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Memo No. 18Date 2019 / 10 / 21Characterizations of plurisubharmonic functions

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Based on works with Ning Wang, Zhang, Zhou.

Outline:

§1. Geometric meaning of p.s.h functions

§2. Outline of the main results

§3. The main results & idea of proof

§4. Some remarks

§1. geometric meaning of p.s.h functions• $(L, h) \rightarrow X$ hermitian line bundle e : hol. section of L on $U \subset X$ $\|e\|_h^2 = e^{-\phi}$, $\phi \in C^\infty(U)$, local weights of L .Curvature of (L, h) :

$$\Theta_{(L, h)} = \frac{i}{2\pi} \partial\bar{\partial}\phi$$

$$\Theta_{(L, h)} \geq 0 \iff \phi \text{ p.s.h.}$$

Remark: important to allow singular metrics ($\phi \in L_{loc}^1$).• $(E, h) \rightarrow X$ hermitian vector bundle

$$\Theta_{(E, h)} \leq 0 \iff \exists s: \text{local hol. section of } E, s \neq 0, \log \|s\|_h \text{ p.s.h.}$$

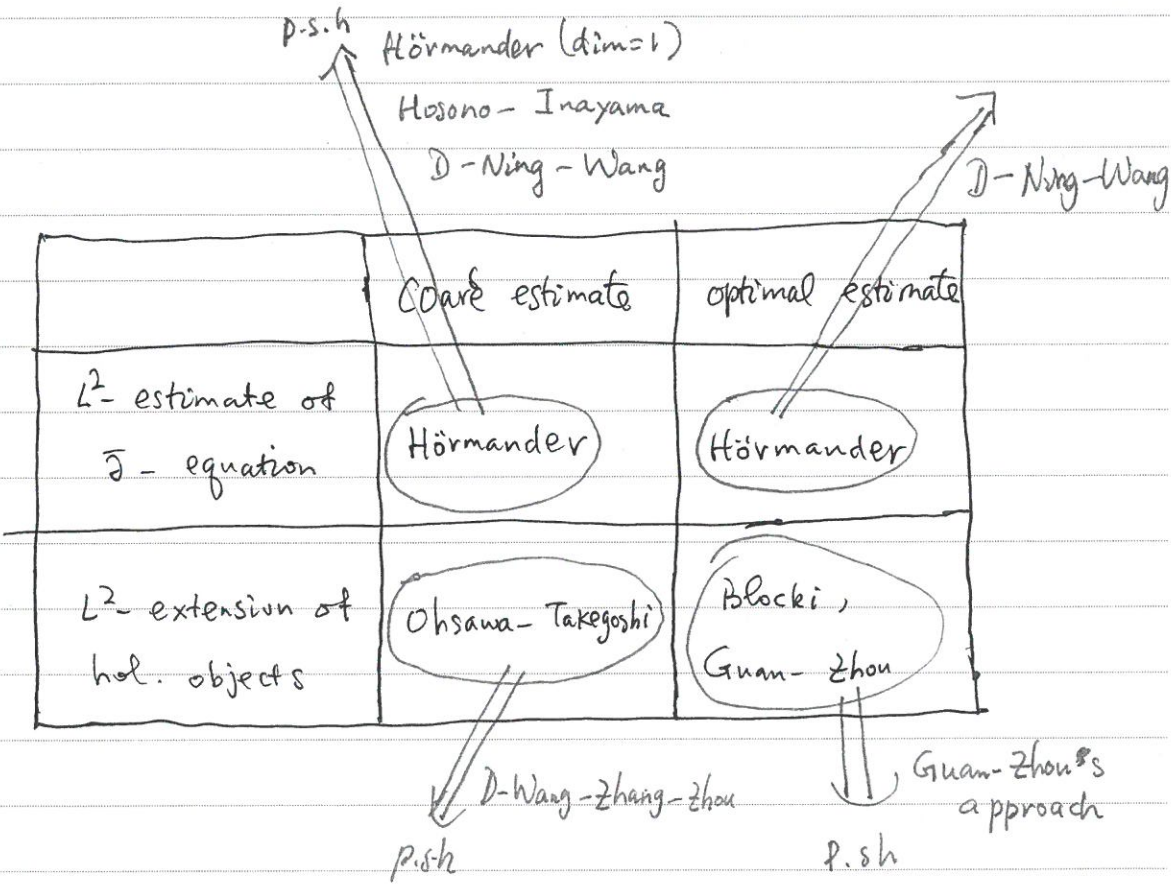
Griffiths



$$\left(\begin{array}{l} (E, h) \geq 0 \\ \text{Griffiths} \end{array} \iff \begin{array}{l} (E^*, h^*) \leq 0 \\ \text{Griffiths.} \end{array} \right)$$

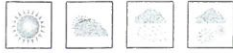
Griffiths positivity (\iff) plurisubharmonicity

§2. Outline of the main results



§3. The main results & idea of proofs

Definition 1: Let $\phi : \mathbb{D} \rightarrow [-\infty, +\infty)$ be u.s.c.,
 we say ϕ satisfies:



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(1) the multiple coarse L^p -extension property if $\forall m \geq 1$,
 $\forall z \in D$ with $\phi(z) \neq -\infty$, $\exists f \in \mathcal{O}(D)$ s.t. $f(z) = 1$ and

$$\int_D |f|^p e^{-m\phi} \leq C_m e^{-m\phi(z)}$$

C_m are uniform constants satisfying $\log C_m / m \rightarrow 0$.
 (introduced by D-Wang-Zhang-Zhou)

(2) the optimal L^p -extension property if $\forall z \in D$ with
 $\phi(z) \neq -\infty$, \forall hole cylinder P with $z+P \subset D$,
 $\exists f \in \mathcal{O}(z+P)$ s.t. $f(z) = 1$ and

$$\frac{1}{\mu(P)} \int_{z+P} |f|^p e^{-\phi} \leq e^{-\phi(z)}$$

(P has the form $A(P_{rs})$, $A \in U(n)$, $P_{rs} = \{|z_1|^2 < r^2, |z_2|^2 + \dots + |z_n|^2 < s^2$

Definition 2: Let $\phi: D \rightarrow [-\infty, +\infty)$ be u.s.c, we say ϕ
 satisfies:

(1) the multiple coarse L^p -estimate property if $\forall m \geq 1$,
 $\forall f \in \mathcal{O}^{0,1}(D)$, $\bar{\partial} f = 0$, and $\forall \psi$: strict p.s.h on D ,
 can solve the equation $\bar{\partial} u = f$ with

$$\int_D |u|^2 e^{-m\phi - \psi} \leq C_m \int_D |f|_{i,\bar{\partial}}^p e^{-m\phi - \psi}$$

C_m uniform constants with $\log C_m / m \rightarrow 0$.
 (introduced by Hosono-Inayama for $p=2$).



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(2) the optimal L^p -estimate property if $\forall f \in \mathcal{O}^{0,1}(\mathbb{D})$,

$\bar{\partial}f = 0$, $\exists \psi$: strict p.s.h on \mathbb{D} , can solve $\bar{\partial}u = f$

with $\int_{\mathbb{D}} |u|^p e^{-\phi - \psi} \leq \int_{\mathbb{D}} |f|_{\bar{\partial}\bar{\psi}}^p e^{-\phi - \psi}$

Remark: if $p=2$, ϕ p.s.h, $\mathbb{D} \subset \subset \mathbb{C}^n$ pseudoconvex.

① ϕ satisfies (1) and (2) in Definition 2 by Hörmander;

② ϕ satisfies (1) in Definition 1 by Ohsawa-Takegoshi;
and satisfies (2) in by Blocki, Guan-Zhou.

Theorem 1: (D-Wang-Zhang-Zhou)

Let $\phi: \mathbb{D} \rightarrow [-\infty, +\infty)$ be u.s.c. If ϕ satisfies the multiple coarse L^p -extension property for some $p > 0$, then ϕ is p.s.h.

Remark:

① Original proof was motivated by Demailly's regularization of p.s.h functions.

② (D-Ning-Wang) New proof by modifying Guan-Zhou's approach.



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② Theorem 1 produces quite different method to positivity of direct image sheaves established by Berndtsson and others.

Theorem 2: (D-Ning-Wang).

Let $\phi: D \rightarrow [-\infty, +\infty)$ be u.s.c. If ϕ satisfies the optimal L^p -extension property for some $p > 0$, then ϕ is p.s.h.

Remark: proved by modifying Guan-Zhou's approach.

Theorem 3: (D-Ning-Wang):

Let $\phi: D \rightarrow [-\infty, +\infty)$ be continuous. If ϕ satisfies the multiple coarse L^p -estimate property for some $p > 1$, then ϕ is p.s.h.

Remark:

① $n=1, p=1$: Berndtsson proved a related result.

② $p=2, \phi$ Hölder continuous: proved by Hosono-Inayama, by reducing the multiple coarse L^2 -estimate property to the multiple coarse L^2 -extension property and applying Theorem 1.

③ Our proof vs by modifying H-I idea.



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Theorem 4: (D - Ning - Wang).

Let $\phi: D \rightarrow \mathbb{R}$ be a C^2 function. If ϕ satisfies the optimal L^2 -estimate property, then ϕ is p.s.h.

Remark:

① The proof uses a Bochner-type identity and a localization technique to produce contradiction.

$$\left[\begin{aligned} &\forall f \in C^2(D), \forall \alpha \in \mathcal{D}^{0,1}(D), \alpha = \sum \alpha_j d\bar{z}_j \\ &\int_D \sum \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-f} + \int_D \sum \left| \frac{\partial \alpha_j}{\partial \bar{z}_k} \right|^2 e^{-f} \\ &= \int_D |\bar{\partial} \alpha|^2 e^{-f} + \int_D |\bar{\partial}_f^* \alpha|^2 e^{-f} \end{aligned} \right]$$

② Can be used to characterize strict plurisubharmonicity.

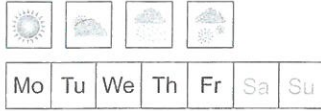
if $\exists \phi_0 \in \text{psh}(D) \cap C^2$ s.t. the estimate in Thm 4 can

be strengthened to

$$\int_D |u|^2 e^{-\phi - \psi} \leq \int_D |f|^2 e^{-\phi - \psi} \quad \text{on } i\partial\bar{\partial}(\psi + \phi_0)$$

then $\phi - \phi_0$ is p.s.h.

③ question: does Thm 4 holds for continuous ϕ ?



§4. Some remarks

Properties in Definition 1, 2 can be generalized to hermitian bundles of rank > 1 , but they are not equivalent, unlike in the case of line bundles.

We have (work in progress):

Rk: (*) can be used to prove (strict) Nakano positivity of direct image sheaves.

