The power of string theory in $T\bar{T}$ and related theories

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Introduction

The recent progress on $T\bar{T}$ deformed CFT's and related systems started as work on 2d QFT's, but from the beginning there were hints that it is related to string theory.

For example, these theories have a Hagedorn spectrum in regions in coupling space where they make sense. Also, the free field Lagrangian $L = \partial X^i \bar{\partial} X^i, i = 1, \dots, d$, turns after $T\bar{T}$ deformation to the Nambu-Goto Lagrangian for a stretched string in $d + 2$ dimensions.

It is interesting to understand this relation further. The work I will describe today is intended to do that by using holography.

It leads to a rich interplay between the dynamics of deformed CFT's which do not approach an RG fixed point in the UV, and string theory in geometries that are not asymptotically AdS.

The insights flow both ways:

- String theory gives rise to an efficient tool for calculating observables such as the spectrum and torus partition sum of deformed CFTs, and provides hints for the behavior of others, such as correlation functions and entanglement entropy.

- Understanding the deformed CFTs leads to new holographic dualities in string backgrounds such as asymptotically linear dilaton spacetimes, that figure prominently in string theory as geometries near NS5-branes and singularities of Calabi-Yau manifolds.

- Understanding these dualities leads to a relation between properties of the spectrum of the deformed CFT’s and properties of the dual geometries.

- It also provides new tests of AdS_3/CFT_2 duality.

Today, I will mainly discuss two technical problems from the above point of view. The starting point of the analysis will be a CFT_2 with a left-moving current J and a right-moving current $\bar{J}$.

We will deform the Lagrangian of the CFT by

$$\delta L = tT\bar{T} + \mu_+ J\bar{T} + \mu_- T\bar{J}$$

and ask what is the spectrum of the deformed theory and what is its torus partition sum.

Of course, once we know the spectrum, we in principle know the torus partition sum. Indeed, for a torus of size R and modulus $\zeta = \zeta_1 + i\zeta_2$, the partition sum can be interpreted as the trace

$$\mathcal{Z}(\zeta, \overline{\zeta}, R) = \mathrm{Tr} e^{-2\pi\zeta_2 R\mathcal{E} + 2\pi i \zeta_1 RP}$$

where the trace runs over all the eigenstates of the Hamiltonian and momentum operator.

However, it is hard to do anything useful with the resulting expression. E.g., one knows on general grounds that the partition sum should be modular invariant, with R transforming non-trivially, but it is hard to see this using the explicit formula.

It is also hard to use this expression to study the partition sum on small tori.

So, it is useful to have a more user friendly expression for the partition sum, and string theory provides one.

The basic ideas we will use are the following:

- **Universality of the spectrum:** from the definition of the deformed theories, it is clear that the deformed energies have the form

$$\mathcal{E}_n = \mathcal{E}(E_n, p_n, q_n, \overline{q}_n, t, \mu_\pm)$$

Here n labels the different states, $\mathcal{E}_n$ is the deformed energy, E_n is the corresponding undeformed energy, and the other variables are the momentum and charges, and the coupling.

The key point is that the function $\mathcal{E}$ is universal, i.e. the deformed energies are given by a universal function of the undeformed charges, and the couplings.

Thus, if we have a class of examples of these theories that is understood, we can use it to find the universal function $\mathcal{E}$, and use it for any other deformed CFT.

- AdS_3/CFT_2 provides precisely such a class of examples, via the dynamics of long strings in AdS_3. Therefore, we can read off from it the function $\mathcal{E}$.

A very similar story can be told for the torus partition sum: if we know the torus partition sum $Z(\zeta,\bar{\zeta},R)$ as a functional of the undeformed partition sum $Z_{cft}(\tau,\bar{\tau})$ for a class of examples, the resulting expression must be true for any undeformed CFT.

Again, AdS_3/CFT_2 provides precisely such a class, and gives an expression for the partition sum which is guaranteed to be general due to universality.

The goal of the rest of this talk is to explain these comments and the resulting expressions for the spectrum and partition sum. At the end, I will comment on other observables.

Aspects of string theory on AdS_3

The worldsheet theory for a string propagating on AdS_3 is described by a WZW model on the SL(2,R) group manifold. It is invariant under left and right moving SL(2,R) current algebras at level k. This level is related to the radius of curvature of AdS_3, $R_{AdS} = \sqrt{k} l_s$.

The current algebra plays an important role in analyzing the spectrum, symmetries, and correlation functions of this theory.

The AdS/CFT correspondence relates string theory on AdS_3 to a 2d CFT living on the boundary. For pure NS H-flux, this theory has the following properties:

- It has an SL(2,R) invariant vacuum, the NS vacuum in the dual CFT. In the bulk, it corresponds to AdS_3 in global coordinates. The Ramond vacuum of the boundary CFT corresponds in the bulk to the M=0 BTZ black hole.

- In the NS sector, the spectrum of excitations includes discrete states followed by a continuum of long string states. The continuum starts at dimension $\sim \frac{k}{2}$ (MO 2001). In the Ramond sector one finds a continuum of long strings above a gap of order $\frac{1}{k}$.

- The theory on a single long string in global AdS_3 was analyzed in SW 1999 . In string theory on $AdS_3 \times N$, it is described by a sigma model on $\mathcal{M}_{6k}^{(L)} = \mathbb{R}_\phi \times \mathcal{N}$. The central charge of this theory is $c_M = 6k$.

- The theory on R_ϕ has a linear dilaton with slope $Q^{(L)} = (k-1)\sqrt{\frac{2}{k}}$

- Example: for string theory on $AdS_3 \times S^3 \times T^4$, which has (4,4) superconformal symmetry, $\mathcal{M}_{6k}^{(L)} = \mathbb{R}_\phi \times SU(2)_k \times T^4$.

- The effective string coupling on the long string, $g_s \sim \exp(Q^{(L)}\phi)$, increases as the string moves towards the boundary. Thus, the physics of long strings is strongly coupled there. This observation plays an important role in studying the theory (GKRS 2005).

An interesting open problem is what is the boundary CFT dual to a given AdS_3 vacuum. There is strong evidence for the conjecture that long strings are described by the symmetric product CFT

$$\left(\mathcal{M}_{6k}^{(L)}\right)^{p}/S_p$$

Some of the evidence for this is:

- Matrix string theory logic (Motl, DVV, 1997): if the theory living on a single string winding around a circle is M, the symmetric product theory M^N/S_N provides a description of the Hilbert space of N free strings. Untwisted sector states describe N strings each winding once around the circle; Z_w twisted states describe strings with winding w; general states of n strings with windings $(w_1, \cdots, w_n)$ are described in terms of conjugacy classes of the symmetric product.

The long strings on AdS_3 are weakly coupled in a wide range of positions in the radial direction, so the symmetric product description should be a good description of their dynamics in this regime.

- Spectrum of long strings: for example, in the Ramond sector (massless BTZ) one finds

$$E_{L,R} = \frac{1}{w}\left[-\frac{j(j+1)}{k} + N_{L,R} - \frac{1}{2}\right]$$

$$E_L = \frac{R}{2}\,(E+P)\,; \quad E_R = \frac{R}{2}\,(E-P)\,; \quad P \in \frac{1}{R}Z$$

$$j = -\frac{1}{2} + is; \quad s \in \mathbb{R}$$

R=radius of boundary circle; $N_{L,R}$ = left and right-moving excitation levels; s ∝ radial momentum of the string.

To make contact with $\left(\mathcal{M}_{6k}^{(L)}\right)^p/S_p$, one notes that in a symmetric product CFT, states in the Z_w twisted sector have energies

$$E_L = h_w - \frac{kw}{4}; \quad E_R = \overline{h}_w - \frac{kw}{4}$$

$w = 1$ corresponds to the original CFT. For every state with dimension h_1 in that CFT, there is a state in the Z_w twisted sector, with dimension h_w given by

$$h_w = \frac{h_1}{w} + \frac{k}{4}\left(w - \frac{1}{w}\right)$$

The string spectrum on the previous page has the same form.

- Another piece of strong evidence for the long string CFT comes from the study of irrelevant deformation such as $T\bar{T}$, to which we turn next.

Some solvable irrelevant deformations of string theory on AdS_3

String theory on AdS_3 always contains an operator $D(x,\bar{x})$, found in KS 1999, that has many properties in common with the operator $T\bar{T}$ in the boundary CFT. In particular, it is quasi-primary and has the same OPEs with the stress-tensor as $T\bar{T}$. However, while $T\bar{T}$ is a double trace operator, $D(x,\bar{x})$ is a single trace one. In this context this means that $D(x,\bar{x})$ corresponds to a vertex operator integrated over the worldsheet, while $T\bar{T}$ is a product of such integrated vertex operators.

It is natural to ask what happens when we add the operator D to the Lagrangian of the boundary CFT. One can show (IGK 2017) that this is the same as adding to the worldsheet Lagrangian

$$\delta\mathcal{L} = \lambda J^{-}\overline{J}^{-}$$

Where J^{-} is the left-moving $SL(2,R)$ current, whose zero mode gives the boundary Virasoro generator L_{-1}.

Already at this level, we see a number of parallels between the deformation corresponding to D and $T\bar{T}$ in the spacetime CFT.

- The coupling has boundary dimension (-1,-1) in both cases.

- Both are solvable. In the case of D, this is related to the fact that while the deformation is irrelevant in the boundary CFT, it is exactly marginal on the worldsheet. Being a current-current deformation, it is solvable.

To understand better the relation between the two types of deformations, we appeal to our discussion of string theory on AdS_3 above. In particular, we would like to understand the action of this deformation on long string states.

We saw that these states are described by a symmetric product CFT. Thus, we need to understand the role of the coupling λ in that CFT.

Some useful observations:

❖ Local single trace operators in string theory correspond in the symmetric product to operators in the block, symmetrized over all copies. Thus, we can write

$$D(x,\overline{x})=\sum_{j=1}^{p} D_j(x,\overline{x})$$

where D_j is a dimension (2,2) quasi-primary operator living in the j'th copy of $\mathcal{M}_{6k}^{(L)}$.

- A natural candidate for D_j is $D_j = T_j\overline{T_j}$. In general, there could be other operators with the right properties, but since the operator D is universal, the above identification is likely. In fact one can prove that it is correct, by showing that in the string theory there in general aren't other operators with the right properties.

- If the boundary CFT has (2,2) supersymmetry, one can show that D is a top component of a superfield, whose bottom component has dimension (1,1), and corresponds (by similar arguments) to $\sum_i J_i\overline{J_i}$.

Therefore, we conclude that turning on the single trace coupling λ in string theory on AdS_3 corresponds in the symmetric product

$$\left(\mathcal{M}_{6k}^{(L)}\right)^{p} / S_p$$

to a $T\bar{T}$ deformation in the block.

In particular, the energies of long string states must be deformed at finite λ in the same way as they would be via the above $T\bar{T}$ deformation.

The above discussion has a generalization that involves Kac-Moody currents. As before, we start with string theory on $AdS_3 \times N$, but now we take N to contain a left-moving conserved current $K(z)$.

From this data, one can construct a holomorphic conserved current in the boundary CFT, $K(x)$. Following the discussion of $T\bar{T}$ before, we can deform the boundary theory by the double trace operator $K(x)\bar{T}(\bar{x})$, or by the dimension (1,2) single trace operator $A(x,\bar{x})$, which corresponds in the bulk string theory to the worldsheet deformation

$$\delta\mathcal{L} = \lambda' K(z)\overline{J}^{-}$$

Many of the comments above are valid here as well. In particular:

- While the single trace deformation by $A(x,\bar{x})$ is irrelevant in the boundary CFT, it is truly marginal (and solvable) in the bulk string theory.

- In the symmetric product theory of the long strings, it corresponds to a $K\bar{T}$ deformation of the block of the symmetric product, $A(x,\overline{x}) = \sum_{j=1}^{p} K_j(x)\overline{T}_j(\overline{x})$.

- Thus, one can use the string theory construction to calculate the deformed energies in $K\bar{T}$ deformed CFT.

Another prediction of the string theory picture is that if the dual CFT has left and right-moving currents $K(x), \overline{K}(\bar{x})$, we can add to the boundary Lagrangian an arbitrary combination of the single trace $T\overline{T}, K\overline{T}, T\overline{K}$ couplings and get a solvable theory. On the worldsheet, this corresponds to studying deformed Lagrangians of the form

$$\delta\mathcal{L} = \lambda J^{-}\overline{J}^{-} + \lambda' K(z)\overline{J}^{-} + \lambda'' J^{-}\overline{K}$$

Since the perturbation is truly marginal and has a current-current form, the theory remains solvable for all values of the couplings.

Why would we want to consider the theory for general couplings?

- In general, to better understand any theory, it is useful to understand all its deformations.

- Two special cases that were already considered are single trace $T\bar{T}$, $\lambda' = \lambda'' = 0$, and single trace $K\bar{T}$, $\lambda = \lambda'' = 0$. In the latter, it was found that the energies are complex for all real values of the couplings. Therefore, to construct a sensible theory that includes the couplings λ', λ'', one presumably needs to turn on λ as well.

- In the cases that were already considered, there was an interesting correspondence between the appearance of complex energies in the deformed field theory, and the appearance of pathologies, like closed timelike curves and curvature singularities in the deformed bulk geometry. It is interesting to extend the discussion to a wider class of theories, to see what features of the geometry are related to the appearance of complex energies in the field theory.

Therefore, we will next:

1. Determine the deformed background at a general point in the three dimensional coupling space discussed above.

2. Determine the spectrum at a general point in coupling space using the ideas described above.

3. Compare the conditions for the spectrum to be real to those for not having pathologies in the dual bulk geometry.

4. Compute the corresponding torus partition sum.

The deformed bulk geometry

Consider first the sigma model on $AdS_3 \times S^1$ (the S^1 gives the left and right-moving U(1)'s that we will need for the construction). In Poincare coordinates, one has

$$S = \frac{k}{2\pi} \int d^2 z \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma + \frac{1}{k}\partial y\bar{\partial}y \right)$$

The exact deformed action takes the form

$$S(\lambda,\epsilon_+,\epsilon_-)=\frac{k}{2\pi}\int d^2z\left(\partial\phi\overline{\partial}\phi+h\overline{\gamma}\overline{\partial}\gamma+\frac{2\epsilon_+h}{\sqrt{k}}\partial y\overline{\partial}\gamma+\frac{2\epsilon_-h}{\sqrt{k}}\partial\overline{\gamma}\overline{\partial}y+\frac{f^{-1}h}{k}\partial y\overline{\partial}y\right)$$

where

$$\begin{aligned} f^{-1} =& \lambda+e^{-2\phi}, \\ h^{-1} =& \lambda-4\epsilon_+\epsilon_-+e^{-2\phi} \end{aligned}$$

$\epsilon_\pm$ provide a parametrization of the space labeled by λ', λ'' above.

To study the geometry, it is convenient to KK reduce this background on the circle labeled by y. This gives

$$
\begin{aligned}
ds^2 =& k\left(d\phi^2 + hd\gamma d\overline{\gamma} - fh\left(\epsilon_+ d\gamma + \epsilon_- d\overline{\gamma}\right)^2\right) , \\
e^{2\Phi} =& g_s^2 e^{-2\phi} h \ , \\
B_{\gamma\overline{\gamma}} =& \frac{kh}{2} \ , \qquad A_\gamma = 2\sqrt{k}\epsilon_+ f \ , \qquad A_{\overline{\gamma}} = 2\sqrt{k}\epsilon_- f
\end{aligned}
$$

The spectrum

We start with the massless BTZ background

$$\mathcal{L} = 2k(\partial\phi\overline{\partial}\phi + e^{2\phi}\partial\overline{\gamma}\overline{\partial}\gamma)$$

where the boundary coordinates are identified as follows:

$$\gamma = \gamma_1 + \gamma_0 \ , \qquad \overline{\gamma} = \gamma_1 - \gamma_0 \ ; \qquad \gamma_1 \simeq \gamma_1 + 2\pi R$$

Since we are interested in long strings propagating at large ϕ, it is convenient to rewrite this in the Wakimoto form

$$\mathcal{L}_W = \partial\phi\overline{\partial}\phi + \beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma} - \exp\left(-\sqrt{\frac{2}{k}}\phi\right)\beta\overline{\beta} - \sqrt{\frac{2}{k}}\widehat{R}\phi$$

where we have rescaled some of the fields. This description is free at large ϕ, which makes it easier to analyze the spectrum. One finds the spectrum quoted earlier in this talk.

Now we turn on the general current-current perturbations discussed above. Using standard techniques one finds that the energies change as follows:

$$h_\omega - \frac{k\omega}{4} = E_L + \frac{\widehat{\lambda}}{2\omega} E_L E_R - \frac{1}{\omega}\left(\widehat{\epsilon}_+ q_L E_R + \widehat{\epsilon}_- q_R E_L + \frac{1}{2}(\widehat{\epsilon}_+ E_R + \widehat{\epsilon}_- E_L)^2\right)$$
$$\overline{h}_\omega - \frac{k\omega}{4} = E_R + \frac{\widehat{\lambda}}{2\omega} E_L E_R - \frac{1}{\omega}\left(\widehat{\epsilon}_+ q_L E_R + \widehat{\epsilon}_- q_R E_L + \frac{1}{2}(\widehat{\epsilon}_+ E_R + \widehat{\epsilon}_- E_L)^2\right)$$
$$h_\omega - \overline{h}_\omega = E_L - E_R = n,$$

The hatted variables are dimensionless couplings. For $\hat{\epsilon} = 0$ one finds the familiar $T\bar{T}$ spectrum. Turning on only $\hat{\epsilon}_+$, gives the $J\bar{T}$ one.

For $w = 1$, i.e. for states in the block of the symmetric product, one has

$$ER = E_L + E_R = n + \frac{1}{2A}\left(-B - \sqrt{B^2 - 4AC}\right)$$

with

$$\begin{aligned}
A =& \frac{1}{4}\left((\widehat{\epsilon}_+ + \widehat{\epsilon}_-)^2 - \widehat{\lambda}\right), \\
B =& -1 + \widehat{\epsilon}_+ q_L + \widehat{\epsilon}_- q_R + n\widehat{\epsilon}_-(\widehat{\epsilon}_+ + \widehat{\epsilon}_-) - \frac{\widehat{\lambda} n}{2}, \\
C =& 2\left(\overline{h}_1 - \frac{c}{24} - \frac{q_R^2}{2}\right) + (q_R + n\widehat{\epsilon}_-)^2;
\end{aligned}$$

A few comments on these formulae:

- The quantity C appearing in the square root is non-negative. Therefore, if $A \leq 0$, all energies are real. This is a generalization of known constraints. For $\hat{\epsilon}_{\pm} = 0$ it is the statement that the $T\bar{T}$ coupling must be positive. For $\hat{\lambda} = 0$ it is the statement that complex energies appear for all real $\hat{\epsilon}_{\pm}$.

- For a given negative A, the high energy spectrum has Hagedorn entropy, $S = \beta_H E$, with inverse Hagedorn temperature

$$\beta_H = 4\pi R\sqrt{\frac{c|A|}{12}}$$

In particular, $\beta_H \to 0$ as $|A| \to 0$. The theory has multiple scales.

- The limiting theory obtained as $|A| \to 0$ is interesting. The Hagedorn temperature vanishes. The behavior of the energies depends on the sign of B. States with $B > 0$ decouple (their energies go to infinity) . States with $B < 0$ survive; their energies approach $ER = n + \frac{2C}{|B|}$.

- For fixed charges, the entropy has, roughly, Cardy behavior, but the coefficient of $\sqrt{E}$ depends on the charges (through B). This is an intermediate behavior between Cardy and Hagedorn. Our construction provides a bulk dual to this theory.

Torus partition sum

As mentioned above, now that we have the spectrum of the theory, in principle we can compute the partition sum by explicitly evaluating the trace. However, using our construction one can get a more useful formula.

The strategy is as before:

- For the large class of examples arising from long strings in AdS_3, we compute the partition sum of the long strings by compactifying Euclidean time on a circle. This gives the partition sum of the deformed CFT as a functional of that of the undeformed CFT.

- Using universality, the expression one gets must be true for any undeformed CFT.

Results:

For $T\bar{T}$ deformed CFT one finds

$$\mathcal{Z}(\zeta,\bar{\zeta},\lambda) = \frac{\zeta_2}{2\lambda}\int_{\mathcal{H}_+}\frac{d^2\tau}{\tau_2^2}e^{-\frac{\pi}{2\lambda\tau_2}|\tau-\zeta|^2}Z_{\rm cft}(\tau,\bar{\tau})$$

where ζ is the modulus of the target space torus, τ the modulus of the worldsheet one, $\mathcal{H}_+$ the upper half plane, and λ the $T\bar{T}$ coupling evaluated at the KK scale.

(see also Dubovsky et al)

For the general deformed CFT with all three couplings turned on:

$$\mathcal{Z}(\zeta,\overline{\zeta},\lambda,\epsilon_+,\epsilon_-) = \int_{\mathcal{H}_+} d^2\tau \int_{\mathcal{C}} d^2\chi \; I(\zeta,\overline{\zeta},\tau,\overline{\tau},\chi,\overline{\chi}) Z_{\text{inv}}(\tau,\overline{\tau},\chi,\overline{\chi})$$

where the kernel of the integral I is

$$I(\zeta,\overline{\zeta},\tau,\overline{\tau},\chi,\overline{\chi}) = \frac{\zeta_2}{4\epsilon_+\epsilon_-\tau_2^3} e^{-\frac{\pi h|\tau-\zeta|^2}{2\tau_2} - \frac{\pi}{2\epsilon_+\epsilon_- h\tau_2}\left(\overline{\chi}+\epsilon_- h(\overline{\tau}-\overline{\zeta})\right)(\chi-\epsilon_+ h(\tau-\zeta))}$$

and

$$Z_{\text{inv}}(\tau,\overline{\tau},\nu,\overline{\nu}) = Z_{\text{cft}}(\tau,\overline{\tau},\nu,\overline{\nu}) e^{\pi(\nu-\overline{\nu})^2/2\tau_2}$$

$$h = \frac{1}{\lambda - 4\epsilon_+\epsilon_-}$$

One can check explicitly that the integral formula gives the correct spectrum of the deformed CFT (by construction), and is modular invariant:

$$\mathcal{Z}\left(|c\zeta+d|^2 h, \frac{\epsilon_+}{c\overline{\zeta}+d}, \frac{\epsilon_-}{c\zeta+d}, \frac{a\zeta+b}{c\zeta+d}, \frac{a\overline{\zeta}+b}{c\overline{\zeta}+d}\right) = \mathcal{Z}(h, \epsilon_+, \epsilon_-, \zeta, \overline{\zeta})$$

The string construction is also useful for calculating the partition sum for the symmetric product of deformed CFT's. It gives the following. Define

$$\mathcal{Z}_N(\zeta,\overline{\zeta},\lambda) = \frac{\zeta_2}{2\lambda} \sum_{m_i,w_i|\, N} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} e^{-S_{\{m_i,w_i\}}} Z_\perp(\tau,\overline{\tau})$$

where

$$S_{\{m_i,w_i\}} = \frac{\pi(w_2\zeta+w_1)(w_2\overline{\zeta}+w_1)}{2\lambda\tau_2}\left(\tau - \frac{m_2\zeta+m_1}{w_2\zeta+w_1}\right)\left(\overline{\tau} - \frac{m_2\overline{\zeta}+m_1}{w_2\overline{\zeta}+w_1}\right)$$

is the action of a map from the worldsheet torus with modulus τ to the target space torus with modulus ζ, and $N = w_1m_2 - w_2m_1$ is the winding number of the map.

Then define

$$\Xi(\zeta,\overline{\zeta},\lambda,\eta) \equiv \exp\left[\sum_{N=1}^{\infty} \eta^N \mathcal{Z}_N(\zeta,\overline{\zeta},\lambda)\right]$$

One can show that expanding

$$\Xi(\zeta,\overline{\zeta},\lambda,\eta) = 1 + \sum_{N=1}^{\infty} \eta^N \Xi_N(\zeta,\overline{\zeta},\lambda),$$

$\Xi_N(\zeta,\overline{\zeta},\lambda)$ is the partition sum of the symmetric product of N $T\bar{T}$ deformed CFTs.

The proof relies on our construction and on the fact that

$$\mathcal{Z}_N(\zeta,\overline{\zeta},\lambda) = T_N\left[\mathcal{Z}_1(\zeta,\overline{\zeta},\lambda)\right]$$

where T_N is the Hecke operator

$$T_N[\mathcal{Z}_1(\zeta,\overline{\zeta},\lambda)] = \frac{1}{N}\sum_{a,b,d}\mathcal{Z}_1\left(\frac{a\zeta+b}{d},\frac{a\overline{\zeta}+b}{d},\frac{\lambda}{d^2}\right)$$

associated with the matrix

$$T_N = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \qquad a,b,d\in\mathbf{Z} \qquad ad=N, \qquad 0\le b<d\right\}$$

Discussion

So far, most of the concrete results on these theories had to do with their spectrum. It is interesting to consider other observables, such as deformed correlation functions, entanglement entropy, etc. For these, the string theory construction is not directly applicable, since as we discussed earlier, the string deformation only agrees with $T\bar{T}$ and generalizations for the spectrum in the long string sector.

Nevertheless, these deformations are closely related, so it is interesting to compute such observables in string theory. This was done, for correlation functions in AGIK (2017), and for the EE in CGIK (2018).

Some insights were obtained from these analyses, but more work is needed.

The main conclusion from the study of these theories so far is that they are very rich and understanding them better is likely to teach us a lot about field theory, string theory and holography.