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# Characterizations of plurisubharmonic functions

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Based on works with Ning. Wang, Zhang, Zhou.

## Outline:



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## §1. geometric meaning ~~of~~ of p.s.h functions

- $(L, h) \longrightarrow X$ hermitian line bundle

  e: holo. section of $L$ on $U \subset X$

  $|e|_h^2 = e^{-\phi}$, $\phi \in C^\infty(U)$, local weight of $L$.

Curvature of $(L, h)$:

$$\Theta_{(L,h)} = \frac{i}{2\pi} \partial\bar\partial \phi$$

$$\Theta_{(L,h)} \geq 0 \iff \phi \text{ p.s.h}$$

Remark: importantant ~~to~~ allow singular metrics ($\phi \in L^1_{loc}$).

- $(E, h) \rightarrow X$ hermitian vector bundle

$\Theta_{(E,h)} \underset{\text{Griffiths}}{\leq} 0 \iff \forall$ s: local hol. section of E, $s \not\equiv 0$, $\log|s|_h$ p.s.h.

$\left( (E, h) \underset{\text{Griffiths}}{\geq 0} \iff (E^*, h^*) \underset{\text{Griffiths}}{\leq 0} \right).$

$\therefore$ **Griffiths positivity $\Leftarrow$ plurisubharmonicity**

## §2. Outline of the main results

p.s.h ⟸ Hörmander (dim = 1), Hosono – Inayama, D – Ning – Wang (from Hörmander, coarse estimate)

⟸ D – Ning – Wang (from Hörmander, optimal estimate)

| | Coare estimate | optimal estimate |
|---|---|---|
| $L^2$- estimate of $\bar{\partial}$- equation | Hörmander | Hörmander |
| $L^2$- extension of hol. objects | Ohsawa – Takegoshi | Blocki, Guan – Zhou |

Ohsawa – Takegoshi ⟹ D-Wang-Zhang-Zhou ⟹ p.s.h

Blocki, Guan – Zhou ⟹ Guan-Zhou's approach ⟹ p.s.h

## §3. The main results & idea of proofs

**Definition 1**: Let $\phi : D \longrightarrow [-\infty, +\infty)$ be u.s.c, $D \subset \mathbb{C}^n$,
we say $\phi$ satisfies :

(1) <u>the multiple coarse $L^p$-extension property</u> if $\forall m \geq 1$, $\forall z \in D$ with $\phi(z) \neq -\infty$, $\exists f \in \mathcal{O}(D)$ s.t $f(z) = 1$ and

$$\int_D |f|^p e^{-m\phi} \leq C_m e^{-m\phi(z)}.$$

$C_m$ are uniform constants satisfying $\log C_m / m \to 0$.

(introduced by D-Wang-Zhang-Zhou)

(2) <u>the optimal $L^p$-extension property</u> if $\forall z \in D$ with $\phi(z) \neq -\infty$, $\forall$ holo. cylinder $P$ with $z + P \subset D$, $\exists f \in \mathcal{O}(z+P)$ s.t $f(z) = 1$ and

$$\frac{1}{\mu(P)} \int_{z+P} |f|^p e^{-\phi} \leq e^{-\phi(z)}$$

($P$ has the form $A(P_{rs})$, $A \in U(n)$, $P_{rs} = \{|z_1|^2 < r^2, |z_2|^2 + \cdots + |z_n|^2 < s^2\}$.

<u>Definition 2</u>: Let $\phi: D \to [-\infty, +\infty)$ be u.s.c, we say $\phi$ satisfies:

(1) <u>the multiple coarse $L^p$-estimate property</u> if $\forall m \geq 1$, $\forall f \in \mathcal{D}^{0,1}(D)$, $\bar\partial f = 0$, and $\forall \psi$: strict p.s.h on $D$, can solve the equation $\bar\partial u = f$ with

$$\int_D |u|^2 e^{-m\phi - \psi} \leq C_m \int_D |f|^p_{i\partial\bar\partial\psi} e^{-m\phi - \psi},$$

$C_m$ uniform constants with $\log C_m / m \to 0$.
(introduced by Hosono-Inayama for $p=2$).

(2) <u>the optimal $L^p$ estimate property</u> if $\forall f \in \mathcal{D}^{0,1}(D)$, $\bar\partial f = 0$, $\forall \psi$: strict p.s.h on $D$, can solve $\bar\partial u = f$ with

$$\int_D |u|^p e^{-\phi-\psi} \leq \int_D |f|^p_{i\partial\bar\partial\psi} e^{-\phi-\psi}.$$

<u>Remark</u>: if $p=2$, $\phi$ p.s.h, $D \subset\subset \mathbb{C}^n$ pseudoconvex.

① $\phi$ satisfies (1) and (2) in Definition 2 by Hörmander;

② $\phi$ satisfies (1) in Definition 1 by Ohsawa-Takegoshi; and satisfies (2) in ...... by Blocki, Guan-Zhou.

<u>Theorem 1</u>: (D - Wang - Zhang - Zhou)

Let ~~...~~ $\phi: D \longrightarrow [-\infty, +\infty)$ be u.s.c. If $\phi$ satisfies <u>the multiple coarse $L^p$ extension property</u> for some $p>0$, then $\phi$ is p.s.h.

<u>Remark</u>:

① Original proof was motivated by Demailly's regularization of p.s.h functions.

② (D - Ning - Wang) New proof by modifying Guan-Zhou's approach.

③ Theorem 1 produces quite different method to positivity of direct image sheaves establised by Berndtsson and others...

**Theorem 2**: (D−Ning−Wang).

Let $\phi: D \to [-\infty, +\infty)$ be u.s.c. If $\phi$ satisfies the optimal $L^p$-extension property for some $p>0$, then $\phi$ is p.s.h.

**Remark**: proved by modifying Guan-Zhou's approach.

**Theorem 3**: (D−Ning−Wang):

Let $\phi: D \longrightarrow (-\infty, +\infty)$ be continuous. If $\phi$ satisfies the multiple coarse $L^p$-estimate property for some $p>1$, then $\phi$ is p.s.h.

**Remark**:

① $n=1$, $p=1$: Berndtsson proved a related result.

② $p=2$, $\phi$ Hölder continuous: proved by Hosono-Inayama, by reducing the multiple coarse $L^2$-estimate property to the multiple coarse $L^2$-extension property and applying Theorem 1.

③ Our proof is by modifying H-I idea.

**Theorem 4**: (D - Ning - Wang).

Let $\phi : D \to \mathbb{R}$ be a $C^2$ function. If $\phi$ satisfies the optimal $L^2$ estimate property, then $\phi$ is p.s.h.

Remark:

(1) The proof uses a Bochner-type identity and a localization technique to produce contradiction.

$\forall f \in C^2(D)$, $\forall \alpha \in \mathcal{D}^{0,1}(D)$, $\alpha = \sum \alpha_j d\bar{z}_j$

$$\left[ \int_D \sum \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-f} + \int_D \sum \left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2 e^{-f} = \int_D |\bar{\partial}\alpha|^2 e^{-f} + \int_D |\bar{\partial}_f^* \alpha|^2 e^{-f} \right]$$

(2) Can be used to characterize strict plurisubharmonicity:

If $\exists \phi_0 \in psh(D) \cap C^2$ s.t the estimate in Thm 4 can be strenghened to

$$\int_D |u|^2 e^{-\phi-\psi} \leq \int_D |f|^2_{i\partial\bar{\partial}(\psi+\phi_0)} e^{-\phi-\psi}$$

then $\phi - \phi_0$ is p.s.h.

(3) question: does Thm 4 holds for continuous $\phi$?

§4. Some remarks

Properties in Definition 1, 2 can be generalized to hermitian bundles of rank > 1, but they are not equivalent, unlike in the case of line bundles.

We have (work in progress):

Rk: (*) can be used to prove (strict) Nakano positivity of direct image sheaves.