eigenvalue rigidity, and applications to locally symmetric spaces

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Remark. These reformulations show that weak commensurability is *independent* of matrix realizations of G_i's.

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Recall: If Γ_1 and Γ_2 are *arithmetic* then

$$\mathcal{G}_1 \simeq \mathcal{G}_2 \text{ over } K \Rightarrow \Gamma_1 \,\&\, \Gamma_2 \text{ commensurable.}$$

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(Additionally, one expects that $r = 1$ in certain situations ...)

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- Similar consequences for orthogonal groups of quadratic forms etc.

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General case is work in progress ...

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Remark. Types excluded in (1) are *honest exceptions.*

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If $\Gamma_2 \subset G_2(\mathbb{R})$ *is a lattice weakly commensurable to* Γ_1,

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Remark. Above results were proved in a more general context of S-arithmetic subgroups.

(cont.)

(3) **If** Γ_1 *and* Γ_2 *are weakly commensurable, and* $K = K_{\Gamma_1} = K_{\Gamma_2}$,

then $\mathrm{rk}_K\, \mathcal{G}(\Gamma_1) = \mathrm{rk}_K\, \mathcal{G}(\Gamma_2)$.

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Remark. Above results were proved in a more general context of S-arithmetic subgroups. (4) is valid for S-arithmetic lattices over any locally compact field F.

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Together, Theorems 3 and 4 cover all situations where Zarsiki-dense S-arithmetic subgroups of absolutely almost simple groups can be weakly commensurable.

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Now, let G_1 and G_2 be *absolutely almost simple* $\mathbb{R}$-groups,

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if α and β are algebraic numbers $\neq 0, 1$, then

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A finite volume locally symmetric space $\mathfrak{X}_\Gamma$ of a simple real group is automatically *arithmetically defined* unless $\mathfrak{X}$ is either real hyperbolic space $\mathbb{H}^n$ or complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$.

(Margulis + Corlette + Gromov-Shoen)

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It consists of single commensurability class if G_1 *and* G_2 *are of same type different from* A_n*,* D_{2n+1} $(n > 1)$*, or* E_6*.*

Corollary

Let M_1 *and* M_2 *be arithmetically defined hyperbolic d-manifolds, where* $d \neq 3$ *is even or* $\equiv 3 (\mathrm{mod}\, 4)$.

If M_1 *and* M_2 *are length-commensurable,* **then** *they are commensurable.*

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- A *complex* hyperbolic manifold cannot be length-commensurable to a *real* or *quaternionic* hyperbolic manifold, etc.

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Assume that both G_1 and G_2 are of one of following types: A_n, D_{2n+1} $(n > 1)$ or E_6, subgroups Γ_1 and Γ_2 are arithmetic, and in addition $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$.

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(1) *If* $d_1 > d_2$ *then* (T_1) *and* (N_1) *hold.*

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Assume that G_1 and G_2 are either of same type or one of them is of type B_ℓ and other of type C_ℓ, and let $M_i = \mathfrak{X}_{\Gamma_i}$ $(i = 1, 2)$ be arithmetically defined locally symmetric spaces.

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$\Rightarrow$ $\mathrm{GL}_n(\mathbb{Q})$ contains $\mathbb{Q}$-generic elements.

Let $A \in \mathrm{GL}_n(F)$, and let $\chi_A(t)$ = characteristic polynomial of A.

A is generic over F **if**

- A is diagonalizable,
- $\chi_A(t)$ is irreducible over F, and
- Galois group of $\chi_A(t)$ over F is symmetric group S_n.

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We will now generalize notion of generic elements and existence theorem to arbitrary semi-simple groups.

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(1) T is generic over F if $\mathrm{Im}\,\theta_T$ contains Weyl group $W(\Phi)$.

(2) A semi-simple element $\gamma \in G(F)$ is generic over F if $T := Z_G(\gamma)^\circ$ is a torus (i.e., γ is *regular*) which is generic over F.

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Remarks. "Components" in (1) refer to almost direct product $G = G_1 \cdots G_r$ of simple groups.

(2) means that set of F-regular elements is open in Γ for profinite topology.

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Such elements were used to study dynamics of actions, rigidity, Auslander problem about properly discontinuous groups of affine transformations, etc.

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This is false for dense subgroups of compact tori!

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We will see a statement about *arbitrary* Riemann surfaces later, but first let us analyze situation in detail.

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Set $A_\Gamma = \mathbb{Q}[\tilde{\Gamma}^{(2)}] \subset M_2(\mathbb{R})$, $\tilde{\Gamma}^{(2)} \subset \tilde{\Gamma}$ generated by squares.

One shows: A_Γ is a *quaternion algebra* with center

$$K_\Gamma = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \Gamma^{(2)})$$

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- **If** Γ is *arithmetic*, **then** A_Γ is the quaternion algebra involved in its description;
- In general, A_Γ **does not** determine Γ, **but** is an invariant of the commensurability class of Γ.

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(Not all – but we will ignore it for now ...)

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We will see what can be said about A_{Γ}'s for length-commensurable Riemann surfaces.

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What happens if one allows only splitting fields of finite degree, or just maximal subfields?

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Question (G. Prasad-A.R.)

Are quaternion algebras over $K = \mathbb{Q}(x)$ determined by their maximal subfields?

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Are quaternion algebras over $K = \mathbb{Q}(x)$ determined by their maximal subfields?

- Yes – D. Saltman

- Amitsur's Theorem is no longer true in this setting.
  (Counterexamples can be found using cubic algebras over number fields.)

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  (S. Garibaldi - D. Saltman)

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(Follows from Albert-Hasse-Brauer-Noether Theorem.)

Theorem 10 (Stability Theorem, Chernousov-I. Rapinchuk, A.R.)

Let $\mathrm{char}\, k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for every quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

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(E.g., $\mathbb{Q}(\sqrt{11}) \hookrightarrow D_1$ but $\mathbb{Q}(\sqrt{11}) \not\hookrightarrow D_2$.)

- Find K_1/K such that

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Then (2) is obvious, and (1) follows from the fact that

$$x_0^2 + x_1^2 - 21x_2^2 - 21x_3^2$$

remains anisotropic over K_1.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

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Note that $\mathcal{K}$ is infinitely generated.

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(When n is divisible by char $K^{(v)}$, we need some additional assumptions)

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If $(n\,, \mathrm{char}\, K^{(v)}) = 1$ or $K^{(v)}$ is perfect, there is a *residue map*

$$r_v \colon {}_n\mathrm{Br}(K) \longrightarrow H^1(\mathcal{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}),$$

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Given a set V of discrete valuations of K, one defines corresponding *unramified Brauer group:*

$$\mathrm{Br}(K)_V = \{\, x \in \mathrm{Br}(K) \mid x \text{ unramified at all } v \in V \,\}.$$

- To prove Theorem 1 (Stability Theorem) we use: *if* $K = k(x)$ *and* V = *set of geometric places, then*

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- More recent argument works in all characteristics, **but** gives no estimate of size of $\mathbf{gen}(D)$.

- Earlier argument works when $(n\,,\, \mathrm{char}\, K) = 1$, gives finiteness of ${}_n\mathrm{Br}(K)_V$ and estimate

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- One can construct examples where ${}_2\mathrm{Br}(K)_V$ is "large."

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  $\mathbf{gen}_K(G) =$ set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori.

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This brings us to groups with good reduction.

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1. generic fiber $\mathcal{G} \otimes_{\mathcal{O}_v} K_v$ is isomorphic to $G \otimes_K K_v$;
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$$q \sim \lambda(a_1x_1^2 + \cdots + a_nx_n^2) \quad \text{with} \quad \lambda \in K_v^{\times}, \ a_i \in \mathcal{O}_v^{\times}$$

(assuming that $\mathrm{char}\, K^{(v)} \neq 2$).

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Most popular case: K field of fractions of Dedekind ring R, and V consists of places associated with maximal ideals of R.

Basic case $R = \mathbb{Z}$:

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Proposition

Let G be an absolutely almost simple simply connected algebraic group over a number field K, and assume that V contains almost all places of K. **Then** *number of K-forms of G that have good reduction at all $v \in V$ is finite.*

Case $R = k[x]$, $K = k(x)$, and $V = \{ v_{p(x)} \mid p(x) \in k[x] \text{ irreducible} \}$.

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Theorem (Raghunathan–Ramanathan, 1984)

Let k be a field of characteristic zero, and let G_0 be a connected reductive group over k. If G' is a K-form of $G_0 \otimes_k K$ that has good reduction at all $v \in V$ then $G' = G'_0 \otimes_k K$ for some k-form G'_0 of G_0.

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This was used to prove conjugacy of Cartan subalgebras in some infinite-dimensional Lie algebras.

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- Two divisorial sets differ only in *finitely many* valuations.

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- V_0 consists of extensions of p-adic valuations ("constant" valuations), and
- V_1 of discrete valuations associated with irreducible polynomials in $\mathbb{Q}[x]$, i.e. with closed points of $\mathbf{A}^1_{\mathbb{Q}}$ ("geometric" valuations).

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True if

- K is a global field;
- G is an inner form of type A_n;
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V.I. Chernousov, A.S. Rapinchuk, I.A. Rapinchuk, *Spinor groups with good reduction*, Compos. Math. **155**(2019), no. 3, 484-527.

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Then *<u>every</u>* $G' \in \mathbf{gen}_K(G)$ *has good reduction at* v, *and reduction* $\underline{G}'^{(v)} \in \mathbf{gen}_{K^{(v)}}(\underline{G}^{(v)})$.

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Since $V \setminus S$ is also divisorial,

Corollary.

Let G be an absolutely almost simple simply connected algebraic group over a finitely generated field K, and V be a divisorial set of places of K.

There exists a finite subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.

Since $V \setminus S$ is also divisorial, finiteness of $\mathbf{gen}_K(G)$ would follow from Finiteness Conjecture for Groups with Good Reduction.

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I. RAPINCHUK, A.R. (2019): True for tori over finitely generated fields of characteristic zero.

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- It is not known how to classify forms by cohomological invariants.
- Even when such description is available (e.g. for type G_2), one needs to prove finiteness of unramified cohomology in degrees > 2, which is a difficult problem.

Challenges in analysis of **Finiteness Conjecture for Groups with Good Reduction**:

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Let A_Γ be the *associated* quaternion algebra.

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Replacing length-commensurability with much stronger relation of isospectrality we have:

Compact Riemann surfaces isospectral to a given one consist of finitely many isometry classes $\Rightarrow$ there are finitely many isomorphism classes of associated quaternion algebras.

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This is one of the first examples of application of techniques from arithmetic geometry to length-commensurable non-arithmetic Riemann surfaces.

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Example. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and set

$$u^+(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ and } u^-(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

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Then for $m \geqslant 3$, subgroup

$$\Delta_m := \langle u^+(m), u^-(m) \rangle$$

is of infinite index in Γ, **but** is weakly commensurable to it.

Weak commensurability follows from inclusion

$$\Gamma(m^2) \subset \bigcup_{g \in \mathrm{GL}_2(\mathbb{Q})} g\,\Delta_m\, g^{-1},$$

where

$$\Gamma(m^2) = \{\, x \in \Gamma \mid x \equiv I_2 \pmod{m^2} \,\}$$

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So, we would like to propose the following

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Problem can be stated for higher-rank S-arithmetic subgroups, but is wide-open even for $\mathrm{SL}_2(\mathbb{Z}[1/p])$.

Problem 2. Let G_1 and G_2 be simple groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let Γ_i be a (finitely generated) Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable.

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The answer is 'yes' for a nonarchimedean locally compact field F, but archimedean case is open.

Problem 3. Let G_1 and G_2 be simple algebraic groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\Gamma_i \subset G_i(F)$ be a lattice for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable.

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Recall: The answer is 'yes' if one space is arithmetically defined.

Problem 4. Develop notion of weak commensurability for Zariski-dense (and particularly arithmetic) subgroups of general semi-simple groups.

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Problem 5. For inner and outer forms of types A_n $(n > 1)$, D_{2n+1} $(n > 1)$ and E_6, construct examples of isospectral compact arithmetically defined locally symmetric spaces that are not commensurable.

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Currently, such construction is available only for inner forms of type A_n.