ARITHMETIC AND ZARISKI-DENSE SUBGROUPS:
weak commensurability, eigenvalue rigidity, and applications to locally symmetric spaces

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Results

1. First signs of eigenvalue rigidity
2. Weakly commensurable arithmetic groups
3. Geometric applications

Generic elements

Division algebras with the same maximal subfields
1. Algebraic and geometric motivations
2. Genus of a division algebra
3. Generalizations

Groups with good reduction
1. Basic definitions and examples
2. Finiteness Conjecture for Groups with Good Reduction
3. Implications of the Finiteness Conjecture for Groups with Good Reduction
4. Application to Nonarithmetic Riemann Surfaces

Some open problems
1 Results

- First signs of eigenvalue rigidity
  - Weakly commensurable arithmetic groups
  - Geometric applications

2 Generic elements

3 Division algebras with the same maximal subfields
  - Algebraic and geometric motivations
  - Genus of a division algebra
  - Generalizations

4 Groups with good reduction
  - Basic definitions and examples
  - Finiteness Conjecture for Groups with Good Reduction
  - Implications of the Finiteness Conjecture for Groups with Good Reduction
  - Application to Nonarithmetic Riemann Surfaces

5 Some open problems
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**Definition.**

(1) Let $\gamma_1 \in \text{GL}_{n_1}(F)$ and $\gamma_2 \in \text{GL}_{n_2}(F)$ be semi-simple (i.e., diagonalizable) matrices,
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$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \ (\in \bar{F})$$

be their eigenvalues.
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$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$
Let $G_1 \subset \text{GL}_n_1$ and $G_2 \subset \text{GL}_n_2$ be reductive $F$-groups, $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be Zariski-dense subgroups.

(2) Subgroups $\Gamma_1$ and $\Gamma_2$ are weakly commensurable if every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.
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$(1) \iff$ there exists maximal $F$-tori $T_i$ of $G_i$ such that $\gamma_i \in T_i(F)$

and characters $\chi_i \in X(T_i)$ ($i = 1, 2$) for which

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1;$$
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Theorem 1

If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable, then $G_1$ and $G_2$ have same Killing-Cartan type, or one of them is of type $B_\ell$ and the other of type $C_\ell$ ($\ell \geq 3$).
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Let $\mathfrak{g}(\Gamma)$ denote algebraic hull of $\Gamma$, i.e. Zariski-closure of $\text{Ad}_G(\Gamma) \subset \text{GL}(g)$. 
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**Recall:** $\mathcal{G}(\Gamma)$ is adjoint group defined over $K_\Gamma$, (i.e., an $F/K_\Gamma$-form of adjoint group $\overline{G}$)
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$\mathcal{G}(\Gamma)$ is an important characteristic of $\Gamma$; it determines $\Gamma$ if it is arithmetic.
First signs of eigenvalue rigidity

To summarize: if $\Gamma_1$ and $\Gamma_2$ as above are weakly commensurable, then

- Their algebraic hulls $G_1 = G(\Gamma_1)$ and $G_2 = G(\Gamma_2)$ are defined over the same field $K_{\Gamma_1} = K_{\Gamma_2} = K$;
- Apart from ambiguity between types $B_\ell$ and $C_\ell$, $G_1$ and $G_2$ have the same type, (i.e., are isomorphic over the closure $K$ or $C$).

Thus, $G_1$ and $G_2$ are $K/K$-forms of one another.

Critical question: How are $G_1$ and $G_2$ related over $K$?

Recall: If $\Gamma_1$ and $\Gamma_2$ are arithmetic then $G_1 \simeq G_2$ over $K \Rightarrow \Gamma_1$ and $\Gamma_2$ commensurable.
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Recall: If $\Gamma_1$ and $\Gamma_2$ are arithmetic then

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Results

First signs of eigenvalue rigidity

More specifically:

If we fix $\Gamma_1$, what are possibilities for $G_2$?

Finiteness conjecture for weakly commensurable groups.

Let

- $G_1$ and $G_2$ be absolutely simple algebraic $F$-groups, $\text{char} F = 0$;
- $\Gamma_1 \subset G_1 (F)$ be a finitely generated Zariski-dense subgroup, $K_{\Gamma_1} = K$.

Then there exists a finite collection $G_1(1), \ldots, G_1(r)$ of $F/K$-forms of $G_2$ such that

if $\Gamma_2 \subset G_2 (F)$ is a finitely generated Zariski-dense subgroup weakly commensurable to $\Gamma_1$,

then $\Gamma_2$ can be conjugated into some $G_i(K) (\subset G_2 (F))$.

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**Finiteness Conjecture** $\Rightarrow$ There are only finitely many c.s.a. $A'$ such that for $G' = \text{PSL}_{1,A'}$, there exists a f.g. Zariski-dense subgroup $\Gamma' \subset G'(K)$ weakly commensurable to $\Gamma$. 
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- Similar consequences for orthogonal groups of quadratic forms etc.
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General case is work in progress ...
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If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable, then they are commensurable.

Remark. Types excluded in (1) are honest exceptions.

Andrei Rapinchuk (University of Virginia)
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Let

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**Remark.** Types excluded in (1) are **honest exceptions**.
If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable, and $K = K_{\Gamma_1} = K_{\Gamma_2}$, then $\text{rk}_K(G)(\Gamma_1) = \text{rk}_K(G)(\Gamma_2)$.

In particular, $\Gamma_1$ contains nontrivial unipotents $\iff \Gamma_2$ does.

Remark. Above results were proved in a more general context of $S$-arithmetic subgroups. (4) is valid for $S$-arithmetic lattices over any locally compact field $F$. 

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Theorem 4 (R. Garibaldi, A.R.)

Let $G_1$ and $G_2$ be absolutely almost simple $F$-groups of types $B_\ell$ and $C_\ell$ ($\ell \geq 3$); $\Gamma_i \subset G_i (F)$ be a Zariski-dense $(K, G_i)$-arithmetic subgroup, $i = 1, 2$.

Then $\Gamma_1$ and $\Gamma_2$ are weakly commensurable iff $G_1$ and $G_2$ are twins, i.e.

- $G_1$ and $G_2$ are both split over all nonarchimedean places of $K$;
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Together, Theorems 3 and 4 cover all situations where Zariski-dense $F$-arithmetic subgroups of absolutely almost simple groups can be weakly commensurable.
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Together, Theorems 3 and 4 cover all situations where Zarsiki-dense $S$-arithmetic subgroups of absolutely almost simple groups can be weakly commensurable.
1 Results

- First signs of eigenvalue rigidity
- Weakly commensurable arithmetic groups

2 Geometric applications

2.1 Generic elements

3 Division algebras with the same maximal subfields

- Algebraic and geometric motivations
- Genus of a division algebra
- Generalizations

4 Groups with good reduction

- Basic definitions and examples
- Finiteness Conjecture for Groups with Good Reduction
- Implications of the Finiteness Conjecture for Groups with Good Reduction
- Application to Nonarithmetic Riemann Surfaces

5 Some open problems
Let $G$ be a semi-simple algebraic $\mathbb{R}$-group; $G = G(\mathbb{R})$.

- $K$ - maximal compact subgroup of $G$;
- $X = K \backslash G$ - corresponding symmetric space.

- For $\Gamma \subset G$ discrete torsion free subgroup, $X_\Gamma = X / \Gamma$ - corresponding locally symmetric space.

$\operatorname{rk} X_\Gamma$: $\operatorname{rk} R G$.

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Now, let $G_1$ and $G_2$ be absolutely almost simple $\mathbb{R}$-groups, $\Gamma_i \subset G_i = G_i(\mathbb{R})$ be a discrete torsion-free subgroup, $X_{\Gamma_i}$ - corresponding locally symmetric space, $i = 1, 2$. Andrey Rapinchuk (University of Virginia)
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Now, let $G_1$ and $G_2$ be \textit{absolutely almost simple} $\mathbb{R}$-groups,

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Proposition (G. Prasad, A.R.)

Assume that $X_{\Gamma_1}$ and $X_{\Gamma_2}$ have finite volume (i.e., $\Gamma_1$ and $\Gamma_2$ are lattices). If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are length-commensurable, then $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.

For rank one locally symmetric spaces different from non-arithmetic Riemann surfaces, proof uses result of Gel'fond and Schneider (1934): if $\alpha$ and $\beta$ are algebraic numbers $\neq 0, 1$, then

$$\log\alpha \log\beta$$

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A finite volume locally symmetric space $\mathcal{X}_\Gamma$ of a simple real group is automatically *arithmetically defined* unless $\mathcal{X}$ is either real hyperbolic space $\mathbb{H}^n$ or complex hyperbolic space $\mathbb{H}^n_\mathbb{C}$.

(Margulis + Corlette + Gromov-Shoen)
Theorem 5

Let (as above)

• $\mathcal{X}_{\Gamma_1}$ be an arithmetically defined locally symmetric space,

• $\mathcal{X}_{\Gamma_2}$ be a locally symmetric space of finite volume.

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Corollary

Let $M_1$ and $M_2$ be arithmetically defined hyperbolic $d$-manifolds, where $d \neq 3$ is even or $\equiv 3 \pmod{4}$.

If $M_1$ and $M_2$ are length-commensurable, then they are commensurable.
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Let $M_1$ and $M_2$ be arithmetically defined hyperbolic $d$-manifolds, where $d \neq 3$ is even or $\equiv 3 \pmod{4}$.

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- A complex hyperbolic manifold cannot be length-commensurable to a real or quaternionic hyperbolic manifold, etc.
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\textbf{(T_i) Compositum $\mathcal{F}_1 \mathcal{F}_2$ has infinite transcendence degree over $\mathcal{F}_{3-i}$.}

So, $L(M_i)$ contains "many" elements that are algebraically independent from all elements of $L(M_{3-i})$. 
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Assume that \(G_1\) and \(G_2\) are of same type different from \(A_n\), \(D_{2n+1}\) \((n > 1)\) and \(E_6\), and that \(\Gamma_1\) and \(\Gamma_2\) are arithmetic.
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**Theorem 6**

Assume that $G_1$ and $G_2$ are of same type different from $A_n$, $D_{2n+1}$ ($n > 1$) and $E_6$, and that $\Gamma_1$ and $\Gamma_2$ are arithmetic.

Then either $M_1 = \mathcal{X}_{\Gamma_1}$ and $M_2 = \mathcal{X}_{\Gamma_2}$ are commensurable (in particular, length-commensurable),
Note that \((T_i)\) implies

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Using Shanuel’s conjecture, we prove

**Theorem 6**

*Assume that* \(G_1\) *and* \(G_2\) *are of same type different from* \(A_n, D_{2n+1}\) \((n > 1)\) *and* \(E_6\), *and that* \(\Gamma_1\) *and* \(\Gamma_2\) *are arithmetic. Then either* \(M_1 = \mathfrak{X}_{\Gamma_1}\) *and* \(M_2 = \mathfrak{X}_{\Gamma_2}\) *are commensurable (in particular, length-commensurable), or* \((T_i)\) *and* \((N_i)\) *hold for at least one* \(i \in \{1, 2\} \).
Theorem 7

Assume that both $G_1$ and $G_2$ are of one of following types: $A_n$, $D_{2n+1} \ (n > 1)$ or $E_6$, subgroups $\Gamma_1$ and $\Gamma_2$ are arithmetic, and in addition $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$. 
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Let $M_i$ ($i = 1, 2$) be quotients of real hyperbolic space $\mathbb{H}^{d_i}$ with $d_i \neq 3$ by a torsion free discrete subgroup $\Gamma_i$ of $G_i(\mathbb{R})$ where $G_i = \text{PSO}(d_i, 1)$. 
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Theorem 8

Assume that $G_1$ and $G_2$ are either of same type or one of them is of type $B_\ell$ and other of type $C_\ell$, and let $M_i = X_{\Gamma_i}$ ($i = 1, 2$) be arithmetically defined locally symmetric spaces.
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Results for isospectral locally symmetric spaces are derived from those for length-commensurable spaces.
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Let $A \in \text{GL}_n(F)$, and let $\chi_A(t) =$ characteristic polynomial of $A$. 
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3. Galois group of $\chi_A(t)$ over $F$ is symmetric group $S_n$. 

It is well-known how to construct irreducible polynomials of degree $n$ over $\mathbb{Q}$ with Galois group $S_n$ for any $n \geq 2$. Therefore, $\text{GL}_n(\mathbb{Q})$ contains $\mathbb{Q}$-generic elements.

We will now generalize notion of generic elements and existence theorem to arbitrary semi-simple groups.
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**Definition.**

1. $T$ is **generic over** $F$ if $\text{Im} \theta_T$ contains Weyl group $W(\Phi)$.
2. A semi-simple element $\gamma \in G(F)$ is **generic over** $F$ if $T := Z_G(\gamma)^{\circ}$ is a torus (i.e., $\gamma$ is regular) which is generic over $F$. 
A field $F \subset \mathbb{C}$ is **finitely generated** if it is obtained by adjoining to $\mathbb{Q}$ **finitely many** elements (algebraic or transcendental).
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Theorem 9 (G. Prasad, A.R.)

1. $\Gamma$ contains an $F$-generic element $\gamma \in \Gamma$ without components of finite order;

2. If $\gamma \in \Gamma$ is $F$-generic then there exists a finite index subgroup $\Delta \subset \Gamma$ such that $\gamma \Delta$ consists of $F$-generic elements.
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"Components" in (1) refer to almost direct product $G = G_1 \cdots G_r$ of simple groups. (2) means that set of $F$-regular elements is open in $\Gamma$ for profinite topology.
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(2) means that set of $F$-regular elements is **open** in $\Gamma$ for profinite topology.
For a semi-simple \( \mathbb{R} \)-group \( G \), an element \( \gamma \in G(\mathbb{R}) \) is \( \mathbb{R} \)-regular if number of eigenvalues of modulus 1 of \( \text{Ad}_G(\gamma) \), is minimal possible.
For a semi-simple $\mathbb{R}$-group $G$, an element $\gamma \in G(\mathbb{R})$ is $\mathbb{R}$-regular if number of eigenvalues of modulus 1 of $\text{Ad}_G(\gamma)$, is minimal possible.

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For a semi-simple \( R \)-group \( G \), an element \( \gamma \in G(\mathbb{R}) \) is \( R \)-regular if number of eigenvalues of modulus 1 of \( \text{Ad}_G(\gamma) \), is minimal possible.

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- If $F \subset \mathbb{R}$ then $\gamma$ in (1) can be selected to be $\mathbb{R}$-regular.

Such elements were used to study dynamics of actions, rigidity, Auslander problem about properly discontinuous groups of affine transformations, etc.
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Any dense subgroup of compact semi-simple Lie group contains a **Kronecker element**, i.e. an element such that closure of cyclic subgroup generated by it is a maximal torus.

This is **false** for dense subgroups of compact tori!
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2 Generic elements

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Any two isospectral/iso-length-spectral arithmetic Riemann surfaces are commensurable.

Underlying algebraic fact:
Let $D_1$ and $D_2$ be two quaternion division algebras over a number field $K$. If $D_1$ and $D_2$ have the same maximal subfields, then $D_1 \cong D_2$.

However, most Riemann surfaces are not arithmetic $\Rightarrow$ One needs to understand to what degree this fact extends to more general fields.

We will see a statement about arbitrary Riemann surfaces later, but first let us analyze the situation in detail.
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Let \( H = \{ x + iy | y > 0 \} \).

"Most" Riemann surfaces are of the form:
\[ M = H / \Gamma \]
where \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) is a discrete torsion free subgroup.

Some properties of \( M \) can be understood in terms of the associated quaternion algebra.

Let
\[
\pi: \text{SL}_2(\mathbb{R}) \to \text{PSL}_2(\mathbb{R})
\]

\[ \tilde{\Gamma} = \pi^{-1}(\Gamma) \subset M_2(\mathbb{R}) \]

Set \( A_\Gamma = \mathbb{Q}[\tilde{\Gamma}^2] \subset M_2(\mathbb{R}) \), \( \tilde{\Gamma}^2 \subset \tilde{\Gamma} \) generated by squares.
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Set $A_\Gamma = \mathbb{Q}[\tilde{\Gamma}^{(2)}] \subset \text{M}_2(\mathbb{R})$, $\tilde{\Gamma}^{(2)} \subset \tilde{\Gamma}$ generated by squares.
Division algebras with the same maximal subfields

Algebraic and geometric motivations

One shows: $A \Gamma$ is a quaternion algebra with center $K \Gamma = \mathbb{Q}(\text{trace } \gamma | \gamma \in \Gamma)^{(2)}$ (trace field).

(Note that for general Fuchsian groups, $K \Gamma$ is not necessarily a number field.)

• If $\Gamma$ is arithmetic, then $A \Gamma$ is the quaternion algebra involved in its description;
• In general, $A \Gamma$ does not determine $\Gamma$, but is an invariant of the commensurability class of $\Gamma$. 

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Division algebras with the same maximal subfields

Algebraic and geometric motivations

To a (nontrivial) semi-simple γ ∈ \tilde{\Gamma}(2)

there corresponds:

• geometrically: a closed geodesic c_γ ⊂ M, if γ ∼ ±(t_γ 0 0 t_γ −1) (t_γ > 1) then length ℓ(c_γ) = 2 \log t_γ;

• algebraically: a maximal etale subalgebra K_Γ[γ] ⊂ A_Γ.

Let M_i = H/Γ_i (i = 1, 2) be Riemann surfaces.

Assume that M_1 and M_2 are length-commensurable, i.e. Q·L(M_1) = Q·L(M_2).

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So, $A_{\Gamma_1}$ and $A_{\Gamma_2}$ share “lots” of maximal etale subalgebras.

(Not all – but we will ignore it for now ...)
Division algebras with the same maximal subfields

Algebraic and geometric motivations

For $M_1$ and $M_2$ to be commensurable, $A_{\Gamma_1}$ and $A_{\Gamma_2}$ must be isomorphic. Thus, proving that length-commensurable $M_1$ and $M_2$ are commensurable must involve answering a version of question (\ast), at least implicitly. We will see what can be said about $A_{\Gamma}$'s for length-commensurable Riemann surfaces.
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Algebra

Amitsur's Theorem

Let $D_1$ and $D_2$ be central division algebras over $K$. If $D_1$ and $D_2$ have the same splitting fields, i.e., for $F/K$ we have $D_1 \otimes_K F \cong M_n_1(F)$ $\iff$ $D_2 \otimes_K F \cong M_n_2(F)$, then $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\text{Br}(K)$.

Proof of Amitsur's Theorem uses generic splitting fields (function fields of Severi-Brauer varieties), which are infinite extensions of $K$.

What happens if one allows only splitting fields of finite degree, or just maximal subfields?
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Division algebras with the same maximal subfields

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Amitsur's Theorem is no longer true in this setting. (Counterexamples can be found using cubic algebras over number fields.)

This leads to question (∗) and its variations.

Question (G. Prasad-A.R.)

Are quaternion algebras over $K = \mathbb{Q}(x)$ determined by their maximal subfields?

• Yes – D. Saltman
• Same over $K = k(x)$, $k$ a number field (S. Garibaldi - D. Saltman)
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1 Results
- First signs of eigenvalue rigidity
- Weakly commensurable arithmetic groups
- Geometric applications

2 Generic elements

3 Division algebras with the same maximal subfields
- Algebraic and geometric motivations
- Genus of a division algebra
- Generalizations

4 Groups with good reduction
- Basic definitions and examples
- Finiteness Conjecture for Groups with Good Reduction
- Implications of the Finiteness Conjecture for Groups with Good Reduction
- Application to Nonarithmetic Riemann Surfaces

5 Some open problems
**Definition.**

Let $D$ be a finite-dimensional central division algebra over $K$. The genus of $D$ is 

$$\text{gen}(D) = \left\{ [D'] \in \text{Br}(K) \mid D' \text{ has same maximal subfields as } D \right\}$$

**Question 1.** When does $\text{gen}(D)$ reduce to a single element? (This means that $D$ is uniquely determined by maximal subfields.)

**Question 2.** When is $\text{gen}(D)$ finite?

Over number fields:
- Genus of every quaternion algebra reduces to one element;
- Genus of every division algebra is finite.

(Follows from Albert-Hasse-Brauer-Noether Theorem.)
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Theorem 10 (Stability Theorem, Chernousov-I. Rapinchuk, A.R.)

Let $\text{char } k \neq 2$. If $|\text{gen}(D)| = 1$ for every quaternion algebra $D$ over $k$, then $|\text{gen}(D')| = 1$ for any quaternion algebra $D'$ over $k$. 

• Same statement is true for division algebras of exponent 2.

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(Indeed, $[D^\text{op}] \in \text{gen}(D)$ and $[D^\text{op}] \neq [D].$)

• $\text{gen}(D)$ can be infinite. (For quaternions - J.S. Meyer (2014), for algebras of prime degree $p > 2$ - S.V. Tikhonov (2016).)

Construction yields examples over fields that are infinitely generated (in fact, huge).
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Theorem 10 (Stability Theorem, Chernousov-I. Rapinchuk, A.R.)

Let $\text{char } k \neq 2$. If $|\text{gen}(D)| = 1$ for every quaternion algebra $D$ over $k$, then $|\text{gen}(D')| = 1$ for any quaternion algebra $D'$ over $k(x)$.

- Same statement is true for division algebras of exponent 2.
- $|\text{gen}(D)| > 1$ if $D$ is not of exponent 2.
  (Indeed, $[D^{\text{op}}] \in \text{gen}(D)$ and $[D^{\text{op}}] \neq [D]$.)
- $\text{gen}(D)$ can be infinite.
  (For quaternions - J.S. Meyer (2014), for algebras of prime degree $p > 2$ - S.V. Tikhonov (2016).)

Construction yields examples over fields that are infinitely generated

( in fact, HUGE )
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- Start with nonisomorphic quaternion algebras $D_1$ and $D_2$ over $K$ (char $K \neq 2$) having a common maximal subfield.
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• If $D_1$ and $D_2$ already have same maximal subfields, we are done.

Otherwise, pick $K(\sqrt{d_1}) \hookrightarrow D_1$ such that $K(\sqrt{d_1}) \nleftrightarrow D_2$. 
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- If $D_1$ and $D_2$ already have same maximal subfields, we are done.

  Otherwise, pick $K(\sqrt{d_1}) \hookrightarrow D_1$ such that $K(\sqrt{d_1}) \not\hookrightarrow D_2$.

  (E.g., $\mathbb{Q}(\sqrt{11}) \hookrightarrow D_1$ but $\mathbb{Q}(\sqrt{11}) \not\hookrightarrow D_2$.)
• Find $K_1/K$ such that

1. $D_1 \otimes_K K_1 \not\cong D_2 \otimes_K K_1$;

2. $K_1(\sqrt{d_1}) \twoheadrightarrow D_2 \otimes_K K_1$. 

For $K_1$ one can take the function field of a quadric. In our example, $K_1$ is the function field of $-x^2_1 + 7x_2^2 + 7x_3^2 = 11x_4^2$. Then (2) is obvious, and (1) follows from the fact that $x_0^2 + x_1^2 - 21x_2^2 - 21x_3^2$ remains anisotropic over $K_1$. 

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Note that $\mathcal{K}$ is infinitely generated.
Theorem 11 (C+R):

Let $K$ be a finitely generated field. Then for any central division $K$-algebra $D$, the genus $\operatorname{gen}(D)$ is finite.

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Proofs of both theorems use analysis of ramification and info about unramified Brauer group.

ASIC FACT:

Let $v$ be a discrete valuation of $K$, and $n$ be prime to characteristic of residue field $K$ ($v$).

If $D_1$ and $D_2$ are central division $K$-algebras of degree $n$ having the same maximal subfields, then either both algebras are ramified or both are unramified. (When $n$ is divisible by char $K$ ($v$), we need some additional assumptions.)
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(When $n$ is divisible by $\text{char} K^{(\nu)}$, we need some additional assumptions)
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If $(n, \text{char } K^{(v)}) = 1$ or $K^{(v)}$ is perfect, there is a *residue map*

$$r_v: n\text{Br}(K) \longrightarrow H^1(\mathcal{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}),$$

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Given a set $V$ of discrete valuations of $K$, one defines corresponding **unramified Brauer group**:

$$\text{Br}(K)_V = \{ x \in \text{Br}(K) \mid x \text{ unramified at all } v \in V \}.$$
To prove Theorem 1 (Stability Theorem) we use: if $K = k(x)$ and $V$ = set of geometric places, then
\[ n\text{Br}(K)_V = n\text{Br}(k) \]
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  More recent argument works in all characteristics, but gives no estimate of size of \( \text{gen}(D) \).

  Earlier argument works when \( (n, \text{char } K) = 1 \), gives finiteness of \( n\Br(K)_V \) and estimate
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Question. Does there exist a quaternion division algebra $D$ over $K = k(\mathbb{C})$, where $\mathbb{C}$ is a smooth geometrically integral curve over a number field $k$, such that $|\text{gen}(D)| > 1$?

• The answer is not known for any finitely generated $K$.

• One can construct examples where $2\text{Br}(K)$ is "large."

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1. Results
   - First signs of eigenvalue rigidity
   - Weakly commensurable arithmetic groups
   - Geometric applications

2. Generic elements

3. Division algebras with the same maximal subfields
   - Algebraic and geometric motivations
   - Genus of a division algebra
   - Generalizations

4. Groups with good reduction
   - Basic definitions and examples
   - Finiteness Conjecture for Groups with Good Reduction
   - Implications of the Finiteness Conjecture for Groups with Good Reduction
   - Application to Nonarithmetic Riemann Surfaces

5. Some open problems
• To define the **genus of an algebraic group**, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.
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• Let \(G_1\) and \(G_2\) be semi-simple groups over a field \(K\).
• To define the **genus of an algebraic group**, we replace maximal subfields with **maximal tori** in the definition of genus of division algebra.

• Let $G_1$ and $G_2$ be semi-simple groups over a field $K$. $G_1$ & $G_2$ have **same isomorphism classes of maximal $K$-tori** if every maximal $K$-torus $T_1$ of $G_1$ is $K$-isomorphic to a maximal $K$-torus $T_2$ of $G_2$, and vice versa.
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• Let $G$ be an absolutely almost simple $K$-group.
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• Let $G$ be an absolutely almost simple $K$-group.

  $\text{gen}_K(G) =$ set of isomorphism classes of $K$-forms $G'$ of $G$ having same $K$-isomorphism classes of maximal $K$-tori.
Question 1'. When does $\text{gen}_K(G)$ reduce to a single element?
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Theorem 12 (G. Prasad-A.R.)

- Let $G$ be an absolutely almost simple simply connected algebraic group over a number field $K$.
  - $\text{gen}_K(G)$ is finite;
  - If $G$ is not of type $A_n$, $D_{2n+1}$ or $E_6$, then $|\text{gen}_K(G)| = 1$.

Conjecture.

- For $K = k(x)$, $k$ a number field, and $G$ an absolutely almost simple simply connected $K$-group with $|Z(G)| \leq 2$, we have $|\text{gen}_K(G)| = 1$;
- If $G$ is an absolutely almost simple group over a finitely generated field $K$ of "good" characteristic then $\text{gen}_K(G)$ is finite.
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(2) If \( G \) is an absolutely almost simple group over a finitely generated field \( K \) of “good” characteristic then \( \text{gen}_K(G) \) is finite.
Results for division algebras do not automatically imply results for $G = \text{SL}_m, D$. 

**Theorem 13 (C+R^2)**

1. Let $D$ be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where $k$ is a number field or a finite field of characteristic $\neq 2$. Then for $G = \text{SL}_m, D (m \geq 1)$ we have $|\text{gen}_K(G)| = 1$.

2. Let $G = \text{SL}_m, D$, where $D$ is a central division algebra over a finitely generated field $K$. Then $\text{gen}_K(G)$ is finite.
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Theorem 14 (C + $R^{2}$)

Let $K = k(C)$ where $C$ is a geometrically integral smooth curve over a number field $k$, and let $G$ be either

- $\text{Spin}_n(q)$, $q$ a quadratic form over $K$ and $n$ is odd,
- $\text{SU}_n(h)$, $h$ a hermitian form over quadratic extension $L/K$.

Then $\text{gen}_K(G)$ is finite.

Theorem 15 (C + $R^{2}$)

Let $G$ be a simple algebraic group of type $G_2$.

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Let $K = k(C)$ where $C$ is a geometrically integral smooth curve over a number field $k$, and let $G$ be either

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1. generic fiber $G \otimes_{\mathcal{O}_v} K_v$ is isomorphic to $G \otimes_K K_v$;

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Examples.

0. If $G$ is $K$-split then $G$ has a good reduction at any $v$, given by Chevalley construction.

1. $G = \text{SL}_1, A$ has good reduction at $v$ if there exists an Azumaya algebra $A$ over $O_v$ such that $A \otimes K \cong A \otimes O_v$ (in other words, $A$ is unramified at $v$).

2. $G = \text{Spin}_n(q)$ has good reduction at $v$ if $q \sim \lambda (a_1 x_1^2 + \cdots + a_n x_n^2)$ with $\lambda \in K^\times_v$, $a_i \in O^\times_v$ (assuming that char $K(v) \neq 2$).
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Groups with good reduction

Basic definitions and examples

General problem: Let $V$ be a set of discrete valuations of $K$. What can one say about those $K/K$-forms of $G$ that have good reduction at all $v \in V$? To make this problem meaningful one needs to specify $K$, $V$ and/or $G$.

Most popular case: $K$ field of fractions of Dedekind ring $R$, and $V$ consists of places associated with maximal ideals of $R$. 

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**Proposition**

Let $G$ be an absolutely almost simple simply connected algebraic group over a number field $K$, and assume that $V$ contains almost all places of $K$. Then number of $K$-forms of $G$ that have good reduction at all $v \in V$ is finite.
Case $R = k[x]$, $K = k(x)$, and $V = \{ v_{p(x)} \mid p(x) \in k[x] \text{ irreducible} \}$. 

Theorem (Raghunathan–Ramanathan, 1984)

Let $k$ be a field of characteristic zero, and let $G_0$ be a connected reductive group over $k$. If $G'$ is a $K$-form of $G_0 \otimes k K$ that has good reduction at all $v \in V$ then $G' = G'_0 \otimes k K$ for some $k$-form $G'_0$ of $G_0$.

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This was used to prove conjugacy of Cartan subalgebras in some infinite-dimensional Lie algebras.
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Finiteness Conjecture

Analysis of Finiteness conjecture for weakly commensurable groups has led us to consider higher-dimensional version of problem, never treated before.

Every finitely generated field $K$ has an almost canonical set of discrete valuations $V$ called divisorial.

Geometrically: Let $X$ be a normal model for $K$ of finite type over $\mathbb{Z}$. Then $v \in V$ correspond to prime divisors on $X$.

Algebraically: Choose an integrally closed $\mathbb{Z}$-subalgebra $A \subset K$ of finite type with fraction field $K$.

Then $v \in V$ correspond to height one prime ideals of $A$.  

• Two divisorial sets differ only in finitely many valuations.
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- $V_1$ of discrete valuations associated with irreducible polynomials in $\mathbb{Q}[x]$, i.e. with closed points of $A^1_\mathbb{Q}$ ("geometric" valuations).
Let $G$ be an absolutely simple simply connected algebraic group over a finitely generated field $K$, and $V$ be a divisorial set of valuations of $K$. Then the number of $K$-isomorphism classes of $(inner) K/K$-forms of $G$ that have good reduction at all $v \in V$ is finite. (One may need to assume that $\text{char } K$ is "good" for $G$.)

True if $K$ is a global field; $G$ is an inner form of type $A_n$; $G$ is spinor group of a quadratic form, certain unitary group, or a group of type $G_2$ over $K = k(C)$, function field of a curve over a global field $k$. 

Finiteness Conjecture for Groups with Good Reduction

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Let $G$ be an absolutely simple simply connected algebraic group over a finitely generated field $K$, and $V$ be a divisorial set of valuations of $K$. 

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Groups with good reduction

Results
- First signs of eigenvalue rigidity
- Weakly commensurable arithmetic groups
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Generic elements

Division algebras with the same maximal subfields
- Algebraic and geometric motivations
- Genus of a division algebra
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Groups with good reduction
- Basic definitions and examples
- Finiteness Conjecture for Groups with Good Reduction
- Implications of the Finiteness Conjecture for Groups with Good Reduction
- Application to Nonarithmetic Riemann Surfaces

Some open problems
Groups with good reduction

Implications of Finiteness Conjecture

• Finiteness of genus

Theorem 16 (C + R_{2})

Let $G$ be an absolutely almost simple simply connected group over $K$, and $v$ be a discrete valuation of $K$. Assume that $K_v$ is finitely generated, and $G$ has good reduction at $v$. Then every $G' \in \text{gen}_K(G)$ has good reduction at $v$, and reduction $G'(v) \in \text{gen}_{K_v}(G(v))$. 

Andrei Rapinchuk (University of Virginia)
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Groups with good reduction

Implications of Finiteness Conjecture

Corollary. Let $G$ be an absolutely almost simple simply connected algebraic group over a finitely generated field $K$, and $V$ be a divisorial set of places of $K$. There exists a finite subset $S \subset V$ (depending on $G$) such that every $G' \in \text{gen}_K(G)$ has good reduction at all $v \in V \setminus S$. Since $V \setminus S$ is also divisorial, finiteness of $\text{gen}_K(G)$ would follow from Finiteness Conjecture for Groups with Good Reduction.

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Since $V \setminus S$ is also divisorial, finiteness of $\text{gen}_K(G)$ would follow from Finiteness Conjecture for Groups with Good Reduction.
Given an algebraic group $G$ over a field $K$ with a set of valuations $V$, one considers the global-to-local map $\iota_{G,V}: H^1(K,G) \to \prod_{v \in V} H^1(K_v,G)$. BOREL, SERRE (1964): If $K$ is a number field and $V$ consists of almost all valuations of $K$, then $\iota_{G,V}$ is proper, i.e., preimage of any finite set is finite. Finiteness Conjecture for Groups with Good Reduction would imply properness of $\iota_{G,V}$ for any semi-simple adjoint group $G$ over an arbitrary finitely generated field $K$ and any divisorial set $V$. Andrei Rapinchuk (University of Virginia)
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Groups with good reduction

Implications of Finiteness Conjecture

• Finiteness Conjecture for Weakly Commensurable Subgroups

This is derived just as finiteness of genus using the following.

Theorem 17

Let $G$ be an absolutely almost simple simply connected algebraic group over a finitely generated field $K$ of characteristic zero, and let $V$ be a divisorial set of places of $K$.

Given a Zariski-dense subgroup $\Gamma \subset G(K)$ with trace field $K$, there exists a finite subset $V(\Gamma) \subset V$ such that any absolutely almost simple algebraic $K$-group $G'$ with the property that there exists a finitely generated Zariski-dense subgroup $\Gamma' \subset G'(K)$ weakly commensurable to $\Gamma$, has good reduction at all $v \in V \setminus V(\Gamma)$.

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Challenges in analysis of Finiteness Conjecture for Groups with Good Reduction:

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5. Some open problems
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Let \( A_\Gamma \) be the \textit{associated} quaternion algebra.
Groups with good reduction

Application to Nonarithmetic Riemann Surfaces

Question.

How does $A_\Gamma$ vary in families of length-commensurable (compact) Riemann surfaces?

If $\Gamma$ is arithmetic then the associated quaternion algebra remains the same for all Riemann surfaces that are length-commensurable to $M = H/\Gamma$.

What about non-arithmetic surfaces?

Replacing length-commensurability with much stronger relation of isospectrality we have:

Compact Riemann surfaces isospectral to a given one consist of finitely many isometry classes $\Rightarrow$ there are finitely many isomorphism classes of associated quaternion algebras.
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Groups with good reduction  

Application to Nonarithmetic Riemann Surfaces

Theorem 18

Let $M_i = H/\Gamma_i$ ($i \in I$) be a family of length-commensurable Riemann surfaces, where $\Gamma \subset \text{PSL}_2(\mathbb{R})$ is discrete and Zariski-dense. Then quaternion algebras $A_{\Gamma_i}$ ($i \in I$) split into finitely many isomorphism classes over common center (the trace field of all $\Gamma_i$'s).

Proof uses good reduction.

This is one of the first examples of application of techniques from arithmetic geometry to length-commensurable non-arithmetic Riemann surfaces.

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KIAS (Seoul)  

April, 2019  

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Theorem 18

Let $M_i = H/\Gamma_i$ ($i \in I$) be a family of length-commensurable Riemann surfaces, where $\Gamma_i \subset \text{PSL}_2(\mathbb{R})$ is discrete and Zariski-dense. Then quaternion algebras $A_{\Gamma_i}$ ($i \in I$) split into finitely many isomorphism classes over common center ($=$ trace field of all $\Gamma_i$'s).

Proof uses good reduction. This is one of the first examples of application of techniques from arithmetic geometry to length-commensurable non-arithmetic Riemann surfaces.
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Some open problems

Results
- First signs of eigenvalue rigidity
- Weakly commensurable arithmetic groups
- Geometric applications

Generic elements

Division algebras with the same maximal subfields
- Algebraic and geometric motivations
- Genus of a division algebra
- Generalizations

Groups with good reduction
- Basic definitions and examples
- Finiteness Conjecture for Groups with Good Reduction
- Implications of the Finiteness Conjecture for Groups with Good Reduction
- Application to Nonarithmetic Riemann Surfaces

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Example. Let $\Gamma = \text{SL}_2(\mathbb{Z})$, and set

$$u^+(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad u^-(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$
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Then for $m \geq 3$, subgroup

$$\Delta_m := \langle u^+(m), u^-(m) \rangle$$

is of infinite index in $\Gamma$, **but** is weakly commensurable to it.
Weak commensurability follows from inclusion

$$\Gamma(m^2) \subset \bigcup_{g \in \text{GL}_2(\mathbb{Q})} g \Delta_m g^{-1},$$

where

$$\Gamma(m^2) = \{ x \in \Gamma \mid x \equiv I_2 \pmod{m^2} \}$$

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So, we would like to propose the following
Problem 1. Let $G_1$ and $G_2$ be simple algebraic groups over a field $F$ of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be an arithmetic subgroups of rank $\geq 2$. 

If $\Gamma_2 \subset G_2(F)$ is a (finitely generated) Zariski-dense subgroup weakly commensurable to $\Gamma_1$, then is $\Gamma_2$ necessarily arithmetic? Do we need finite generation?

It is not even known if a subgroup $\Delta$ of $\Gamma = SL_n(Z)$, $n \geq 3$, weakly commensurable to $\Gamma$, necessarily has finite index.

Problem can be stated for higher-rank $S$-arithmetic subgroups, but is wide-open even for $SL_2(Z[1/p])$. 
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Problem 2. Let $G_1$ and $G_2$ be simple groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\Gamma_i$ be a (finitely generated) Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$. Assume that $\Gamma_1$ and $\Gamma_2$ are weakly commensurable. Does discreteness of $\Gamma_1$ imply discreteness of $\Gamma_2$? The answer is 'yes' for a nonarchimedean locally compact field $F$, but the archimedean case is open.
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Geometric version: Let $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ be length-commensurable locally symmetric spaces of finite volume.
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Geometric version: Let $X_{\Gamma_1}$ and $X_{\Gamma_2}$ be length-commensurable locally symmetric spaces of finite volume. Does compactness of $X_{\Gamma_1}$ imply compactness of $X_{\Gamma_2}$?

Recall: The answer is ‘yes’ if one space is arithmetically defined.
Problem 4. Develop notion of weak commensurability for Zariski-dense (and particularly arithmetic) subgroups of general semi-simple groups.
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Problem 5. For inner and outer forms of types $A_n \ (n > 1)$, $D_{2n+1} \ (n > 1)$ and $E_6$, construct examples of isospectral compact arithmetically defined locally symmetric spaces that are not commensurable.
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Currently, such construction is available only for inner forms of type $A_n$. 