Symmetry and Its Breaking in Path Integral Approach to Quantum Brownian Motion

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Introduction

- Classical stochastic thermodynamics: thermodynamic quantity ⇔ individual stochastic trajectory
- No unique notion of a quantum trajectory
  - Quantum FT: Two projective energy measurement scheme (TPM)
  - Many other definitions for quantum work
- How far can we take ideas of classical stochastic thermodynamics to quantum regime?
- Take Symmetry in path integral as a guiding principle
- Ref.: J. Yeo, arXiv:1909.08212 to appear in Phys. Rev. E.

- Langevin equation

$$\dot{x} = f(x, \lambda_t) + \xi(t),$$

where $\langle \xi(t)\xi(t') \rangle = 2T\delta(t - t')$

- Transition probability (Onsager-Machlup):

$$P[x_1, \tau | x_0, 0] = \int_{x(0)=x_0}^{x(\tau)=x_1} \mathcal{D}x(t)\; e^{-S_{\rm OM}[x]},$$

where

$$S_{\rm OM}[x] = \int_0^\tau dt\; \frac{1}{4T}(\dot{x}(t) - f)^2.$$

- Path probability $\mathcal{P}$ is obtained from $\exp[-S_{\rm OM}] P_i(x(0))$.

Classical Stochastic Thermodynamics

- Irreversibility:

$$\mathcal{R}[\boldsymbol{x}] = \ln \frac{\mathcal{P}[\boldsymbol{x}]}{\tilde{\mathcal{P}}[\tilde{\boldsymbol{x}}]}$$

  - $\mathcal{P}[\boldsymbol{x}]$: path probability for $\boldsymbol{x}(t)$
  - $\tilde{\mathcal{P}}[\tilde{\boldsymbol{x}}]$: path probability for $\tilde{\boldsymbol{x}}(t)$, e.g. $\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(\tau - t)$.
  - normalization of $\tilde{\mathcal{P}} \Rightarrow$ Fluctuation Theorems (FTs).
- Can we use the similar concept in the quantum case?
- Convenient to use an alternative representation

Martin-Siggia-Rose-Janssen-De Dominicis formalism

- Introduce an auxiliary field $\hat{x}$

$$P[x_1, \tau | x_0, 0] = \int_{x(0)=x_0}^{x(\tau)=x_1} \mathcal{D}x(t) \int \mathcal{D}\hat{x}(t)\, e^{-S_{\rm MSRJD}[x,\hat{x}]},$$

where

$$S_{\rm MSRJD}[x,\hat{x}] = \int_0^{\tau} dt\ \left[T\hat{x}(t)^2 + i\hat{x}(t)\{\dot{x} - f\} \right].$$

Integrate over $\hat{x}$ to get Onsager-Machlup.

- FTs are obtained from transformation properties of fields

$$\begin{cases} x(t) & \rightarrow & \tilde{x}(t) \equiv x(\tau - t) \\ \hat{x}(t) & \rightarrow & \tilde{\hat{x}}(t) \equiv \hat{x}(\tau - t) - \frac{i}{T}\frac{d}{dt}x(\tau - t) \end{cases}$$

- Equilibrium (no λ_t): Symmetry in action. Equilibrium FDR $\leftarrow$ Ward identity
- Nonequilibrium: Broken symmetry. FT follows from the change in action

Symmtery and its Breaking in Classical Systems

- Total action

$$S_{\rm tot}[x,\hat{x};\lambda] \equiv S_{\rm MSRJD}[x,\hat{x};\lambda] - \ln P_i(x(0);\lambda_0)$$

- Equilibrium (no λ): $f(x) = -\boldsymbol{\nabla} U(x)$

$$S_{\rm tot}[\tilde{x},\tilde{\hat{x}}] = S_{\rm tot}[x,\hat{x}]$$

- Nonequilibrium: $f(x;\lambda) = -\boldsymbol{\nabla} U(x;\lambda)$

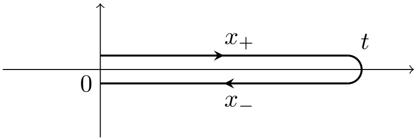
$$S_{\rm tot}[\tilde{x},\tilde{\hat{x}};\tilde{\lambda}] = S_{\rm tot}[x,\hat{x};\lambda] + \frac{1}{T}[F_\tau - F_0] + \frac{1}{T}\underbrace{\int_0^\tau dt\ \dot{\lambda}_t \frac{\partial U}{\partial \lambda_t}}_{\Delta W}.$$

where

$$\tilde{\lambda}_t \equiv \lambda_{\tau - t}.$$

Why MSRJD? Road to Quantum: Schwinger-Keldysh

- MSRJD is classical limit of SK
- Quantum dynamics: $\rho(t) = U(t,0)\rho(0)U^\dagger(t,0)$
- Forward and backward paths



- "Classical" and "Quantum" fields:

$$x_c(t) = \frac{1}{2}(x_+(t) + x_-(t)), \qquad x_q(t) = x_+(t) - x_-(t)$$

- Classical limit (for dissipative quantum systems): Connection with MSR fields

$$x_c(t) \to x(t), \qquad x_q(t) \to \hbar\hat{x}(t)$$

Transformation for Schwinger-Keldysh fields

Q: Can we find transformations for $x_\pm$ that correspond to MSRJD ones in classical limit?

We also want

- The action is symmetric for equilibrium case (no time dependent protocol)
- Symmetry is broken for nonequilibrium case
- The change in the action for nonequilibrium leads to quantum FT
- Identify quantum work as a functional of quantum path? (not based on two-measurement scheme)

Previous Works and What We Do

- Previous Works
  - Funo and Quan, Phys. Rev. Lett. **121**, 040601 (2018) : Path Integral Formalism for TPM work.
  - Sieberer, *et al.*, Phys. Rev. B **92**, 134307 (2015)
    – Established symmetry for equilibrium; Nonequilibrium was not considered
    – Dynamics from $t = -\infty$ to $t = \infty$; Fourier components of fields are considered.
    – Hard to apply to get finite-time changes.
  - Aron, Biroli and Cugliandolo, SciPost Phys. **4**, 008 (2018)
    – Closed system (but with time-dependent protocol)
    – Deformation of Keldysh contour for a finite-time interval
    – No dissipation into a reservoir.
- Present Work
  – Generalize Aron et al. to an open system with a reservoir.
  – Establish quantum formalism whose classical limit is classical Langevin systems

Treat a heat bath as a collection of harmonic oscillators:

- System

$$H_S = \frac{p^2}{2m} + V(x, \lambda_t)$$

- Bath

$$H_B = \sum_n \left(\frac{p_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 q_n^2 \right)$$

- Interaction:

$$H_I = -x \sum_n c_n q_n + \sum_n \frac{c_n^2}{2 m_n \omega_n^2} x^2$$

- Total Hamiltonian $H_{\rm tot} = H_S + H_B + H_I$

$$H_{\rm tot} = \frac{p^2}{2m} + V(x, \lambda_t) + \sum_n \left(\frac{p_n^2}{2m_n} + \frac{m_n \omega_n^2}{2} (q_n - \frac{c_n}{m_n \omega_n^2} x)^2 \right)$$

Reduced Density Matrix

- Density matrix of the total system: $\rho(t) = U(t,0)\rho(0)U^\dagger(t,0)$ where

$$U(t,0) = \mathbb{T}\exp(-\frac{i}{\hbar}\int_0^t H(t')dt')$$

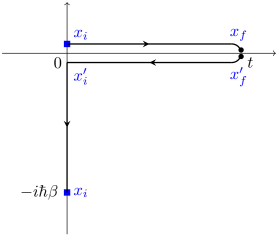
- Reduced density matrix:

$$\rho_r(t) \equiv \mathrm{Tr}_B \rho(t)$$

- In this work, we focus on the case where the system and the bath are initially at equilibrium, i.e.

$$\rho(0) = \frac{1}{Z_\beta} e^{-\beta H(0)}, \qquad Z_\beta = \mathrm{Tr} e^{-\beta H(0)}$$

$$\rho_r(x_f, x_f'; t) = \frac{1}{Z}\int dx_i \int dx_i' \int_{x_i}^{x_f} \mathcal{D}x \int_{x_i'}^{x_f'} \mathcal{D}x' \int_{x_i'}^{x_i} \mathcal{D}\bar{x}$$
$$\times \exp\left[\frac{i}{\hbar}(S_S[x] - S_S[x']) - \frac{1}{\hbar}S_S^E[\bar{x}] - \frac{1}{\hbar}\Psi[x, x', \bar{x}]\right]$$



- $Z = Z_\beta / Z_B$ with $Z_B = \mathrm{Tr}_B e^{-\beta H_B}$
- Ψ: Feynman-Vernon Influence Functional

Path Integral on Deformed Keldysh Contour

- To find desired transformations for $x_\pm$, we need to deform the Kadanoff-Baym contour
- Based on Aron *et al.* (2018).
- Usual approach: Insert a completeness relation at a time slice t_k

$$1 = \int dx_k d\boldsymbol{q}_k \, |x_k, \boldsymbol{q}_k\rangle\langle x_k, \boldsymbol{q}_k|$$

alternative form

$$1 = \int dx_k d\boldsymbol{q}_k \; e^{i\theta(t_k)H(t_k)/\hbar}|x_k, \boldsymbol{q}_k\rangle\langle x_k, \boldsymbol{q}_k|e^{-i\theta(t_k)H(t_k)/\hbar}$$

- Use $\theta_\pm(t)$ for forward and backward paths

Equilibrium

Time-independent Hamiltonian

- In the k-th time slice (forward path)

$$\begin{aligned}&\langle x_{k+1}, \boldsymbol{q}_{k+1} | e^{-i\theta_+(t_{k+1})H_{\rm tot}/\hbar} e^{-iH_{\rm tot}(dt)/\hbar} e^{i\theta_+(t_k)H_{\rm tot}/\hbar} | x_k, \boldsymbol{q}_k\rangle \\ =&\langle x_{k+1}, \boldsymbol{q}_{k+1} | e^{-i(1+\partial_t\theta_+(t_k))(dt)H_{\rm tot}/\hbar} | x_k, \boldsymbol{q}_k\rangle,\end{aligned}$$

- Reparametrization of time

$$\boxed{dz = (1+\partial_t\theta_+(t))dt, \quad or \quad z(t) = t + \theta_+(t)}$$

- Lagrangian

$$\text{(k-th matrix element)} = \exp\left[\frac{i}{\hbar}(dz)\mathcal{L}(x(z), \boldsymbol{q}(z))\right]$$

Equilibrium

Lagrangians

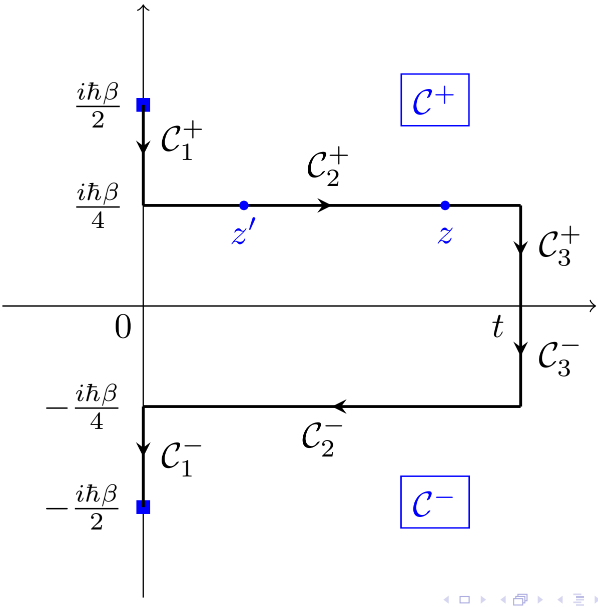
- $\mathcal{L}_{\rm tot}^{\pm} = \mathcal{L}_{\rm S}^{\pm} + \mathcal{L}_{\rm B}^{\pm} + \mathcal{L}_{\rm I}^{\pm}$

$$\mathcal{L}_{\rm S}^{\pm} = \frac{m}{2}\left(\frac{dx_{\pm}}{dz_{\pm}}\right)^2 - V(x_{\pm}(z_{\pm})),$$

$$\mathcal{L}_{\rm B}^{\pm} = \sum_n \left[\frac{m_n}{2}\left(\frac{dq_{\pm,n}}{dz_{\pm}}\right)^2 - \frac{1}{2}m_n\omega_n^2 q_{\pm,n}^2(z_{\pm})\right],$$

$$\mathcal{L}_{\rm I}^{\pm} = x_{\pm}(z_{\pm})\sum_n c_n q_{\pm,n}(z_{\pm}) - \frac{\mu}{2}x_{\pm}^2(z_{\pm}).$$

Deformed Path: $\theta_{\pm}(t) = \pm i\hbar\beta/4$



Integration over Bath Variables

- Upon integrating over $\boldsymbol{q}_\pm$, we obtain

$$1 = \frac{1}{Z(0)} \int dx_i \int dx_f \int_{x_i}^{x_f} \mathcal{D}x_+(z) \int_{x_f}^{x_i} \mathcal{D}x_-(z)$$
$$\times \exp\left(\frac{i}{\hbar} S_+[x_+] + \frac{i}{\hbar} S_-[x_-] - \frac{1}{\hbar} \Psi[x_+, x_-] \right),$$

where

$$S_\pm[x_\pm] = \int_{\mathcal{C}_\pm} dz \, \mathcal{L}_{\mathrm{S}}^{\pm}(x_\pm(z))$$

is the system action

Influence Functional

- The effect of the coupling to the bath is reflected in

$$\Psi[x_+, x_-] = \int_{C_+} dz \int_{C_+,\ z>z'} dz'\ x_+(z)K(z-z')x_+(z') + \int_{C_-} dz \int_{C_-,\ z>z'} dz'\ x_-(z)K(z-z')x_-(z') + \int_{C_-} dz \int_{C_+} dz'\ x_-(z)K(z-z')x_+(z'),$$

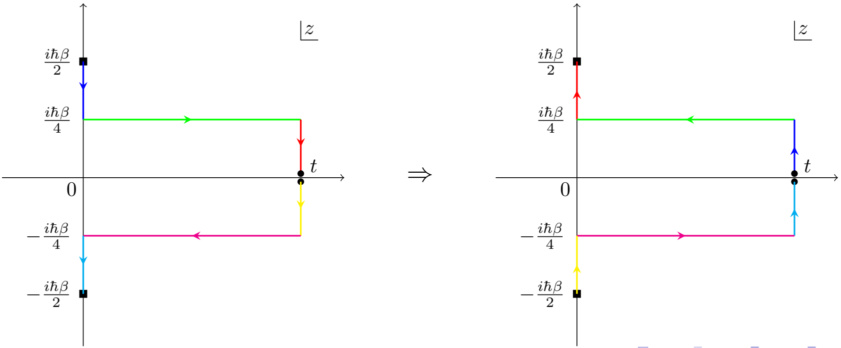
where

$$K(z) \equiv \sum_n \frac{c_n^2}{2m_n\omega_n} \frac{\cosh(\frac{1}{2}\beta\hbar\omega_n - i\omega_n z)}{\sinh(\frac{1}{2}\beta\hbar\omega_n)}$$

Equilibrium Symmetry

Field Transformations

$$x_+(z) \to \tilde{x}_+(z) \equiv x_+\left(t - z + \frac{i\hbar\beta}{2}\right)$$
$$x_-(z) \to \tilde{x}_-(z) \equiv x_-\left(t - z - \frac{i\hbar\beta}{2}\right)$$



Equilibrium Symmetry

One can show

Equilibrium Symmetry

$$S_\pm[\tilde{x}_\pm] = S_\pm[x_\pm], \qquad \Psi[\tilde{x}_+, \tilde{x}_-] = \Psi[x_+, x_-]$$

Consequence of Equilibrium Symmetry

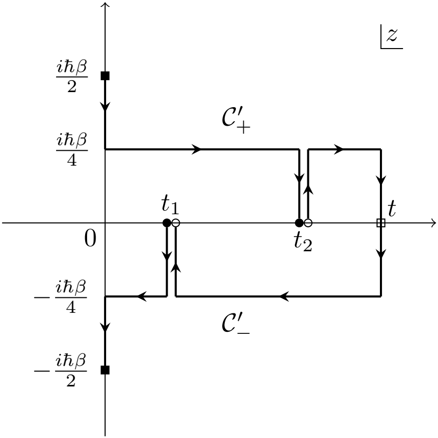
Equilibrium Fluctuation Dissipation Relations

- Apply field transformation to correlation functions (Green's functions)

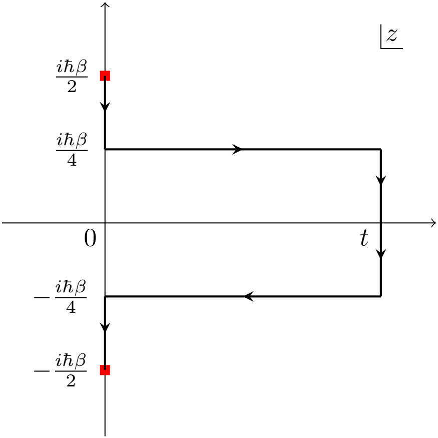
$$iG^{>}(t_1, t_2) = \mathrm{Tr}[\hat{x}(t_1)\hat{x}(t_2)\hat{\rho}(0)] = \langle x_-(t_1)x_+(t_2)\rangle$$
$$iG^{<}(t_1, t_2) = \mathrm{Tr}[\hat{x}(t_2)\hat{x}(t_1)\hat{\rho}(0)] = \langle x_+(t_1)x_-(t_2)\rangle,$$

where $\hat{x}(t) = e^{i\hat{H}_{\rm tot}t/\hbar}\hat{x}e^{-i\hat{H}_{\rm tot}t/\hbar}$.

Equilibrium FDR



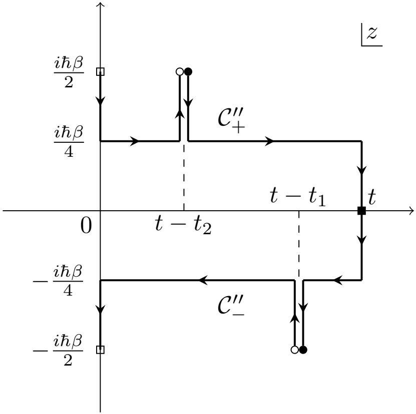
Contour used in $S_\pm$ for $iG^>(t_1, t_2)$



Contour for dissipative part, $\Psi[x_+, x_-]$

Equilibrium FDR

- Contour after field transformations



- $e^{-\beta \hat{H}_{\rm tot}/2} \hat{x} e^{\beta \hat{H}_{\rm tot}/2}$ at $t - t_2$
- $e^{\beta \hat{H}_{\rm tot}/2} \hat{x} e^{-\beta \hat{H}_{\rm tot}/2}$ at $t - t_1$

Equilibrium FDR

- Equilibrium symmetry gives

FDR

$$G^>(t_1, t_2) = G^<(t - t_2 + \frac{i\hbar\beta}{2}, t - t_1 - \frac{i\hbar\beta}{2})$$

Using $\tau \equiv t_1 - t_2$,

$$G^>(\tau) = G^<(\tau + i\hbar\beta)$$

- Relationship with retarded, advanced and Keldysh Green's functions

$$G^R(\tau) - G^A(\tau) = G^>(\tau) - G^<(\tau), \quad G^K(\tau) = G^>(\tau) + G^<(\tau)$$

$$\cosh\left(\frac{i\hbar\beta}{2}\partial_\tau\right)[G^{\mathrm{R}}(\tau) - G^{\mathrm{A}}(\tau)] = 2\sinh\left(\frac{i\hbar\beta}{2}\partial_\tau\right)G^{\mathrm{K}}(\tau).$$

- Evaluate $\langle x_{k+1}, \boldsymbol{q}_{k+1} | M_k^+ | x_k, \boldsymbol{q}_k \rangle$

$$
\begin{aligned}
M_k^+ =& e^{-\frac{i}{\hbar}\theta_+(s_{k+1})H_{\rm tot}(s_{k+1})} e^{-\frac{i}{\hbar}H_{\rm tot}(s_k)ds} e^{\frac{i}{\hbar}\theta_+(s_k)H_{\rm tot}(s_k)} \\
\simeq& 1 - \frac{i}{\hbar} ds (1+\dot{\theta}_+(s_k)) H_{\rm tot}(s_k) \\
& - \frac{i}{\hbar} ds \theta_+(s_k) \int_0^1 d\xi \; e^{-\frac{i}{\hbar}\xi\theta_+(s_k)H_{\rm tot}(s_k)} (\partial_s H_{\rm tot}(s_k)) e^{\frac{i}{\hbar}\xi\theta_+(s_k)H_{\rm tot}(s_k)},
\end{aligned}
$$

- Identity

$$
\frac{d}{dt} e^{O(t)} = \int_0^1 d\xi \; e^{\xi O(t)} \frac{dO(t)}{dt} e^{(1-\xi)O(t)}
$$

- On $\mathcal{C}_2^+$, modified system Lagrangian $\widehat{\mathcal{L}}_{\rm S}^+$ at time slice s_k

$$e^{\frac{i}{\hbar}ds\widehat{\mathcal{L}}_{\rm S}^+} = \langle x_{k+1}|1 - \frac{i}{\hbar}ds(H_{\rm S}(s_k) + F_+(s_k))|x_k\rangle$$

where

$$F_+(s) \equiv i\frac{\beta\hbar}{4}\int_0^1 d\xi\ e^{\frac{1}{4}\xi\beta H_{\rm S}(s)}\partial_s H_{\rm S}(s)e^{-\frac{1}{4}\xi\beta H_{\rm S}(s)}.$$

- On $\mathcal{C}_2^-$ modified system Lagrangian $\widehat{\mathcal{L}}_{\rm S}^-$ at time slice s_k

$$e^{-\frac{i}{\hbar}ds\widehat{\mathcal{L}}_{\rm S}^-} = \langle x_k|1 + \frac{i}{\hbar}ds(H_{\rm S}(s_k) + F_-(s_k))|x_{k+1}\rangle,$$

where

$$F_-(s) \equiv -i\frac{\beta\hbar}{4}\int_0^1 d\xi\ e^{-\frac{1}{4}\xi\beta H_{\rm S}(s)}\partial_s H_{\rm S}(s)e^{\frac{1}{4}\xi\beta H_{\rm S}(s)}.$$

- Modified system action

$$\widehat{S}_{\pm}[x_{\pm};\lambda] = \widehat{S}_1^{\pm}[x_{\pm};\lambda_0] + \widehat{S}_3^{\pm}[x_{\pm};\lambda_t] + \widehat{S}_2^{\pm}[x_{\pm};\lambda],$$

- $\widehat{S}_1$ and $\widehat{S}_3$ use the same Lagrangian

$$\widehat{S}_1^{\pm}[x_{\pm};\lambda_0] = \int_{\mathcal{C}_1^{\pm}} dz\, \mathcal{L}_{\mathrm{S}}^{\pm}(x_{\pm}(z), \dot{x}_{\pm}(z); \lambda_0)$$

$$\widehat{S}_3^{\pm}[x_{\pm};\lambda_t] = \int_{\mathcal{C}_3^{\pm}} dz\, \mathcal{L}_{\mathrm{S}}^{\pm}(x_{\pm}(z), \dot{x}_{\pm}(z); \lambda_t)$$

- Modified Lagrangian for $\widehat{S}_2$:

$$\widehat{S}_2^{\pm}[x_{\pm};\lambda] = \pm \int_0^t ds\, \widehat{\mathcal{L}}_{\mathrm{S}}^{\pm}(x_{\pm}(s \pm \frac{i\hbar\beta}{4}), \dot{x}_{\pm}(s \pm \frac{i\hbar\beta}{4}); \lambda_s)$$

- Behavior of modified actions under field transformations

$$\widehat{S}_1^{\pm}[\tilde{x}_\pm;\lambda_0] = \widehat{S}_3^{\pm}[x_\pm;\lambda_0] \equiv \widehat{S}_3^{\pm}[x_\pm;\tilde{\lambda}_t]$$
$$\widehat{S}_3^{\pm}[\tilde{x}_\pm;\lambda_t] = \widehat{S}_1^{\pm}[x_\pm;\lambda_t] \equiv \widehat{S}_1^{\pm}[x_\pm;\tilde{\lambda}_0],$$

where time-reversed protocol: $\tilde{\lambda}_s \equiv \lambda_{t-s}$

$$\widehat{S}_2^{\pm}[\tilde{x}_\pm;\lambda] = \widehat{S}_2^{\pm}[x_\pm;\tilde{\lambda}] \pm \Sigma_\pm[x_\pm;\tilde{\lambda}],$$

with

$$\Sigma_\pm[x_\pm;\tilde{\lambda}] = -\Sigma_\pm[\tilde{x}_\pm;\lambda].$$

Quantum FT

- Consider

$$\left\langle e^{\frac{i}{\hbar}(\Sigma_+[x_+;\lambda]+\Sigma_-[x_-;\lambda])}\right\rangle = \frac{1}{Z(0)}\int dx_i \int dx_f \int_{x_i}^{x_f}\mathcal{D}x_+ \int_{x_f}^{x_i}\mathcal{D}x_-$$
$$\times\ e^{\frac{i}{\hbar}(\Sigma_+[x_+;\lambda]+\Sigma_-[x_-;\lambda])} e^{\frac{i}{\hbar}\widehat{S}_+[x_+;\lambda]+\frac{i}{\hbar}\widehat{S}_-[x_-;\lambda]-\frac{1}{\hbar}\Psi[x_+,x_-]},$$

- Change the path integral variables from $x_\pm$ to $\tilde{x}_\pm$: Jacobian=1

$$\left\langle e^{\frac{i}{\hbar}(\Sigma_+[x_+;\lambda]+\Sigma_-[x_-;\lambda])}\right\rangle = \frac{1}{Z(0)}\int dx_i \int dx_f \int_{x_f}^{x_i}\mathcal{D}x_+ \int_{x_i}^{x_f}\mathcal{D}x_-$$
$$\times\ e^{\frac{i}{\hbar}\widehat{S}_+[x_+;\tilde{\lambda}]+\frac{i}{\hbar}\widehat{S}_-[x_-;\tilde{\lambda}]-\frac{1}{\hbar}\Psi[x_+,x_-]}$$

- Normalization with the reverse protocol $\tilde{\lambda}$ except for the factor of $1/Z(0)$.

$$\left\langle e^{\frac{i}{\hbar}(\Sigma_+[x_+;\lambda]+\Sigma_-[x_-;\lambda])}\right\rangle = \frac{Z(t)}{Z(0)} \equiv e^{-\beta(\mathcal{F}(t)-\mathcal{F}(0))},$$

- If we identify

QM work-like quantity on fluctuating trajectory

$$\frac{i}{\hbar}(\Sigma_+[x_+;\lambda]+\Sigma_-[x_-,\lambda]) \equiv -\beta\mathcal{W}$$

- we have

Jarzynski-like Equality

$$\left\langle e^{-\beta\mathcal{W}}\right\rangle = e^{-\beta\Delta\mathcal{F}}$$

- To the lowest order of $O(\hbar)$,

$$F_{\pm}(s) \simeq \pm \frac{i\hbar\beta}{4} \dot{\lambda}_s \partial_\lambda V,$$

and

$$\Sigma_{\pm}[x_{\pm}, \lambda] \simeq \frac{i\hbar\beta}{2} \int_0^t ds \, \dot{\lambda}_s \partial_\lambda V(x_{\pm}(s \pm \frac{i\hbar\beta}{4}), \lambda_s).$$

- To the leading order in $\hbar$

$$\mathcal{W} \simeq \frac{1}{2} \int_0^t ds \, \dot{\lambda}_s \partial_\lambda \Big[V(x_+(s); \lambda_s) + V(x_-(s); \lambda_s) \Big].$$

Example

Pulled harmonic oscillator

- System Hamiltonian

$$H_S = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 (x - \lambda_s)^2$$

- Difference

$$\Sigma_\pm[x_\pm; \lambda] = -2im\omega_0 \int_0^t ds\, \dot{\lambda}_s (x_\pm(s \pm \frac{i\hbar\beta}{4}) - \lambda_s) \sinh(\frac{\beta\hbar\omega_0}{4}).$$

QM work: pulled harmonic oscillator

$$\mathcal{W} = -\frac{4}{\beta\hbar\omega_0} \sinh(\frac{\beta\hbar\omega_0}{4}) \int_0^t ds\, m\omega_0^2 \dot{\lambda}_s (x_c(s) - \lambda_s),$$

where

$$x_c(s) \equiv \frac{1}{2}[x_+(s + \frac{i\hbar\beta}{4}) + x_-(s - \frac{i\hbar\beta}{4})]$$

Classical Limit

Saddle point (small $\hbar$) analysis in terms of

classical and quantum fields

$$x_c(s) \equiv \frac{1}{2}\left[x_+(s+\frac{i\hbar\beta}{4}) + x_-(s-\frac{i\hbar\beta}{4})\right]$$

$$x_q(s) \equiv x_+(s+\frac{i\hbar\beta}{4}) - x_-(s-\frac{i\hbar\beta}{4})$$

Connection with MSRJD fields

$$r(s) \equiv \frac{1}{2}(x_+(s)+x_-(s)), \qquad \hat{x}(s) \equiv \lim_{\hbar\to 0}\frac{1}{\hbar}(x_+(s)-x_-(s)).$$

$$x_c(s) = r(s) + O(\hbar^2), \qquad x_q(s) = \hbar(\hat{x}(x) + \frac{i\beta}{2}\dot{r}(s)) + O(\hbar^2).$$

Classical Limit of Influence Functional

- Write $K(s) = N(s) - \frac{i}{2}D(s)$ where

$$N(s) = \sum_n \frac{c_n^2}{2m_n\omega_n} \coth(\frac{1}{2}\beta\hbar\omega_n)\cos(\omega_n s), \quad D(s) = \sum_n \frac{c_n^2}{m_n\omega_n} \sin(\omega_n s).$$

- In the $\hbar \to 0$ limit, $N(s) \to (1/\hbar)(1/\beta)\gamma(s)$, where

$$\gamma(s) \equiv \sum_n \frac{c_n^2}{m_n\omega_n^2} \cos(\omega_n s).$$

- Classical Limit of Influence Functional

$$\lim_{\hbar\to 0} \frac{1}{\hbar}\Psi = \frac{1}{2\beta}\int_0^t ds \int_0^t du\, \hat{x}(s)\gamma(s-u)\hat{x}(u) + i\int_0^t ds\, \hat{x}(s)\int_0^s du\, \gamma(s-u)\dot{r}(u).$$

Classical Limit of Path Integral

- In the classical limit, quantum path integral behaves as a path integral over $Z^{-1}(0)\int \mathcal{D}r(s)\int \mathcal{D}\hat{x}\ \exp[S_{\rm MSRJD}[r,\hat{x}]]$, where

$$
\begin{aligned}
S_{\rm MSRJD} = &-\frac{1}{2\beta}\int_0^t ds \int_0^t du\ \hat{x}(s)\gamma(s-u)\hat{x}(u) \\
&- i\int_0^t ds\ \hat{x}(s)\int_0^s du\ \gamma(s-u)\dot{r}(u) \\
&- i\int_0^t ds\ \hat{x}(s)\{m\ddot{r}(s)+\partial_r V(r(s);\lambda_s)\} - \beta\mathcal{H}_{\rm S}(0).
\end{aligned}
$$

- This is exactly the MSRJD action for the generalized Langevin equation

$$
m\ddot{r}(s)+\partial_r V(r;\lambda_s)+\int_0^s du\ \gamma(s-u)\dot{r}(u)=\xi(s), \tag{1}
$$

where the noise $\xi(s)$ satisfies

$$
\langle\xi(s)\rangle = 0, \qquad \langle\xi(s)\xi(s')\rangle = \frac{1}{\beta}\gamma(s-s'). \tag{2}
$$

Classical Limit of Field Transformations

$$\tilde{x}_c(s) = \frac{1}{2}[x_+(t-s+i\hbar\beta/4) + x_-(t-s-i\hbar\beta/4)] = r(t-s) + O(\hbar^2).$$

$$\tilde{r}(s) = r(t-s).$$

$$\tilde{x}_q(s) = x_+(t-s+i\hbar\beta/4) - x_-(t-s-i\hbar\beta/4)$$
$$= \hbar[\hat{x}(t-s) - \frac{i\beta}{2}\partial_s r(t-s)] + O(\hbar^3).$$

On the other hand, we have $\tilde{x}_q(s) = \hbar[\tilde{\hat{x}}(s) + (i\beta/2)\partial_s \tilde{r}(s)]$.

$$\tilde{\hat{x}}(s) = \hat{x}(t-s) - i\beta\partial_s r(t-s).$$

These are transformations used in classical stochastic systems to obtain FTs

Summary and Conclusion

- Identified the field transformations for open quantum systems which leave the action invariant in equilirium
- FDR follows from the symmetry
- When there is an external time-dependent protocol, the action is not invariant
- In nonequilibrium, using the change in the action under the transformation, we were able to find a version of quantum FT (not based on TPM)
- In the process, we identified quantum work defined on the quantum path.
- Is it possible to identify "entropy production" from these transformations? (e.g. no time-dependent protocol but initial product state of system and bath or a general initial state)