UNIVERSAL OPERATOR GROWTH AND EMERGENT HYDRODYNAMICS IN QUANTUM SYSTEMS

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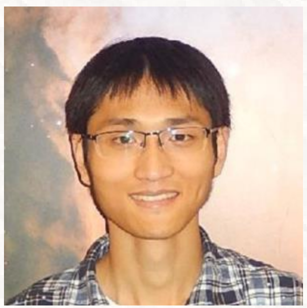
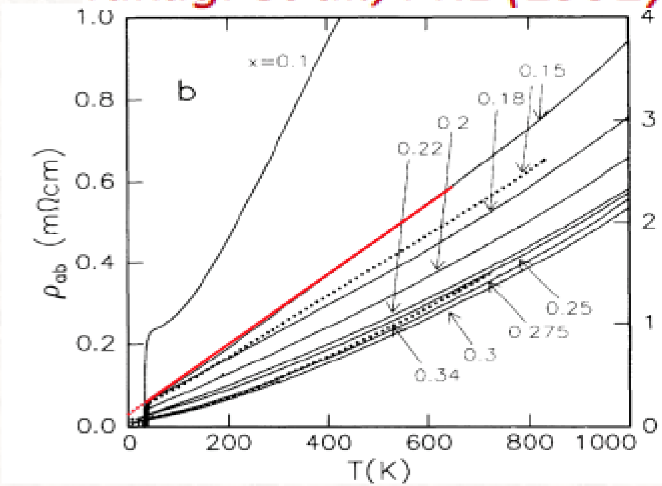


erc UQUAM ultracold quantum matter

How to compute dynamical properties of strongly coupled quantum matter ?

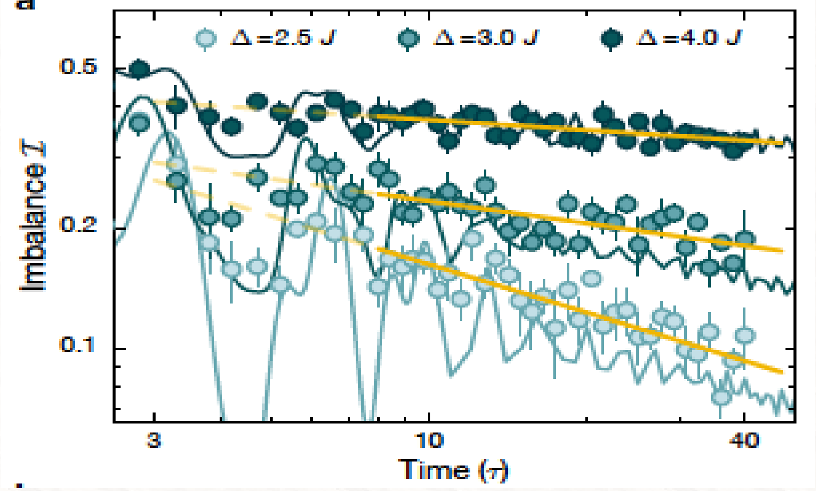
Unconventional transport:
(Strange metals)

Takagi et al., PRL (1992)



Relaxation following a quench in cold atomic systems:

Luechen et. al. PRL 2017



Often boils down to computing time dependent correlations:

$$C(t) = \langle \mathcal{O}(t)\mathcal{O} \rangle$$

Quantum Mechanics → ? → **Hydrodynamics**

Microscopic description of the system.

Example: Chaotic Ising Model

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i$$

Correlation functions:

$$C(t) = \langle \mathcal{O}(t,x)\mathcal{O}(0)\rangle$$

Hard Solution: Hamiltonian dynamics

$$\mathcal{O}(t) = e^{-iHt}\mathcal{O}e^{iHt}.$$

Exact and reversible dynamics.

Macroscopic description of quantum systems as classical PDEs.

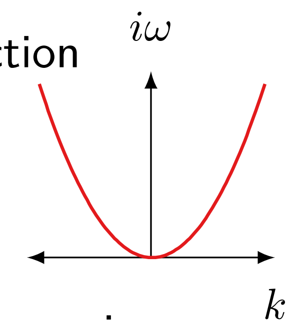
Example: Diffusion of energy

$$\frac{\partial}{\partial t}\varepsilon(t,x) = D\nabla^2\varepsilon(t,x) + \nabla f,$$

with D diffusion, f thermal noise.

Easy Solution: Green's function

$$G(i\omega, k) = \frac{1}{i\omega + Dq^2}$$



Approximate & irreversible dynamics.

Quantum evolution is hard to compute

Operators evolve in a huge Hilbert space:

$$-i\frac{d\hat{A}}{dt} = [H, \hat{A}] \quad \Longleftrightarrow \quad -i\frac{d|A)}{dt} = \mathcal{L}|A)$$

$$(A|B) = \mathrm{tr}(AB) \qquad |A(t)) = e^{-i\mathcal{L}t}|A)$$

Spin-1/2 models:

$$H = \sum_{\langle ij \rangle} h_{\alpha\beta}\, \sigma_i^{\alpha} \sigma_j^{\beta}$$

Basis of “Pauli strings:

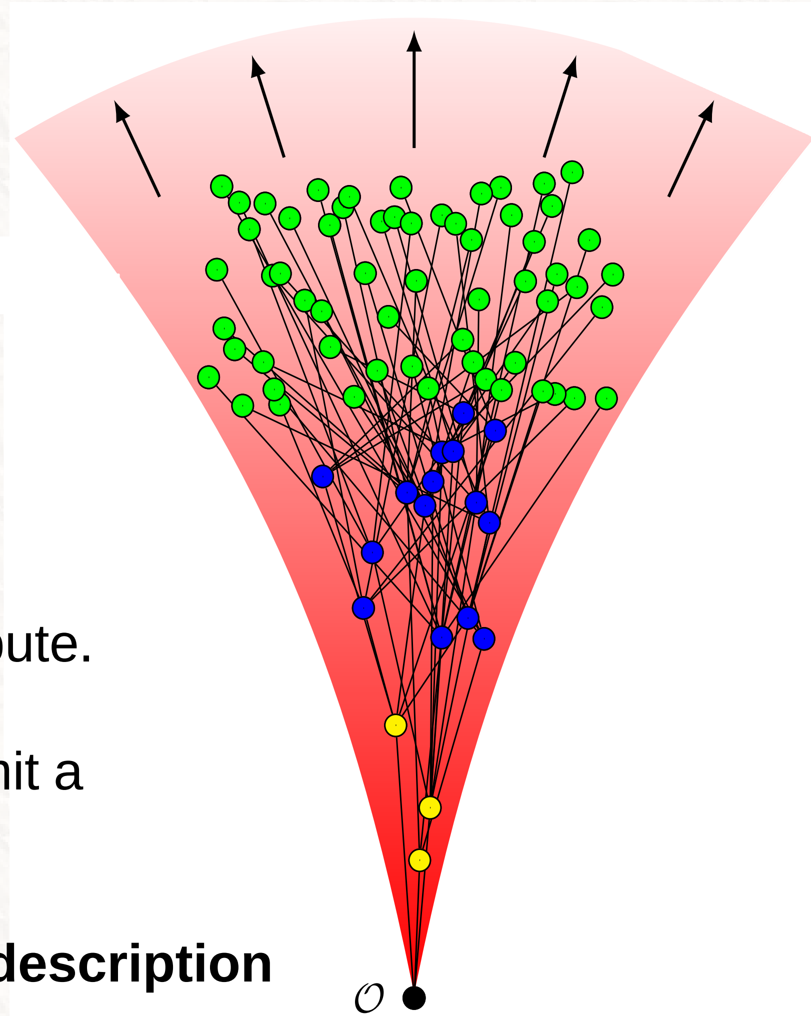
$$\sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \ldots \otimes \sigma^{\alpha_N} \equiv |\alpha)$$

$$\alpha_i = 0, 1, 2, 3$$

Quantum evolution is hard to compute

$$C(t) = (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O})$$

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O} = \mathcal{O} + (it)\mathcal{L}\mathcal{O} + (it)^2\mathcal{L}^2\mathcal{O} + \cdots$$



The basic idea

- Operators flow from simple to complex eventually becoming too complex to compute.
- Sufficiently complex operators should admit a *universal statistical description.*
- **Our goal is to formulate this universal description**

Outline

- Background: Krylov sub-space and operator complexity

- A hypothesis for universal operator growth

- Evidence for the hypothesis:
  (i) Numerical (Spin chains)
  (ii) Analytical (SYK models)

- Application: generalized notion of chaos and the bound

- Application: computation of transport coefficients

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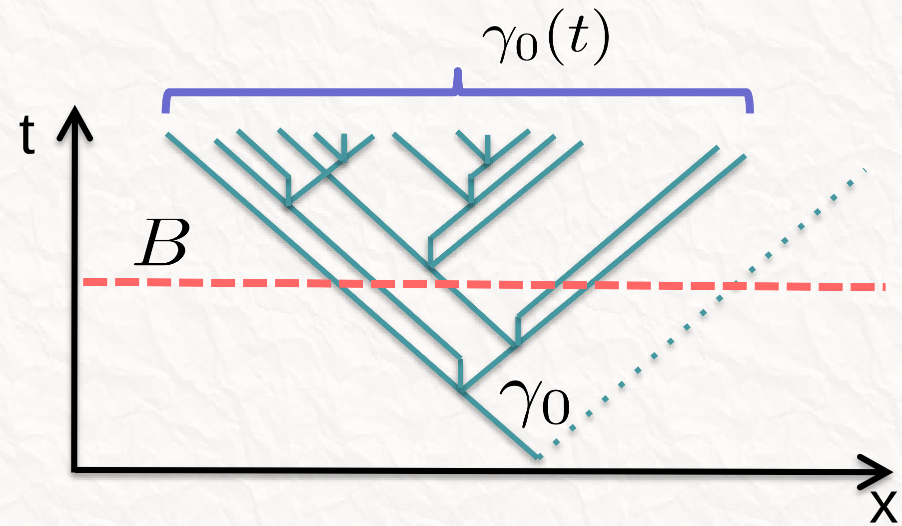
Out of time order correlations (OTOC): a measure for operator growth

$$F(t) \equiv \langle [A(t), B]^2 \rangle$$



Example:

$$A(t) = \gamma_0(t)$$

$$B = \sum_j i\gamma_{2j}\gamma_{2j+1}$$

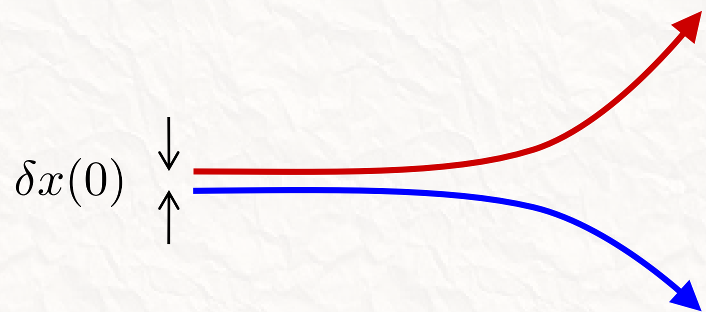
$$\gamma_i(\delta t) = \gamma_i + (\gamma_{i-1} + \gamma_{i+1})\delta t + \lambda\,(\gamma_{i-1}\gamma_i\gamma_{i+1})\,\delta t$$

If $\lambda<<1$ then: $$F(t) \sim \epsilon\, e^{\lambda t}$$

OTOC commonly used as a proxy of many-body quantum chaos.

Connection to classical chaos

$$\langle [\hat{x}(t), \hat{p}]^2 \rangle \longleftrightarrow \langle \{x(t), p\}^2 \rangle = \left\langle \left(\frac{dx(t)}{dx(0)} \right)^2 \right\rangle \sim e^{\lambda_L t}$$

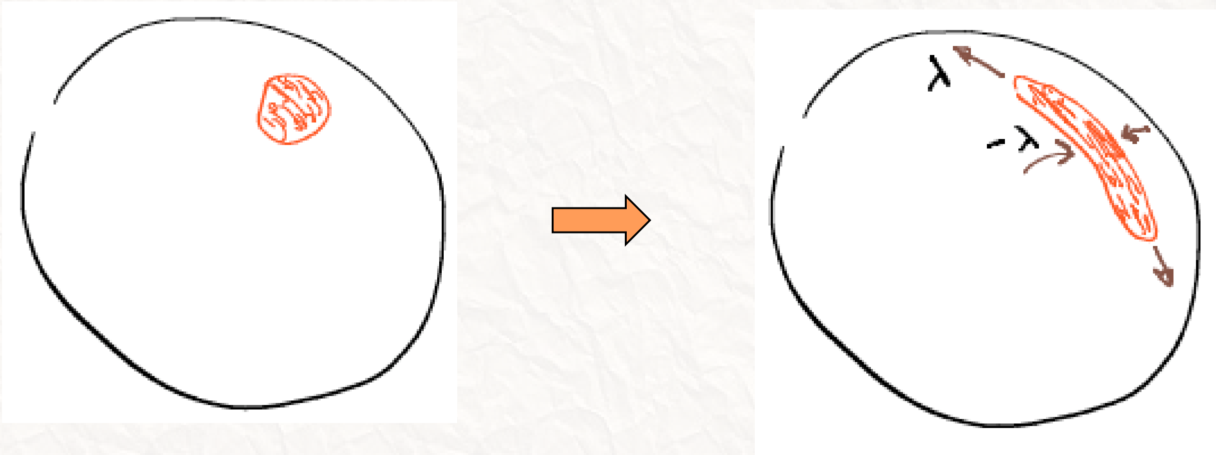


Measures sensitivity to initial conditions in a classical system

Classical "operator" complexity growth

Classical operators = Distribution functions on phase space

$$\frac{\partial f(x,p,t)}{\partial t} = \{\mathcal{H}, f\}$$



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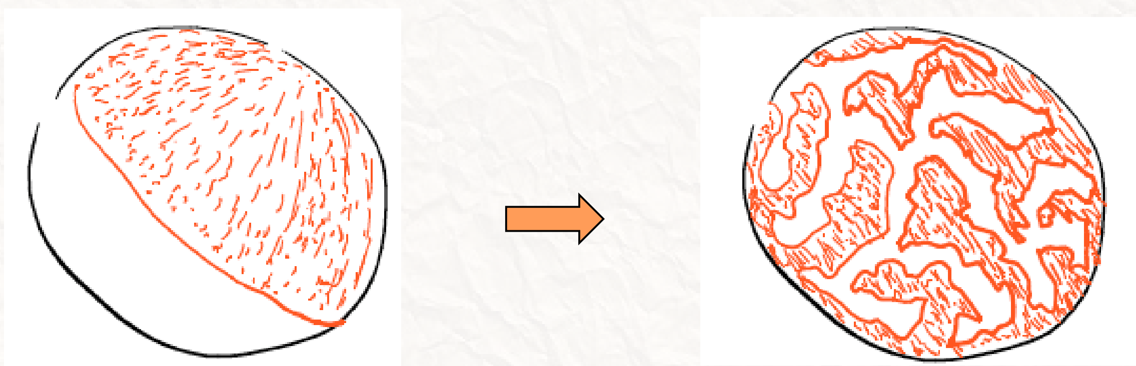


Lyapunov exponents quantify the rate at which increasingly fine structures on phase space are being generated.

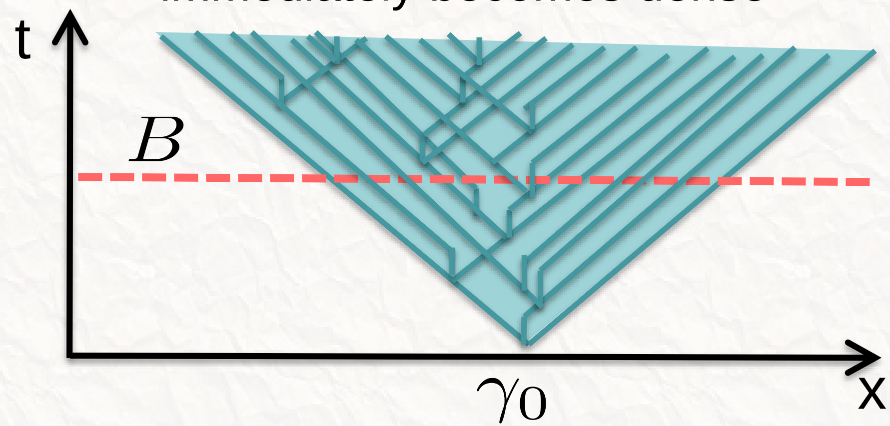
A problem with this measure of operator growth:

OTOCs do not necessarily grow exponentially in generic systems
(i.e. not large N or semiclassical)

ħ limits the resolution of structures on phase space.

$$\lambda^{-1} < t_{\text{saturation}}$$

At strong coupling the operator immediately becomes dense



$$F(t) \equiv \langle [A(t), B]^2 \rangle \sim vt$$

Another way to characterize operator complexity ?

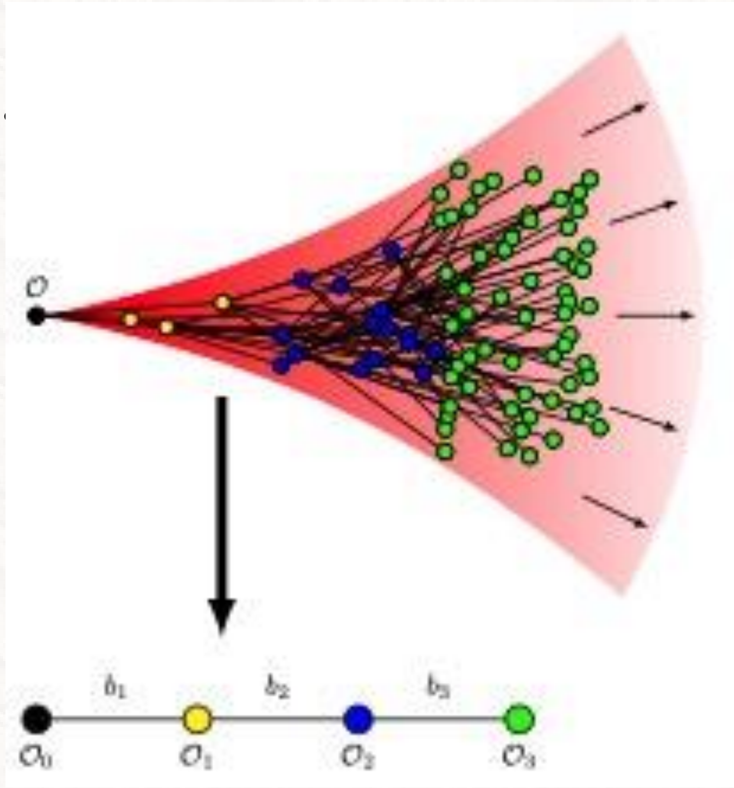
Krylov basis: folding the graph on a line

Generate orthonormal basis from successive application of $\mathcal{L}$

$$|O) \xrightarrow{\mathcal{L}} |O_1) \xrightarrow{\mathcal{L}} |O_2) \xrightarrow{\mathcal{L}} |O_3) \cdots$$

$$(\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

"*Recursion Coefficients*"

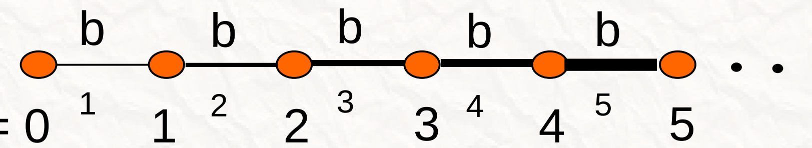


- Problem mapped to single-particle hopping on a semi-infinite chain !
- Krylov index ~ operator complexity

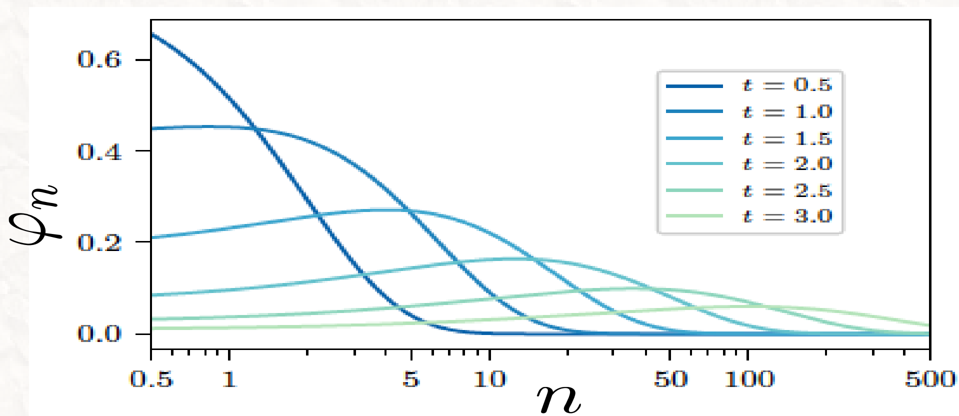
D.C. Mattis, 1981; Viswanath & Müller, *The Recursion Method*, 2008.

"Operator wavefunction" in Krylov space

$$\varphi_n(t) = (\mathcal{O}_n|\mathcal{O}(t))$$



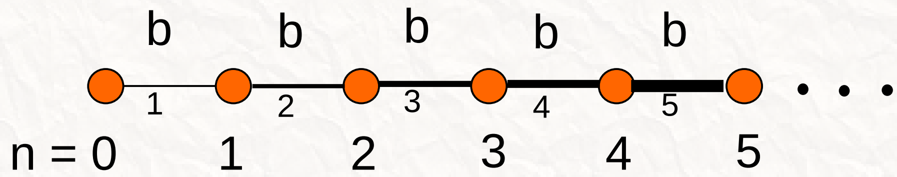
$$\partial_t \varphi_n = -b_{n+1}\varphi_{n+1} + b_n \varphi_{n-1}, \qquad \varphi_n(0) = \delta_{n0}$$



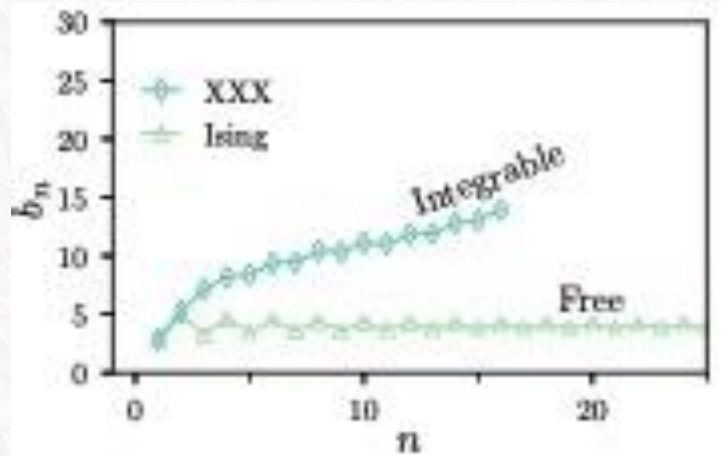
The autocorrelation function: $C(t) = \mathrm{tr}\left[\mathcal{O}(t)\mathcal{O}\right] = \varphi_0(t)$

"Krylov-complexity": $\langle n(t)\rangle = \sum_{n=0}^{\infty} |\varphi_n(t)|^2 n$

How do the recursion coefficients grow with n ?



Asymptotic	Growth Rate	System Type
$b_n \sim O(1)$	constant	Free models
$b_n \sim O(\sqrt{n})$	square-root	Integrable models
$b_n \sim$???	???	Chaotic models
$b_n \gtrsim O(n)$	superlinear	Disallowed



D.C. Mattis, 1981; Viswanath & Müller, *The Recursion Method*, 2008.

The hypothesis for generic models: linear growth of the reursion coefficients

Dan Parker, Xiangyu Cao, Thomas Scaffidi, EA arXiv:1812.08657

$$b_n = \alpha n + \beta, \quad n \to \infty$$

Logarithmic correction in 1d models:

$$b_n = \frac{\alpha n}{\log(n)} + \beta$$

(Theorem by Araki 1969 excludes faster growth. Thanks to Alex Advoshkin)

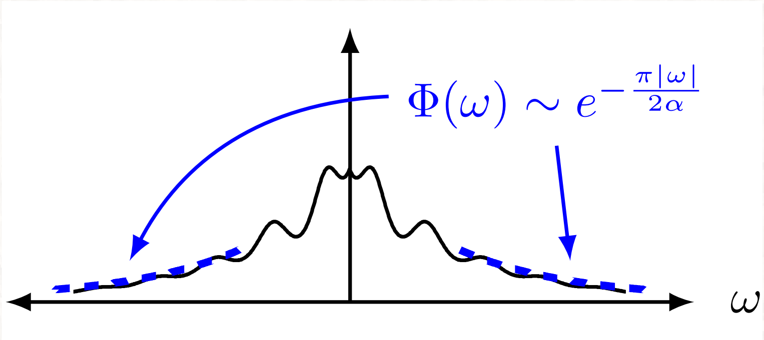
Faster asymptotic growth is not possible. We term α, the "growth rate" of the operator for reasons that will become clear.

Relation to spectral function

$$\Phi(\omega) = \int_{-\infty}^{\infty} dt C(t) e^{-i\omega t} = \int_{-\infty}^{\infty} dt \, \mathrm{tr}\left[\mathcal{O}(t)\mathcal{O}\right] e^{-i\omega t}$$

$$b_n = \alpha\, n + O(1) \iff \Phi(\omega) \sim e^{-\pi \frac{|\omega|}{2\alpha}}$$

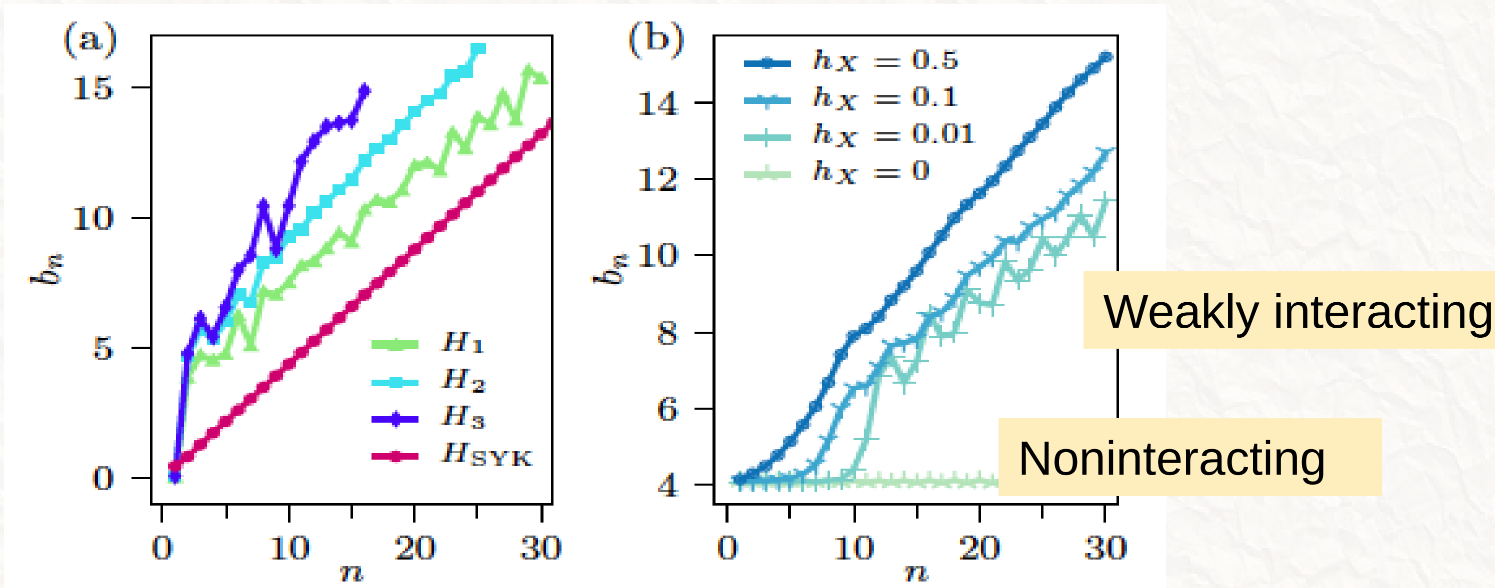
Or in 1d: $$b_n = \frac{\alpha\, n}{\log(n)} + O(1) \iff \Phi(\omega) \sim e^{-\pi \frac{|\omega|}{2\alpha} \log|\omega|}$$



The operator "growth rate" α, is directly related to the high frequency limit of the spectral function

The evidence

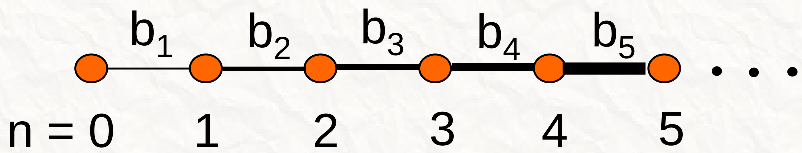
<u>Numerical:</u> Many distinct nonintegrable spin chains, SYK model

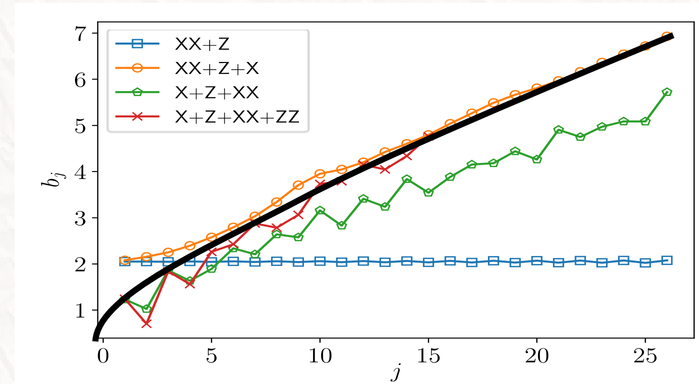


<u>Analytical:</u> SYK model in the limit of large q (infinite T)

$$b_n = J\,2^{(1-q)/2}\sqrt{q\,n(n-1)} + O(1/q) \qquad n \geq 2$$

Phenomenology of the semi-infinite chain

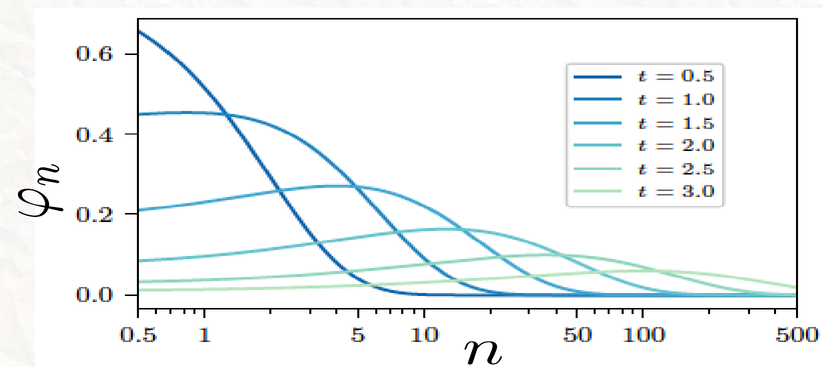




Exactly solvable "universal" model:

$$\widetilde{b}_n = \alpha\sqrt{n(n-1+\eta)} \xrightarrow{n\gg 1} \alpha n + \beta$$

$$\langle n(t)\rangle = \eta \sinh(\alpha t)^2 \sim \eta e^{2\alpha t}$$



A general property of models with linear growth of recursion coeffs.

Operators grow as fast as they can.

* in 1d: $\langle n(t)\rangle \sim e^{\sqrt{4\alpha t}}$

What is the relation to quantum chaos ?

Suggestive result from SYK model at infinite T

Both the recursion coefficients and the
Lyapunov exponent can be computed exactly.
(numerically for finite q and analytically in the large q limit)

q	2	3	4	7	10	∞
$\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
$\lambda_L/(2\mathcal{J})$	0	0.454	0.620	0.799	0.863	1

$$\mathcal{J} = J\, 2^{(1-q)/2} \sqrt{q}$$

Taken from: Roberts, Stanford and Streicher JHEP 2018

We will now establish a precise connection.

Define precise notion of complexity: Q-complexity

Positive semi-definite super-operator: $\mathcal{Q} = \sum_a q_a |q_a)(q_a|,\ q_a \geq 0$

Additional requirements : $(q_b|\mathcal{L}|q_a) = 0 \text{ if } |q_a - q_b| > b$

$$(q_a|\mathcal{O}) = 0 \text{ if } q_a < d$$

(i.e. L affects a bounded change of complexity and the initial operator complexity is small)

Q-complexity: $(\mathcal{Q})_t := (\mathcal{O}(t)|\mathcal{Q}|\mathcal{O}(t))$

Q-complexity - Examples

$$\mathcal{Q} = \sum_a q_a |q_a)(q_a|,\ q_a \geq 0 \qquad (\mathcal{Q})_t := (\mathcal{O}(t)|\mathcal{Q}|\mathcal{O}(t))$$

1. <u>"Krylov" complexity</u>

$$\mathcal{Q} = \sum_{n=0}^{\infty} n\,|\mathcal{O}_n)(\mathcal{O}_n|$$

2. <u>Operator size</u>
   q-eigenvectors are Pauli strings

$$\mathcal{Q}|IXYZI\cdots) = 3|IXYZI\cdots)$$

3. <u>OTOC</u>

$$\mathcal{Q} := \sum_i \mathcal{Q}_i, \quad (A|\mathcal{Q}_i|B) := ([V_i, A] \mid [V_i, B])$$

A generalized bound of chaos

Theorem*: the growth of any Q-complexity is bounded by the growth of the Krylov complexity

$$(\mathcal{Q})_t \leq C\,(n)_t$$

For proof see our paper arXiv:1812.08657

Implication: a generalized bound on chaos

$$\lambda_L \leq 2\alpha$$

Direct connection between the Lyapunov exponent and a property of an observable correlation function!

*Extension of the notion and of the bound to finite T is still a conjecture

Finite temperature

Freedom in defining the scalar product

To recover physical correlations: $(A|B) = \mathrm{tr}[\rho A^\dagger B] \equiv \langle A^\dagger B \rangle_\beta$

We will a use a different scalar product:

$$(A|B)_H := \mathrm{tr}[\sqrt{\rho}A^\dagger \sqrt{\rho}B] = \mathrm{tr}[\rho A^\dagger B(i\beta/2)]$$

➡ $$C_H(t) = \langle A^\dagger(t - i\beta/2)B\rangle_\beta = C(t - i\beta/2)$$

➡ $$\Phi_H(\omega) = \frac{\Phi(\omega)}{\sinh(\beta\omega/2)}$$

Finite T Chaos bound

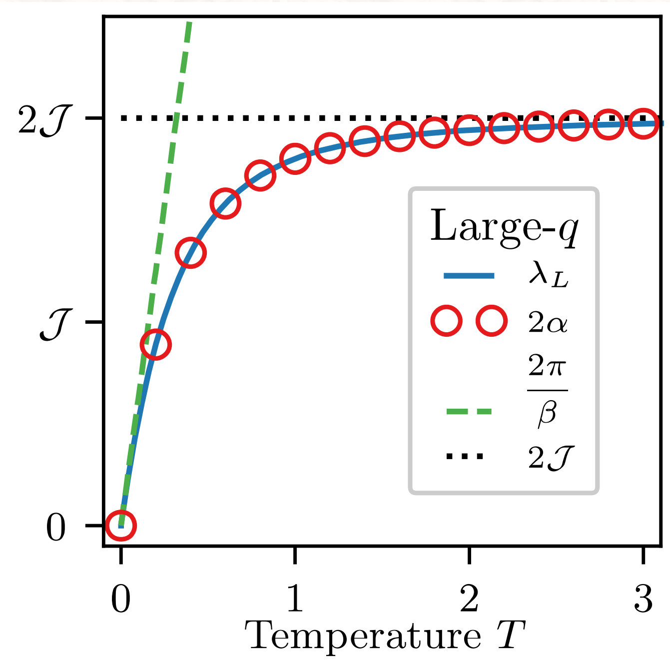
Low T limit

High frequency spectral function dominated by thermal suppression:

$$\Phi_H(\omega) = \frac{\Phi(\omega)}{\sinh(\beta\omega/2)} \to e^{-\omega/\Lambda - \omega/2T} \sim e^{-\omega/2T}$$

$$\lambda_L \leq 2\alpha = 2\pi T$$

Recover the bound of
Maldacena, Shenker & Stanford 2016



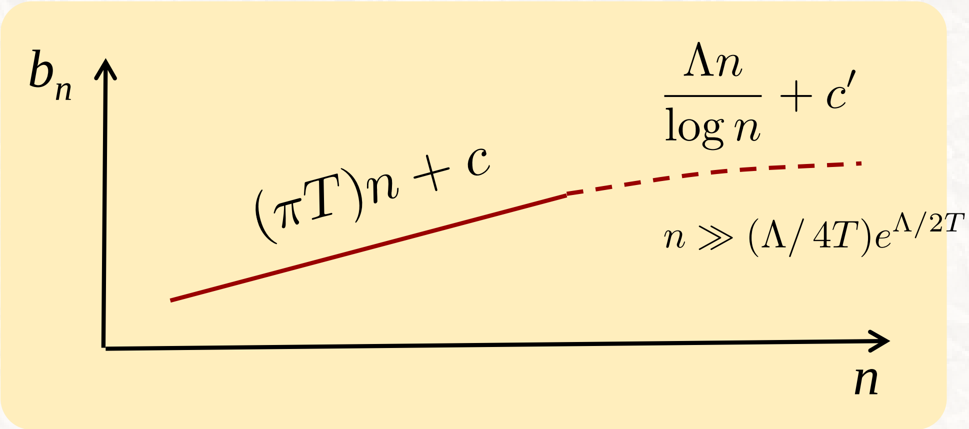
From low to high T in the large q SYK model

Tighter bound on chaos!
SYK model saturates the tighter bound.

Chaos bounds in the low T limit – 1d models

$$\Phi_H(\omega) \sim e^{-\frac{\omega}{\Lambda}\log\left(\frac{\omega}{\Lambda}\right)-\omega/2T}$$

Crossover frequency: $\omega_* \sim \Lambda e^{\Lambda/2T}$



$$\langle n(t)\rangle \sim e^{2\pi T t}$$

for $t < \Lambda/T^2$

So, even for 1d systems we recover: $\lambda_L \leq 2\alpha = 2\pi T$

Application to classical chaos

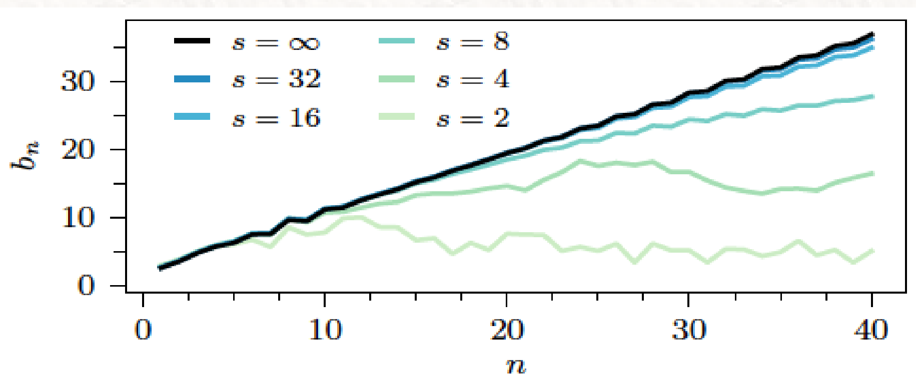
The framework carries over for classical dynamics with:

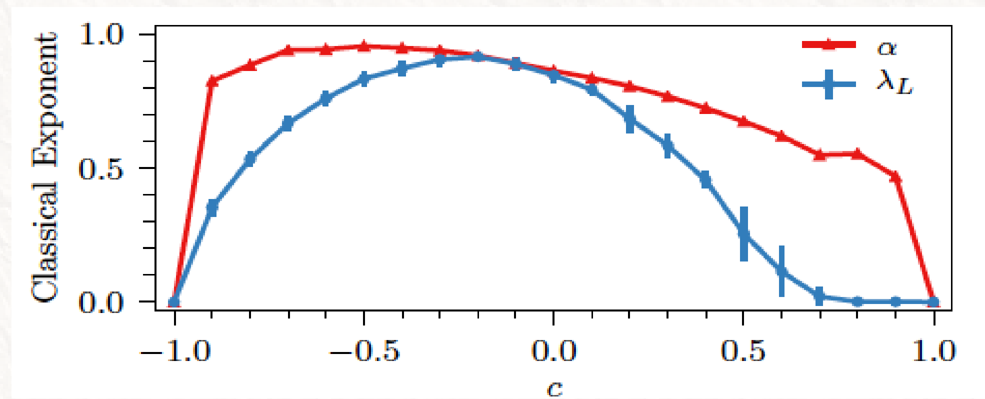
Liouvillian ⟶ $\mathcal{L} = i\{\mathcal{H}, \cdot\}$

Operators ⟶ Functions on the classical phase space

Compare α to λ_L Peres- Feingold model:

$$H_{\mathrm{FP}} = (1+c)\left[S_1^z + S_2^z\right] + 4s^{-1}(1-c)S_1^x S_2^x$$

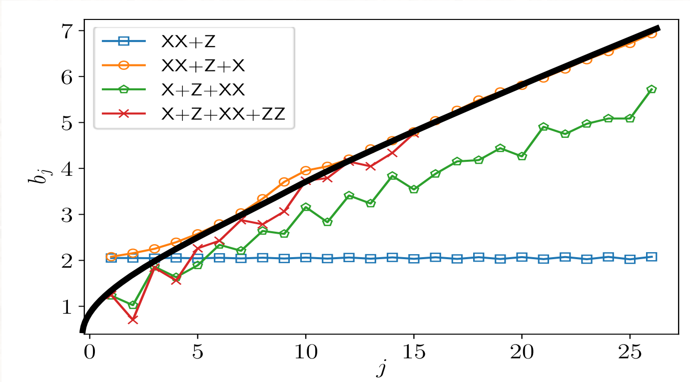




The two exponents coincide where the model is most chaotic.
Otherwise α appears as an upper bound on λ_L

Application: computing operator decay

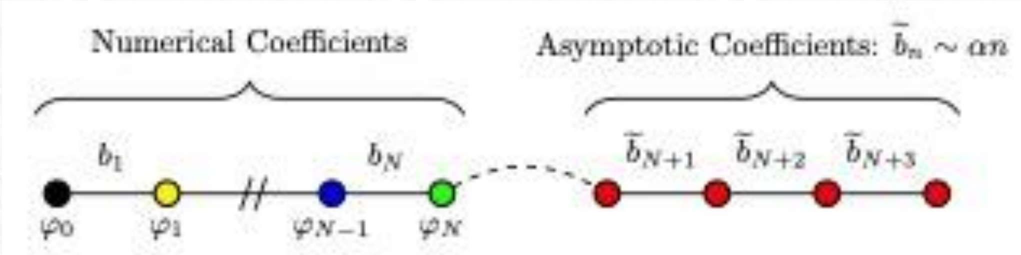
The basic idea:



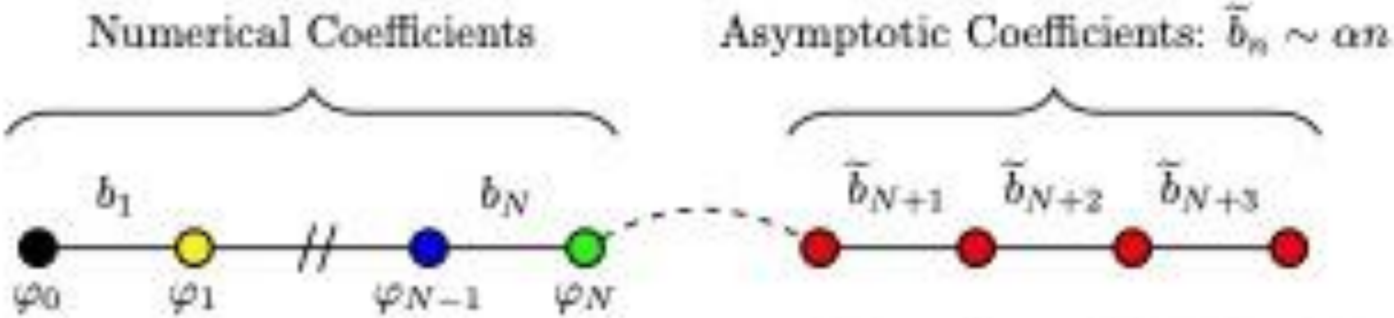
1. Compute the first m recursion coefficients numerically.

2. Complete with the fitted "universal" model at larger values of n

$$\widetilde{b}_n = \alpha\sqrt{n(n-1+\eta)} \xrightarrow{n\gg 1} \alpha n + \beta$$



3. Stich the small to large n wavefunctions to get an approximation of the decay of C(t).



$$L = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & b_2 & 0 \\ 0 & b_2 & 0 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix} \approx \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & b_N \\ 0 & 0 & b_n & \widetilde{G^{(N)}}(z) \end{pmatrix}$$

$$G(z) = \int dt\, e^{izt} \langle \mathcal{O}(t)\mathcal{O}(0) \rangle \approx \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{\cfrac{\ddots}{z - b_N^2 \widetilde{G^{(N)}}(z)}}}}$$

$$\widetilde{G^{(N)}}(z) = \Gamma(N+1)\Gamma\left(\frac{z+1}{2}\right) \times {}_2F_1\left(N+1, \frac{z+1}{2}, \frac{z+2N+3}{2}; -1\right)$$

Diffusion in the Chaotic Ising Model

Chaotic Ising Model

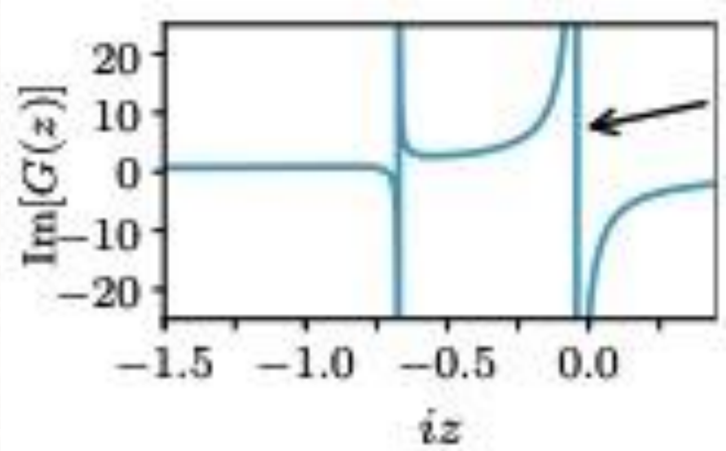
$$H = \sum_j X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j$$

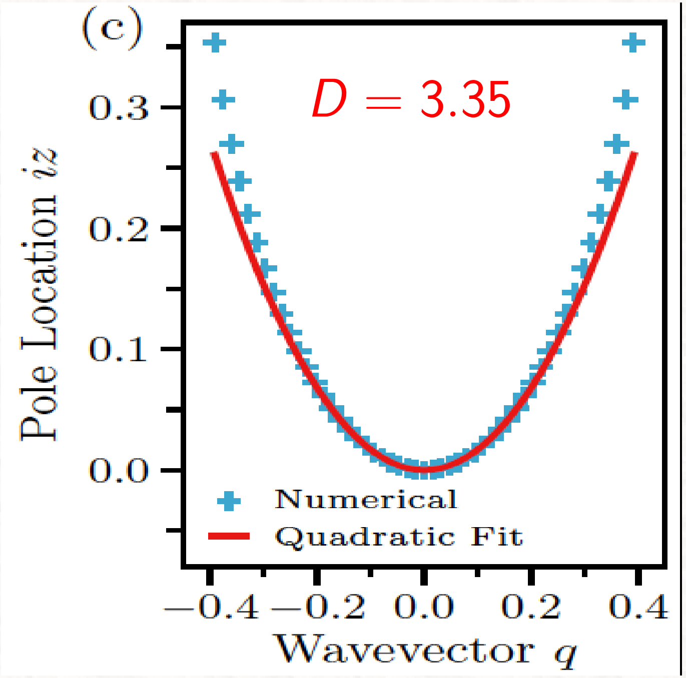
Initial operator at wavevector k:

$$\mathcal{O}_k = \sum_j e^{ikj} \left(X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j \right)$$

We see the dispersion relation for diffusion

$$\frac{d}{dt}\epsilon(t,x) = D\nabla^2 \epsilon(t,x).$$





Summary

- Hypothesis for universal operator dynamics supported by extensive evidence. Linear growth of Lanczos coefficients

$$b_n = \alpha n + \beta + o(1), \quad n \to \infty$$

- Implies exponential growth in operator complexity:

$$(n)_t \sim e^{2\alpha t} \qquad\qquad \lambda_L \leq 2\alpha$$

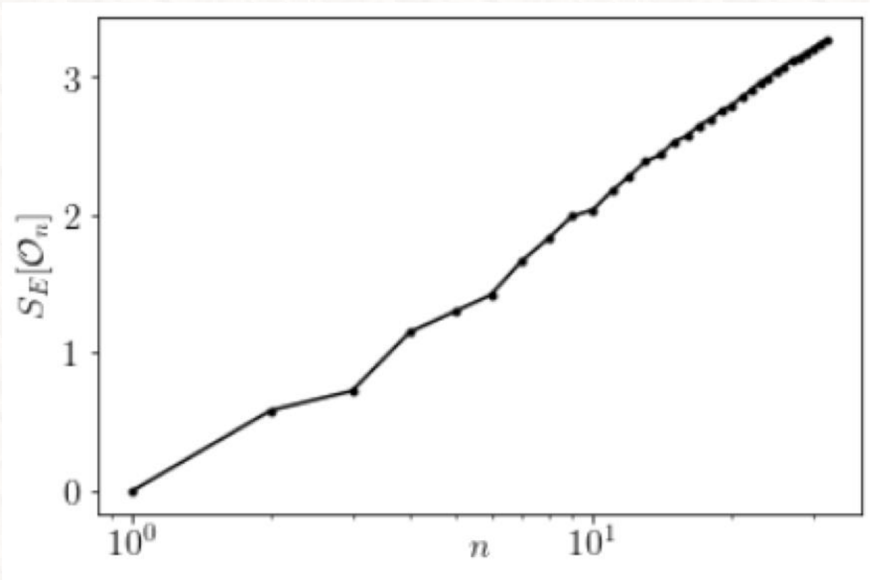
- The hypothesis enables a new numerical scheme to compute dynamical correlations and transport coefficients.

Extensions and outlook

- Does K-complexity provide a more fundamental notion of chaos? Measureable in a two point function (high frequency limit)!

- MPO calculation of the Krylov vectors and recursion coefficients

Operator entanglement appears to grow only logarithmically with n. **Why ?**



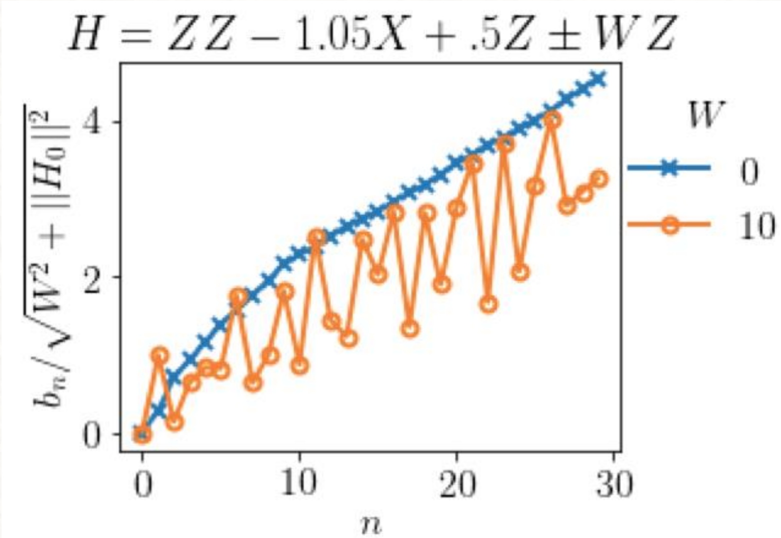
Extensions and outlook

- Behavior of Lanczos coefficients in integrable models?

- Growth of Lanczos coefficcients in MBL?
  Systematic derivation of disorder averaged LIOMs

  - Non trivial violation of the hypothesis

  - Systematic derivation of disorder averaged LIOMs



Extensions and outlook

- Finite size saturation
  - In a finite system, the exponential growth of K-complexity is saturated at a logarithmically long time.
  - Relation to Ehrenfest/Thouless time?

$$t^* \sim \ln L$$

Kos, Ljubotina, Prosen (2018);
Chan, De Luca, Chalker (2018)