IWASAWA THEORY AND THE EXACT BIRCH-SWINNERTON-DYER FORMULA

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The conjectural exact Birch-Swinnerton-Dyer formula for the order of the conjecturally finite Tate-Shafarevich group of any elliptic curve defined over a number field remains one of the great mysteries of number theory, and up until now has been proven in very few cases. In my lectures, I plan to discuss recent ongoing work with Yongxiong Li, Yukako Kezuka, and Ye Tian which we believe will prove both the finiteness of the Tate-Shafarevich group and the exact formula for its order for a wide class of elliptic curves with complex multiplication, whose complex L-series does not vanish at $s = 1$.

Let q be any prime $\equiv 7 \bmod 16$, $K = \mathbb{Q}(\sqrt{-q})$, and let H the Hilbert class field of K. Let A/H be Gross' $\mathbb{Q}$-curve with complex multiplication by the maximal order of K, and whose associated Hecke character has conductor $(\sqrt{-q})$. Let k be any non-negative integer, and let $R = r_1 \dots r_k$, where the r_i are distinct rational primes such that $r_i \equiv 1 \bmod 4$ and r_i is inert in K for $1 \leq i \leq k$. Let $A^{(R)}/H$ be the quadratic twist of A by $H(\sqrt{R})/H$, and write $L(A^{(R)}/H, s)$ for its complex L-series, which is entire by Deuring's theorem. Recently, Yongxiong Li and I found a proof by arguments from Iwasawa theory that always $L(A^{(R)}/H, 1) \neq 0$. When $k = 0$ this is an old result of D. Rohrlich, which he proved using complex methods, but there seems little hope of proving the more general statement by such complex methods.

In my lectures, I shall briefly discuss the proof of this non-vanishing theorem, and then go on to explain how ideas from Iwasawa theory enable one to show that the Tate-Shafarevich group of $A^{(R)}/H$ is indeed finite, and that its order is given by the exact Birch-Swinnerton-Dyer formula.