Anomaly inflow and the η-invariant

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- [1909.08775] with Edward Witten

Introduction

How to characterize anomalies in general ?

Perturbative anomaly:

χ : chiral fermion in d-dimensions

δ : gauge transformation $\delta A_\mu = D_\mu \alpha$

$$Z_\chi = \int [D\chi] e^{-S} \qquad \left(S = \bar{\chi}\gamma^\mu(\partial_\mu + A_\mu)\chi \right)$$

$\delta Z_\chi \neq 0$: anomaly

Introduction

Perturbatively, a well-known characterization is by the anomaly descent equations.

Gauge transformation	$\delta \log Z_\chi = i \int I_d^{(1)}$	: anomaly in d-dimensions
Going to 1-higher dim.	$dI_d^{(1)} = \delta I_{d+1}^{(0)}$	: Chern-Simons in $d+1$-dimensions
Going to 2-higher dim.	$dI_{d+1}^{(0)} = I_{d+2}$	: Anomaly $d+2$-form in $d+2$-dimensions

Introduction

The gauge invariant anomaly $d+2$-form

$$I_{d+2} \sim \mathrm{tr}(F)^{(d+2)/2} + \ldots \qquad \left(F = dA + A^2\right)$$

Perturbatively, it contains all information about anomalies.

Chern-Simons $d+1$-form

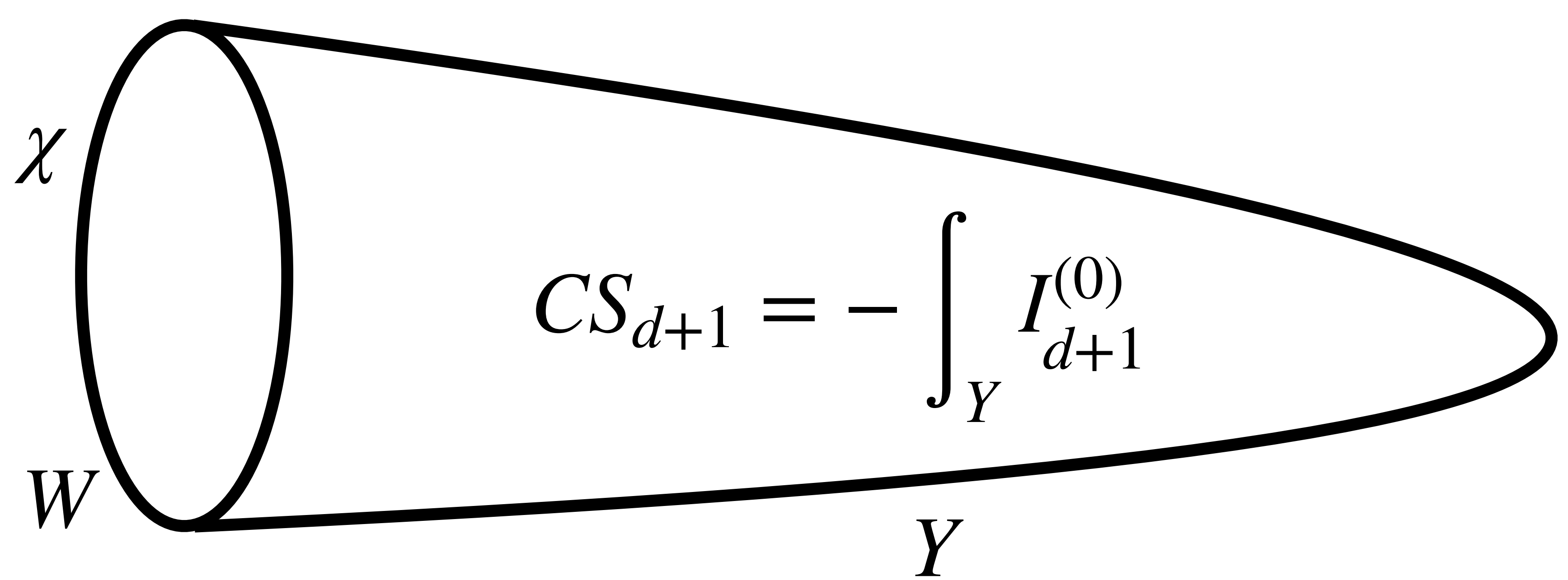
$$I^{(0)}_{d+1} \sim \mathrm{tr}(AF^{d/2}) + \ldots$$

Introduction

Anomaly inflow makes clear the physical meaning of the anomaly descent equations.

Y : $d+1$-dim. spacetime with boundary $\partial Y = W$

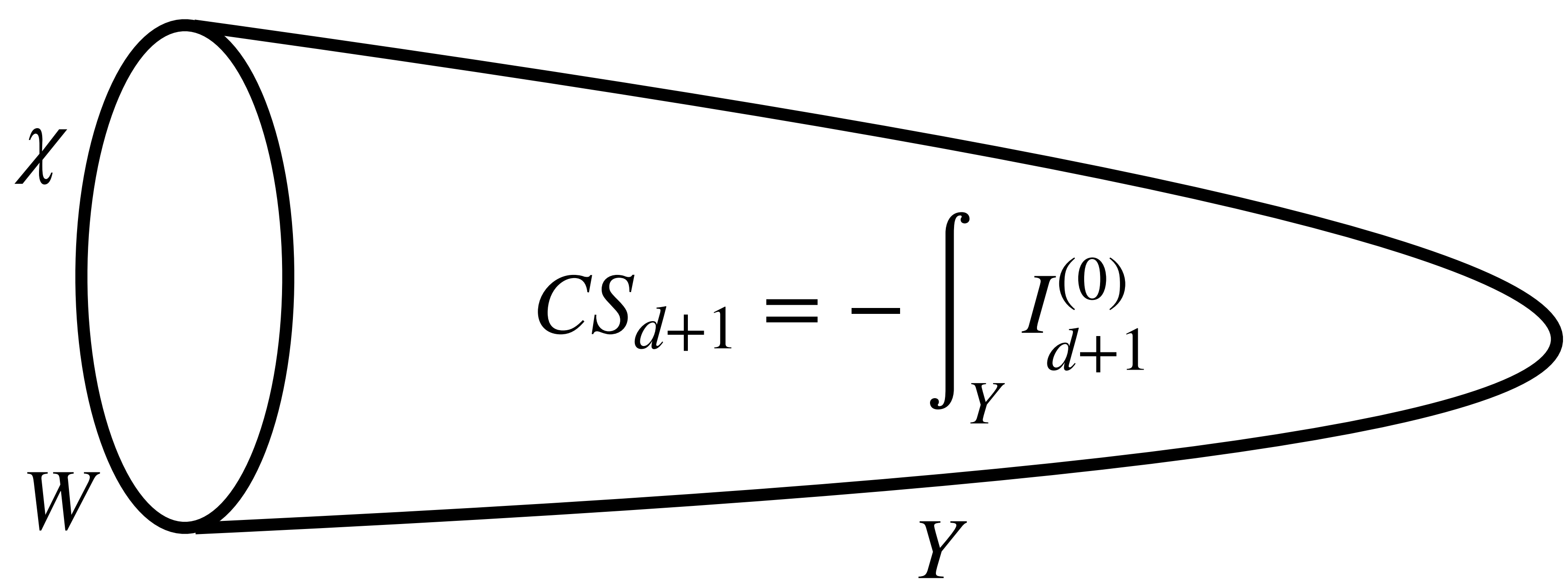
W : d-dim. boundary of Y with the fermion χ



Introduction

The total system has

$$Z(Y) = Z_\chi(W)\exp(iCS_{d+1}(Y)) \qquad \left(CS_{d+1} = -\int I^{(0)}_{d+1} \right)$$



Introduction

The total system has

$$Z(Y) = Z_\chi(W)\exp(iCS_{d+1}(Y)) \qquad \left(CS_{d+1} = -\int I^{(0)}_{d+1}\right)$$

$$\begin{aligned}
i^{-1}\delta\log Z(Y) &= i^{-1}\delta\log Z_\chi(W) + \delta CS_{d+1}(Y) \\
&= \int_W I^{(1)}_d - \delta\int_Y I^{(0)}_{d+1} \\
&= \int_W I^{(1)}_d - \int_Y dI^{(1)}_d \\
&= 0
\end{aligned}$$

Introduction

$$\boxed{\begin{gathered} Z(Y) = Z_\chi(W)\exp(iCS_{d+1}(Y)) \\ \delta \log Z(Y) = 0 \end{gathered}}$$

The anomaly of χ is cancelled by the gauge variation of Chern-Simons in $d+1$-dimensions:

Anomaly inflow from the bulk $d+1$-dimensions to boundary d-dimensions.

Anomalous chiral fermion is well-defined if it is coupled to a 1-higher dimensional theory.

Introduction

Perturbatively, using the Chern-Simons is OK.

However, how can we treat nonperturbative anomaly?

Nonperturbatively, there exist many global anomalies.

Example 1

$d = 4$ chiral fermion in a doublet rep. of SU(2) gauge field is anomalous, although there is no perturbative anomaly.

[Witten,1982]

Introduction

Example 2

$d = 3$ majorana fermions with time-reversal symmetry is anomalous if the number is not a multiple of 16. (Topological superconductors).

[Hsieh-Cho-Ryu,2015]
[Witten,2015]

Example 3

$d = 11$ gravitino in M-theory is anomalous.
(Cancelled in a very subtle way by the 3-form field.)

[Witten,1996]
[Freed-Hopkins,2019]

Example 4

Are you sure that the standard model is anomaly free beyond perturbation theory?

[Freed,2006]
[García-Etxebarria-Montero,2018]

Introduction

I discuss a non-perturbative anomaly inflow formula including global anomalies, which involves

Atiyah-Patodi-Singer (APS) η-invariant

instead of Chern-Simons invariant CS_{d+1}.

Remark

The formula was already expected in the past based on what is called the Dai-Freed theorem. [Witten,2015]

But there was no explicit derivation of the formula.

The Dai-Freed theorem has a physical derivation. [KY,2016]

The idea there can give a concrete physical derivation of the non-perturbative anomaly inflow formula. [Witten-KY,2019]

Contents

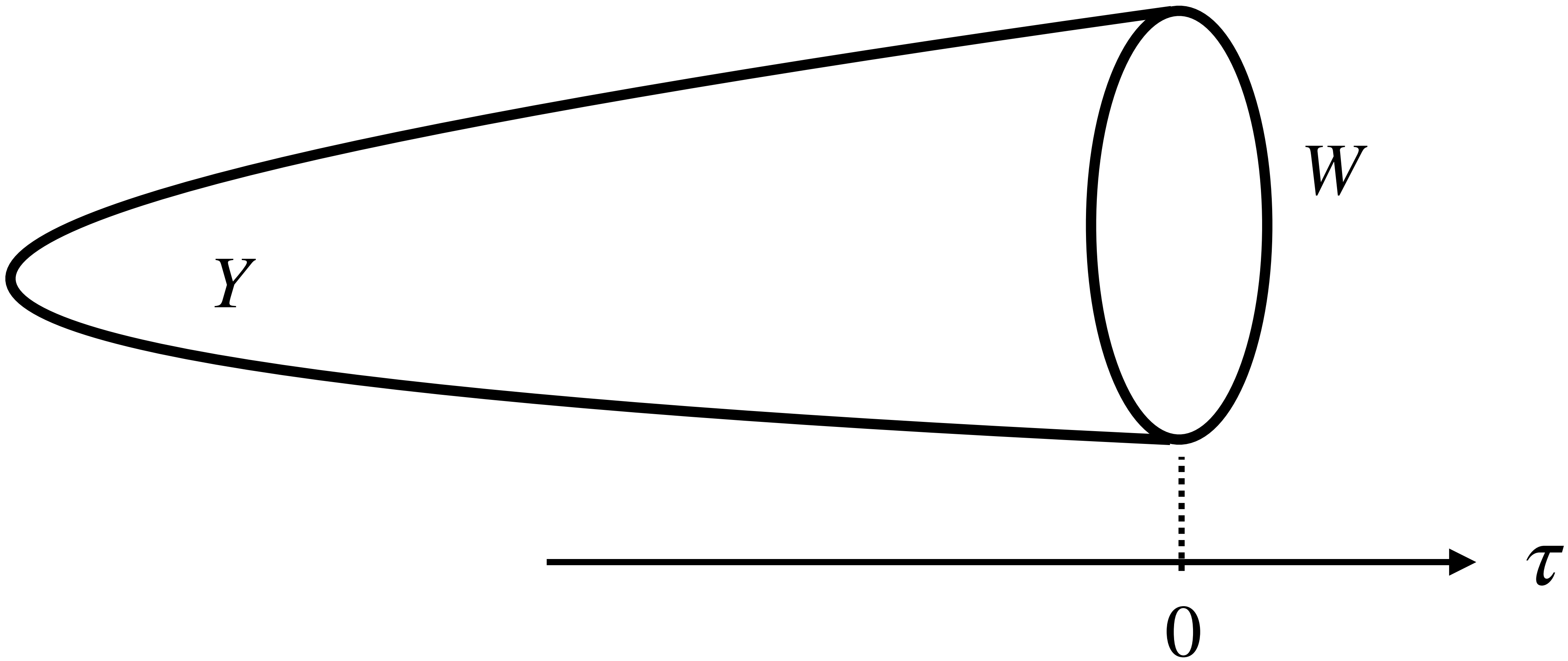
Massive fermion

Let us consider a massive fermion Ψ in $d+1$-dimensions

$$\mathscr{L} = -\bar{\Psi}(\gamma^{\mu} D_{\mu} + m)\Psi$$

It is considered on a spacetime Y with boundary W

Spacetime with boundary



Near the boundary, the spacetime Y is of the form

$$Y \supset (-\epsilon, 0] \times W$$

$\tau \in (-\epsilon, 0]$: coordinate orthogonal to the boundary

Near the boundary

Near the boundary, the Dirac operator in $d+1$-dim.
is represented as

$$\gamma^\mu D_\mu = \gamma^\tau \left(\partial_\tau + \mathscr{D}_W\right) \qquad \left(\partial_\tau = \frac{\partial}{\partial \tau} \right)$$

$$\mathscr{D}_W = \sum_{\mu \neq \tau} \gamma^\tau \gamma^\mu D_\mu$$

(No gauge field in the τ direction, and no τ dependence.)

We impose a local boundary condition

$$\mathrm{L}: \ (1-\gamma^\tau)\Psi|_{\tau=0} = 0$$

Localized chiral fermion

Dirac equation $(\gamma^\mu D_\mu + m)\Psi = 0$:

$$\left(\partial_\tau + \mathscr{D}_W + m\gamma^\tau\right)\Psi = 0$$

Boundary condition :

$$\mathsf{L} : (1-\gamma^\tau)\Psi\,|_{\tau=0} = 0$$

These equations admit a localized solution
if the mass parameter is negative: $m < 0$

Localized chiral fermion

$$\left(\partial_\tau + \mathscr{D}_W + m\gamma^\tau\right)\Psi = 0$$

$$\text{L} : (1-\gamma^\tau)\Psi|_{\tau=0} = 0$$

Solution for $m < 0$:

$$\Psi = \chi\exp(-m\tau), \qquad (1-\gamma^\tau)\chi = 0, \qquad \mathscr{D}_W\chi = 0$$

Recall: $\tau \leq 0$. Exponentially localized near $\tau = 0$

No such localized solution for $m > 0$.

Localized chiral fermion

- The operator $\mathscr{D}_W$ is a Dirac operator on boundary W.

- γ^τ can be regarded as a generalized chirality operator on the boundary W, because of the anticommutation

$$\gamma^\tau \mathscr{D}_W + \mathscr{D}_W \gamma^\tau = 0$$

Example

$d + 1 = 5$ gamma matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$.

The 5th gamma matrix $\gamma^5 = \gamma^\tau$ is the chirality in $d = 4$

Localized chiral fermion

Solution for $m < 0$:

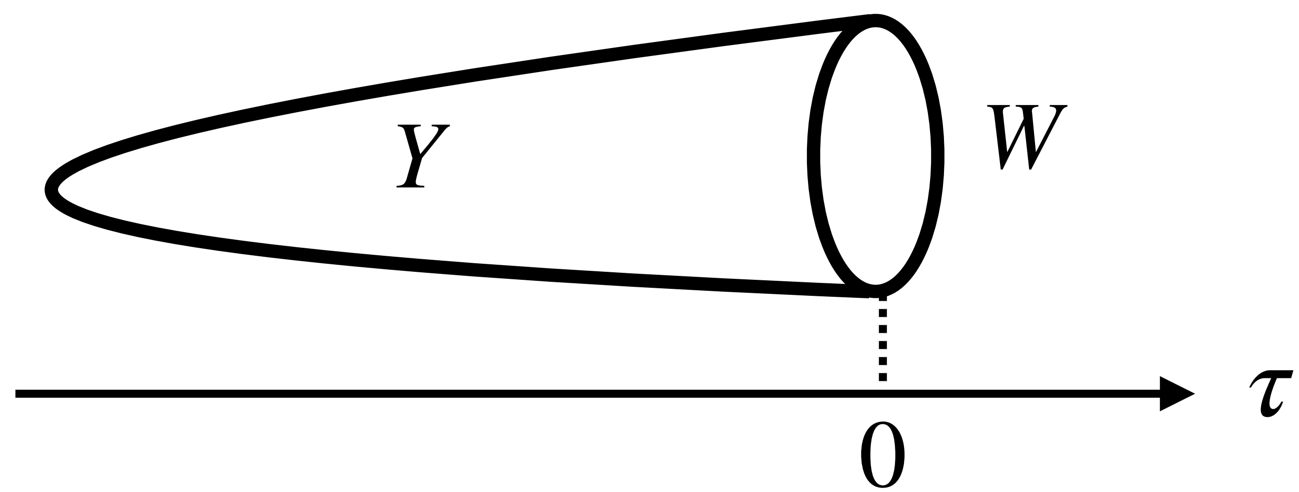
$$\Psi = \chi \exp(-m\tau), \qquad (1 - \gamma^\tau)\chi = 0, \qquad \mathscr{D}_W \chi = 0$$

These equations mean that χ is

- Massless on W : $\mathscr{D}_W \chi = 0$
- Chiral fermion : $\gamma^\tau \chi = +\chi$

Summary of localized modes

$\mathscr{L} = -\bar{\Psi}(\gamma^\mu D_\mu + m)\Psi$ L : $(1 - \gamma^\tau)\Psi|_{\tau=0} = 0$



$m < 0$: localized chiral fermion on the boundary W

$m > 0$: no localized chiral fermion

Pauli-Villars regulator

We regularize the massive fermion theory Ψ by
a Pauli-Villars regulator with mass parameter M.

We don't want the Pauli-Villars field to have a localized massless mode near the boundary. So we take

$M > 0$: no localized mode from the Pauli-Villars

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Bulk path integral on Y

We have seen that two choices $m < 0$ and $m > 0$ are significantly different. How can we see it in the bulk?

Answer:

The path integral of the $m < 0$ theory produces the APS η-invariant, which is equivalent to Chern-Simons invariant CS_{d+1} at the perturbative level.

The theory $m > 0$ does not have η nor CS_{d+1}.

Bulk path integral on Y

For a while, let us consider spacetime Y without boundary.

$$S = -\int \bar{\Psi}(\gamma^\mu D_\mu + m)\Psi$$

$$Z(Y) = \int [D\Psi] e^{-S} = \prod_\lambda \frac{-i\lambda + m}{-i\lambda + M}$$

λ : eigenvalues of the Dirac operator $i\gamma^\mu D_\mu$

Denominator : Pauli-Villars

Bulk path integral on Y

The absolute value $|m|$ is not so important as long as it is very large. Let us take $m = \pm M$ for simplicity.

$m = +M$

$$Z(Y) = \prod_{\lambda} \frac{-i\lambda + M}{-i\lambda + M} = 1$$

Nothing in $d+1$-dim. bulk Y

Bulk path integral on Y

$m = -M$

$$Z(Y) = \prod_{\lambda} \frac{-i\lambda - M}{-i\lambda + M} = \prod_{\lambda} \exp(-i\pi s(\lambda))$$

: pure phase

$$s(\lambda) = -\frac{1}{\pi} \arg\left(\frac{-i\lambda - M}{-i\lambda + M}\right)$$

$$-1 < s(\lambda) \leq 1$$

APS η-invariant

$$s(\lambda) = -\frac{1}{\pi}\arg\left(\frac{-i\lambda - M}{-i\lambda + M}\right) \simeq \mathrm{sign}(\lambda) + \mathcal{O}(\lambda/M)$$

$$\mathrm{sign}(\lambda) = \frac{\lambda}{|\lambda|} \quad (\lambda \neq 0), \qquad \left(\ \mathrm{sign}(0) = 1\ \right)$$

Atiyah-Patodi-Singer (APS) η-invariant

$$\eta(Y) = \sum_{\lambda} \mathrm{sign}(\lambda)_{\mathrm{reg}} = \lim_{M\to\infty} \sum_{\lambda} s(\lambda)$$

Bulk partition function on Y

$m = -M$

$$Z(Y) = \prod_{\lambda} \exp(-i\pi s(\lambda))$$

$$= \exp(-i\pi\eta(Y)) \qquad (M \to \infty)$$

The partition function $Z(Y)$ is given by the η-invariant.

Relation to Chern-Simons

An explicit 1-loop computation shows that perturbatively

$$Z(Y) = \exp(iCS_{d+1}) \quad \text{(perturbatively)}$$

Alternatively, the APS index theorem shows that if Y is a boundary of some $d+2$-dim. manifold X,

$$\exp(-i\pi\eta(Y)) = \exp(-i\int_X I_{d+2})$$

I_{d+2} : the anomaly $d+2$-form which appears in the anomaly descent equations.

Bulk path integral on Y

Perturbatively, or if $Y = \partial X$ for some X

$$\exp(-i\pi\eta(Y)) = \exp(iCS_{d+1})$$

But η contains information of nonperturbative, global anomalies.

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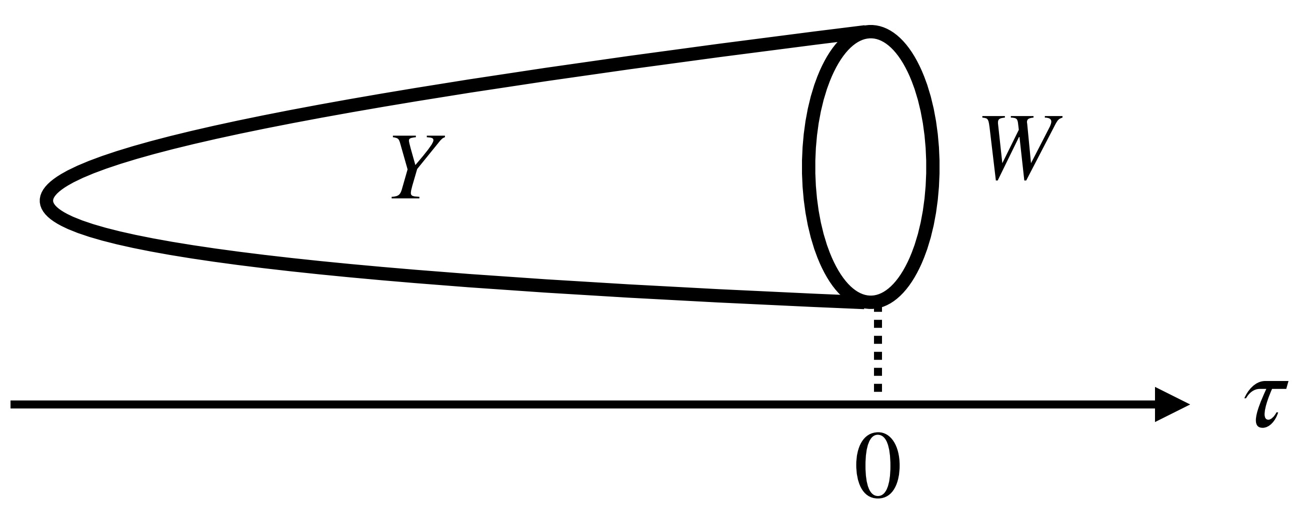
Path integral with local b.c.

Let us return to the path integral on a spacetime Y with boundary W, with local boundary condition L

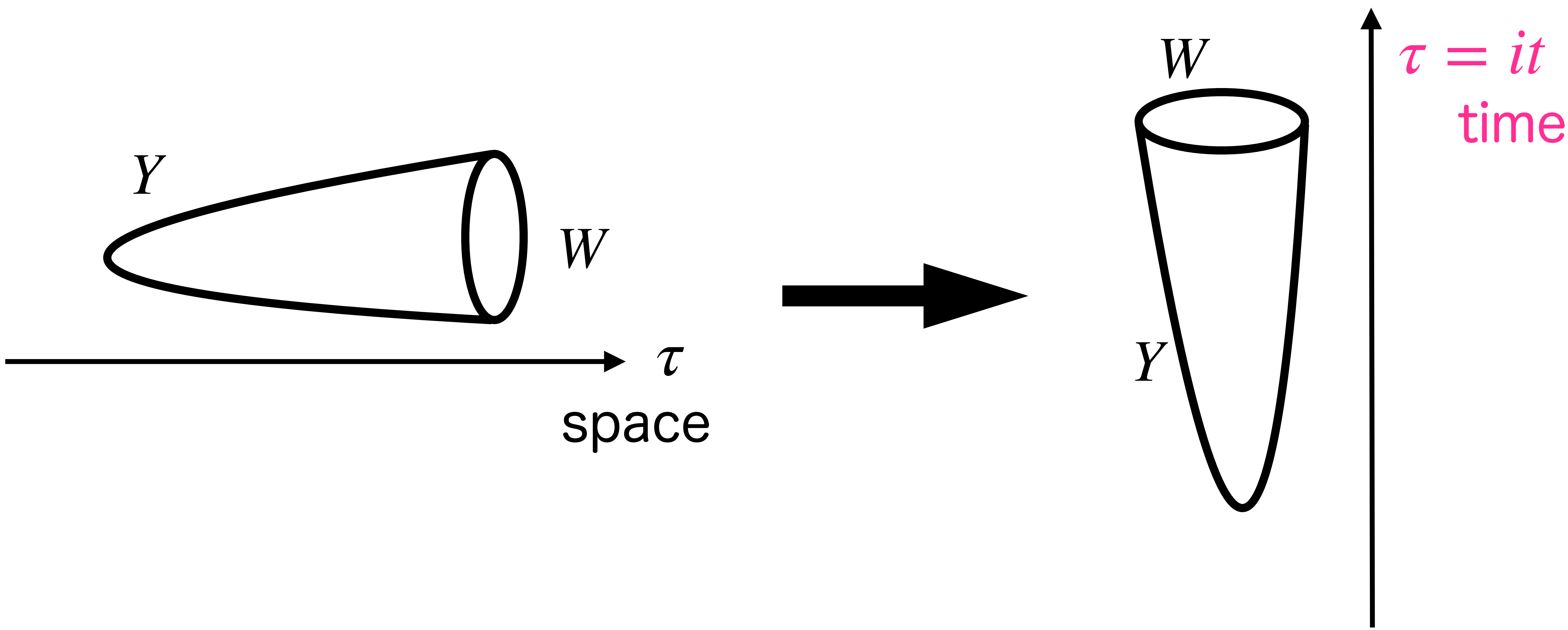
$$\mathsf{L} : (1 - \gamma^\tau)\Psi|_{\tau=0} = 0$$



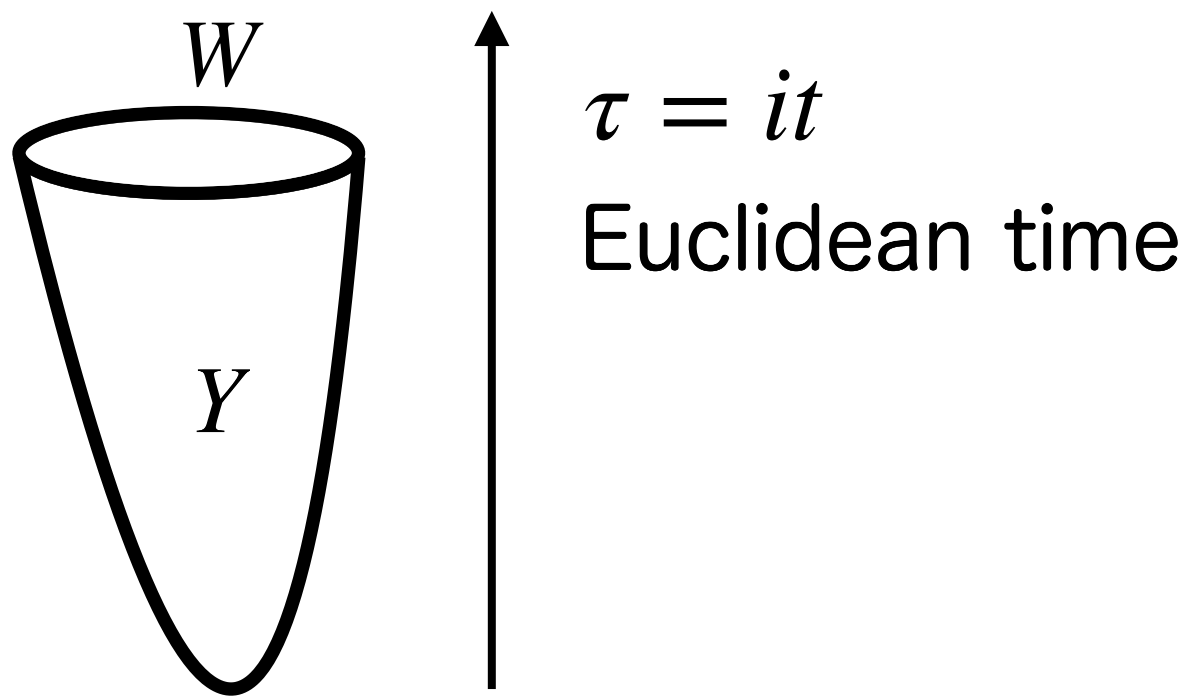
Wick rotation

We can change our point of view :

Perform Wick rotation to see $\tau = it$ as a Wick rotated time.



Transition from empty to W



This can be seen as a transition amplitude from nothing to W. This gives a physical state

$$|Y\rangle \in \mathscr{H}_W$$

$\mathscr{H}_W$: Hilbert space on W

Local b.c. as a physical state

The local boundary condition L can also be seen as defining some physical state after Wick rotation:

$$\mathsf{L} : (1 - \gamma^{\tau})\Psi\,|_{\tau=0} = 0$$

$$\Big\downarrow \ \text{Wick rotation}$$

$$|\,\mathsf{L}\rangle \in \mathscr{H}_W$$

This is specified by

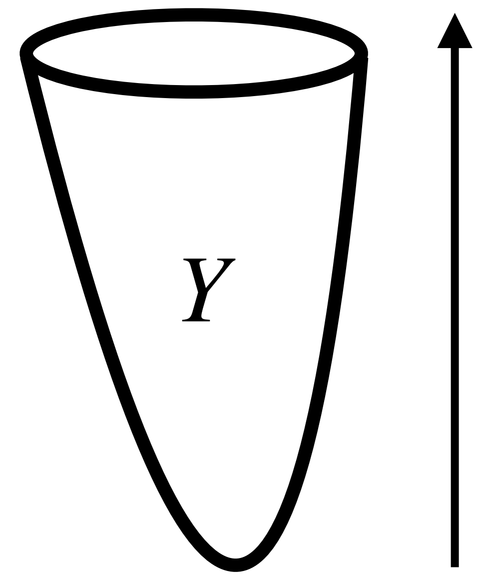
$$\langle \mathsf{L}\,|\,(1 - \gamma^{\tau})\Psi = 0$$

Path integral as state overlap

The partition function with the boundary condition L is now computed as a state overlap

$$
\boxed{\begin{aligned} Z(Y, \mathsf{L}) &= \int [D\Psi] e^{-S} \\ &= \langle \mathsf{L} \,|\, Y \rangle \end{aligned}}
$$

Domination of ground state



$\tau = it$

Euclidean time

Near the boundary,

$$Y \supset (-\epsilon, 0] \times W$$

Euclidean time evolution in this region

$$e^{-\epsilon H} \to |\Omega\rangle\langle\Omega| \qquad (\epsilon|m| \to \infty)$$

$|\Omega\rangle$: the ground state

Splitting bulk and boundary

$$|\,Y\rangle \propto |\,\Omega\rangle \qquad (\epsilon\,|\,m\,|\to\infty)$$

$$\begin{aligned} Z(Y,\mathsf{L}) &= \langle \mathsf{L}\,|\,Y\rangle \\ &= \langle \mathsf{L}\,|\,\Omega\rangle\langle\Omega\,|\,Y\rangle \end{aligned}$$

Determined by boundary W

Determined by bulk Y

Chiral fermion partition function

$$Z(Y, \mathsf{L}) = \langle \mathsf{L} | \Omega \rangle \langle \Omega | Y \rangle$$

It turns out by explicit computation that for $m < 0$

$\langle \mathsf{L} | \Omega \rangle = Z_\chi(W)$: partition function of boundary chiral fermion χ

$|\langle \Omega | Y \rangle| = 1$: pure phase (up to local counterterm)

The path integral of the massive fermion in $d+1$-dim. with local boundary condition really produces the d-dim. chiral fermion partition function, up to phase.

Anomaly as phase ambiguity

$$Z(Y, \mathsf{L}) = \langle \mathsf{L} \,|\, \Omega \rangle \langle \Omega \,|\, Y \rangle$$

$$\langle \mathsf{L} \,|\, \Omega \rangle = Z_\chi(W) \qquad |\langle \Omega \,|\, Y \rangle| = 1$$

- $Z_\chi(W)$ has ambiguity, because $|\Omega\rangle$ has phase ambiguity. (Unavoidable due to Berry phase of $|\Omega\rangle$)

- The ambiguity is cancelled in the product $\langle \mathsf{L} \,|\, \Omega \rangle \langle \Omega \,|\, Y \rangle$.

This explains why the chiral fermion must be coupled to the bulk in 1-higher dimensions.

APS boundary condition

The phase ambiguity of $|\Omega\rangle$ is inconvenient.

Let us try another state instead of $|\Omega\rangle$.

Definition

$|\mathrm{ASP}\rangle$: a state which would be the ground if $m = 0$

(Physical interpretation of APS boundary condition.)

This has no phase ambiguity if we take the ratio between the physical fermion and Pauli-Villars field.

Because $|\mathrm{ASP}\rangle$ is independent of mass parameter.

APS boundary condition

This definition of $|\text{ASP}\rangle$ breaks down when

the $m = 0$ theory has a degenerate ground state.

This happens when we vary background fields on W.

Let us consider the case that $|\text{ASP}\rangle$ is possible.

New splitting

Using $\quad |Y\rangle \propto |\Omega\rangle \quad (\epsilon|m| \to \infty)$

$$Z(Y, \mathsf{L}) = \langle \mathsf{L} | \Omega \rangle \langle \Omega | Y \rangle$$

$$= \underbrace{\frac{\langle \mathsf{L} | \Omega \rangle \langle \Omega | \mathsf{APS} \rangle}{|\langle \Omega | \mathsf{APS} \rangle|^2}}_{\text{Determined by boundary } W} \cdot \underbrace{\langle \mathrm{APS} | Y \rangle}_{\text{Determined by bulk } Y}$$

Bulk contribution

Bulk contribution $\langle \mathrm{APS} \,|\, Y \rangle$ is computed in the same way as before: for $m = -M < 0$,

$$\boxed{\begin{aligned} \langle \mathrm{APS} \,|\, Y \rangle &= \prod_{\lambda} \frac{-i\lambda - M}{-i\lambda + M} \\ &= \exp(-i\pi\eta(Y)) \end{aligned}}$$

Now the eigenvalues λ of $i\gamma^{\mu}D_{\mu}$ are computed under the APS boundary condition.

(Mathematically, APS boundary condition guarantees that the Dirac operator have real eigenvalues.)

Boundary contribution

Massive fermion $\mathscr{L} = -\bar{\Psi}(\gamma^\mu D_\mu + m)\Psi$ is a free theory.

Its quantization on W is straightforward.

<u>The result for $m < 0$:</u>

$$\boxed{\frac{\langle \mathrm{L} | \Omega \rangle \langle \Omega | \mathrm{APS} \rangle}{|\langle \Omega | \mathrm{APS} \rangle|^2} = |Z_\chi(W)| = |\mathrm{Det}(\mathscr{D}_W^+)|}$$

$\mathscr{D}_W^+$: Dirac operator acting on the chiral fermion χ

The anomaly inflow formula

Combining the previous results, we finally get

Nonperturbative anomaly inflow formula

$$Z(Y, \mathsf{L}) = |\mathrm{Det}(\mathscr{D}_W^+)| \exp(-i\pi\eta(Y))$$

- Left-hand-side:

  $d+1$-dim. massive fermion partition function with the local boundary condition L

- Right-hand-side:

  d-dim. chiral fermion partition function coupled to $d+1$-dim. bulk topological phase

The anomaly inflow formula

$$Z(Y, \mathsf{L}) = |\mathrm{Det}(\mathscr{D}_W^+)| \exp(-i\pi\eta)$$

We cannot separate the bulk and boundary contributions.

This decomposition is like polar coordinate $z = re^{i\theta} \in \mathbb{C}$

$|\mathrm{Det}(\mathscr{D}_W^+)|$: not smooth around points $|\mathrm{Det}(\mathscr{D}_W^+)| = 0$

$\exp(-i\pi\eta)$: not well-defined when APS b.c. fails.

Therefore, bulk and boundary must be combined.

Contents

General case

Completely general chiral fermion:

1. The discussion so far was Dirac fermion. But Majorana fermion is more general.
   (Any complex number can be decomposed into two real numbers.)

2. I started from $d+1$-dimensions and then find a localized d-dim. fermion. It is possible to start from arbitrary d-dim. fermion and find a massive $d+1$-dim. fermion.

General case

General formula

$$Z(Y, \mathsf{L}) = |\mathrm{Pf}(\mathscr{D}_W^+)| \exp\left(-i\pi\eta(Y)/2\right)$$

- Pf : Pfaffian (which is relevant for majorana)
- η is multiplied by $1/2$

We claim that this is the general anomaly inflow formula for arbitrary chiral fermions. (Details omitted)

Anomaly

General anomaly of a chiral fermion is characterized by the Y-dependence of the partition function

$$Z(Y, \mathsf{L}) = |\,\mathrm{Pf}(\mathscr{D}_W^+)\,| \exp\left(-i\pi\eta(Y)/2\right)$$

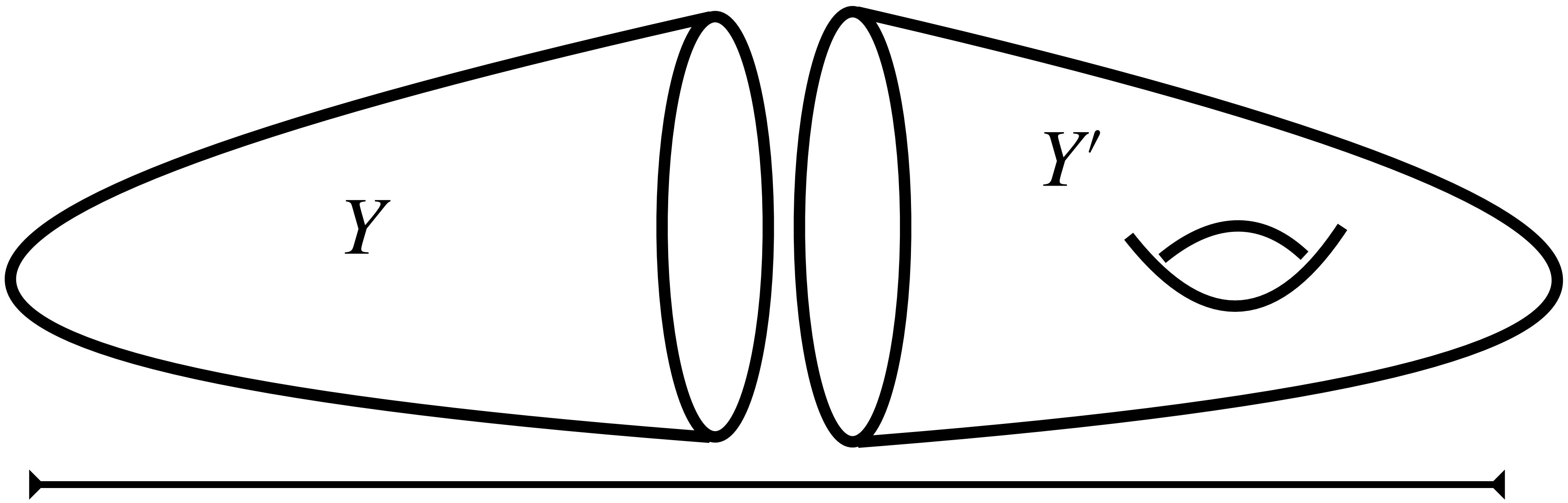
Take another Y'

$$\frac{Z(Y, \mathsf{L})}{Z(Y', \mathsf{L})} = \exp\left(-i\pi(\eta(Y) - \eta(Y'))/2\right)$$

$$= \exp\left(-i\pi\eta(\overline{Y})/2\right)$$

$\overline{Y}$: closed manifold obtained by gluing Y and Y'

Anomaly



glue them together : closed manifold $\overline{Y}$

Gluing theorem of the η-invariant (locality of η)

$$\exp\left(-i\pi(\eta(Y) - \eta(Y')/2)\right) = \exp\left(-i\pi\eta(\overline{Y})/2\right)$$

Anomaly

If $\exp\left(-i\pi\eta(\overline{Y})/2\right) = 1$ for any closed manifold $\overline{Y}$,

$$Z(Y, \mathsf{L}) = Z(Y', \mathsf{L})$$

We can regard $Z(Y, \mathsf{L})$ as a definition of the chiral fermion partition function on the boundary W.

The obstruction for such a definition : Anomaly

$\exp\left(-i\pi\eta(\overline{Y})/2\right)$ $\qquad$ $\overline{Y}$: closed manifolds

(There are some important details which I omit here. Please see our paper.)

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Summary

- Non-perturbative anomaly inflow formula is given by

$$\boxed{Z(Y, \mathsf{L}) = |Z_\chi(W)| \exp\left(-i\pi\eta(Y)/2\right)}$$

Partition function of massive fermion on Y with local boundary condition L

Chiral fermion partition function on the boundary $W = \partial Y$

Exponential of η-invariant in the bulk Y